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WEIGHTED COMPOSITION OPERATOR VALUED INTEGRAL

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In the paper we are going to introduce a weighted composition operator valued integral on $L^2(X)$ with a weakly integrable weight function $u : X \to H$ and we will consider some classic properties of these kind operators. Then we will give the necessary and sufficient condition for uniqueness dual pair of u.

1. Introduction

Let (X,Σ,μ) be a σ-finite measure space and let $\varphi : X \to X$ be a measurable transformation, that is, $\varphi^{-1}(\Sigma) \subseteq \Sigma$. If $\mu(\varphi^{-1}(A)) = 0$ for all $A \in \Sigma$ with $\mu(A) = 0$, then φ is said to be non-singular. This condition means that the measure $\mu \circ \varphi^{-1}$, defined by $(\mu \circ \varphi^{-1})(A) := \mu(\varphi^{-1}(A))$ for $A \in \Sigma$, is absolutely continuous with respect to μ (it is usually called push forward of μ through φ, denoted by $\mu_\sharp$). Here the non-singularity of φ guarantees that the operator $f \to f \circ \varphi$ is well defined as a mapping on $L^0(\Sigma)$ where, $L^0(\Sigma)$ denote the linear space of all equivalence classes of Σ-measurable functions on X. Let h_0 be the Radon-Nikodym derivative $\frac{d\mu\circ\varphi^{-1}}{d\mu}$ and we always assume that h_0 is almost everywhere finite-valued or, equivalently, $\varphi^{-1}(\mathcal{A})$ is σ-finite, for any σ-finite subalgebra $\mathcal{A} \subseteq \Sigma$. The L^p-space $L^p(X,\mathcal{A},\mu_{|\mathcal{A}})$ is abbreviated by $L^p(\mathcal{A})$. The support of

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a measurable function f is defined by $supp(f) = \{x \in X : f(x) \neq 0\}$ and also spectrum of mesurable function f is denoted by $\sigma(f)$. All comparisons between two functions or two sets are to be interpreted as holding up to μ-null set.

The conditional expectation operator associated with sigma-finite algebra $\mathcal{A}$ is the mapping $f \mapsto E^{\mathcal{A}}(f)$ defined for all non-negative $f \in L^0(\Sigma)$ as well as for all $f \in L^p(\Sigma), 1 \leq p \leq \infty$ where $E^{\mathcal{A}}(f)$ is the unique A-measurable function satisfy

$$\int_A f d\mu = \int_A E^{\mathcal{A}}(f) d\mu, \quad A \in \mathcal{A}.$$

For $p = 2$ and $\mathcal{A} = \varphi^{-1}(\Sigma)$ we may interpret the conditional expectation operator $E := E^{\varphi^{-1}(\Sigma)}$ as a contractive orthogonal projection onto $L^2(\varphi^{-1}(\Sigma)) = \overline{R(C_\varphi)}$, the closure of the range of composition operator$C_\varphi(f) = f \circ \varphi$ or $L^2(\Sigma)$ (see [5]).
For each $f \in L^2(\Sigma)$, there exists a unique $g \in L^2(\Sigma)$ with $supp(g) \subseteq supp(h_0)$ such that $E(f) = g \circ \varphi$. We then write $g = E(f) \circ \varphi^{-1}$ though we make no assumptions regarding the invertibility of φ (see [2]).
Those properties of E used in our discussion are summarized below. In all cases f and g are conditionable functions.
(i) For $f \in L^2(\mathcal{A})$ and $g \in L^2(\Sigma)$, $E(fg) = fE(g)$.
(ii) If $f \geq 0$ then $E(f) \geq 0$, if $f > 0$ then $E(f) > 0$.
(iii) For $f \in L^2(\Sigma)$ and $p \geq 1$, $\mid E(f) \mid^p \leq E(\mid f \mid^p)$.

A detailed discussion and verification of most of these properties maybe found in [2, 3]. The authors in [1] introduced an operator-valued integral of a square modulus weakly integrable mapping the ranges of which are Hilbert spaces, as bounded integrals. In the section 2 we will consider some classic properties the *wco*-valued integral operator. Then in section 4 we will give the necessary and sufficient condition for uniqueness dual pair of u. We denote by $\langle .,. \rangle$ inner product in H and $(.,.)$ denotes inner product in $L^2(X,H)$.

2. Basic definitions and preleminiares

For a given complex Hilbert space H, let $L^2(X,H)$ be the class of all measurable mapping $f : X \to H$ such that $\|f\|_2^2 = \int_X \|f(x)\|^2 d\mu < \infty$. It follows that $L^2(X,H)$ is a Hilbert space with the inner product defined by

$$(f,g) = \int_X \langle f(x), g(x) \rangle d\mu,$$

and for each $f,g \in L^2(X,H)$, the mapping $x \mapsto \langle f(x), g(x) \rangle$ is measurable. Recall that $f : X \to H$ is said to be weakly measurable if for each $h \in H$ the mapping

$x \mapsto \langle h, f(x)\rangle$ is measurable. We shall write $L^2(X)$ when $H = \mathbb{C}$.
Let $u : X \to H$ be a weakly measurable function. We say that u has weakly bounded integral(or *wbi*-condition for H) if there exist $A > 0$, $B > 0$ such that

$$A\|h\|^2 \leq \int_X |\langle h, u(x)\rangle|^2 d\mu \leq B\|h\|^2, \quad h \in H.$$

Also u has semi-weakly bounded integral (or *swbi*-condition for H) if for each $h \in H$, $\int_X |\langle h, u(x)\rangle|^2 d\mu \leq B\|h\|^2$, for some $B > 0$. Note that if $u \in L^2(X,H)$, it is easy to see that $\int_X |\langle h, u(x)\rangle|^2 d\mu \leq B\|h\|^2$, for some $B > 0$.

Definition 2.1. Let $\varphi : X \to X$ be a non-singular measurable transformation, $h_0 := \frac{d\mu o\varphi^{-1}}{d\mu}$ is essentially bounded and let $u : X \to H$ satisfies in the *swbi*-condition for H. The *wco*-valued integral operator $T_u^\varphi : L^2(X) \to H$ associated with the pair (u, φ) is defined by

$$\langle T_u^\varphi(f), h\rangle = \int_X \langle u(x), h\rangle (f \circ \varphi)(x) d\mu, \quad h \in H, f \in L^2(X).$$

It is evident that T_u^φ is well defined and linear, and we have

$$\|T_u^\varphi(f)\| = \sup_{h \in H_1} |\langle T_u^\varphi(f), h\rangle|$$

, Which H_1 is the closed unit ball of H. Since h_0 is essentially bounded, we have

$$\begin{aligned}
\|T_u^\varphi(f)\| &= \sup_{h\in H_1} |\langle T_u^\varphi(f), h\rangle| \\
&= \sup_{h\in H_1} |\int_X \langle u(x), h\rangle (f \circ \varphi)(x) d\mu| \\
&\leq \sup_{h\in H_1} (\int_X |(f \circ \varphi)(x)|^2 d\mu)^{\frac{1}{2}} (\int_X |\langle u(x), h\rangle|^2 d\mu)^{\frac{1}{2}} \\
&\leq (\textstyle\int_X h_0 |f(x)|^2 d\mu)^{\frac{1}{2}} \sup_{h\in H_1} (B\|h\|^2)^{\frac{1}{2}} \\
&\leq \|h_0\|_\infty^{\frac{1}{2}} \|f\|_{L^2} B^{\frac{1}{2}} < \infty.
\end{aligned}$$

We shall denote $T_u^\varphi : L^2(X) \to H$ by $T_u^\varphi(f) = \int_X u.f \circ \varphi d\mu$.

Now, since T_u^φ is bounded, then for each $f \in L^2(X)$ and $h \in H$ we obtain

$$\begin{aligned}
(f, (T_u^\varphi)^*(h)) &= \langle T_u^\varphi(f), h\rangle \\
&= \textstyle\int_X \langle u(x), h\rangle (f \circ \varphi)(x) d\mu \\
&= \textstyle\int_X E(\langle u, h\rangle)(x) (f \circ \varphi)(x) d\mu \\
&= \textstyle\int_X h_0(x) f(x) E(\langle u, h\rangle) \circ \varphi^{-1}(x) d\mu \\
&= (f, h_0 E(\langle h, u\rangle) \circ \varphi^{-1}).
\end{aligned}$$

Therefor we can write $(T_u^\varphi)^*(h) = h_0 E(\langle h,u\rangle)\circ\varphi^{-1}$, and so for each $h \in H$, $\|(T_u^\varphi)^*(h)\|^2 = \int_X \|h_0(x)E(\langle h,u\rangle)\circ\varphi^{-1}(x)\|^2 d\mu$. It follows that

$$\|(T_u^\varphi)^*\| = \|T_u^\varphi\| = \sup_{h\in H_1}(\int_X \|h_0(x)E(\langle h,u\rangle)\circ\varphi^{-1}(x)\|^2 d\mu)^{\frac{1}{2}}.$$

We define $S_u^\varphi : H \to H$ by $S_u^\varphi(h) = T_u^\varphi(T_u^\varphi)^*(h)$, then we have

$$\begin{aligned} S_u^\varphi(h) &= (T_u^\varphi)(h_0.E(\langle h,u\rangle)\circ\varphi^{-1}) \\ &= \int_X (h_0\circ\varphi)E(\langle h,u\rangle)u d\mu. \end{aligned}$$

These observations establish the following proposition.

Proposition 2.2. *Let u satisfies in the swbi-condition for H. Then*

(i) If $h_0 \in L^\infty(\Sigma)$, then the operator T_u^φ is bounded.

(ii) If T_u^φ is bounded, then $(T_u^\varphi)^*(h) = h_0 E(\langle h,u\rangle)\circ\varphi^{-1}$.

Lemma 2.3. [1] *Let H be a Hilbert space. Then*

(i) If $dimH < \infty$ then $L^2(X,H)$ is the class of all mappings such that satisfy in the swbi-condition for H.

(ii) Let μ be a σ-finite measure. If there exists $f \in L^2(X,H)$ such that f satisfies in the wbi-condition for H, then $dimH < \infty$.

Theorem 2.4. [1] *Let $u : X \to H$ satisfies in the swbi-condition for H with upper bound B. Then the following assertation are equivalent:*

(i) The operator $S_u^\varphi : H \to H$ is invertible.

(ii) The operator $T_u^\varphi : L^2(X) \to H$ is surjective.

Let $u : X \to H$ satisfies in the *swbi*-condition for H and $E \subseteq X$ be measurable. Then it is clear that $u\chi_E : X \to H$ and $u|_E : E \to H$ satisfy in the *swbi*-condition for H. Also $L^2(E) = L^2(E, \Sigma_{|E}, \mu_{|\Sigma_{|E}}) = \{f|_E : f \in L^2(X)\}$. So we can embed $L^2(E)$ in $L^2(X)$ as a closed subspace. Since for each $h \in H$ and each $f \in L^2(X)$,

$$\int_E f\circ\varphi(x)\langle u(x),h\rangle d\mu = \int_X f\circ\varphi(x)\langle u(x)\chi_E(x),h\rangle d\mu.$$

So we can take $T^\varphi_{u|_E} = T^\varphi_{u\chi_E}$. Therefor for each $F,E \subseteq X$ measurable,

$$T^\varphi_{u|_E} + T^\varphi_{u|_F} = T^\varphi_{u\chi_E} + T^\varphi_{u\chi_F}.$$

Hence for disjoint E,F we have $T^\varphi_{u|_E} + T^\varphi_{u|_F} = T^\varphi_{u\chi_{(E\cup F)}} = T^\varphi_{u|_{(E\cup F)}}$.

3. The main results

Theorem 3.1. *Let u satisfies in the swbi-condition for H, and also let $\{E_i\}$ be a sequence of measurable subsets of X. Then*

(i) $\lim_{n\to\infty} \|T^{\varphi}_{u|_{\cup_{i=1}^n E_i}}\| = \|T^{\varphi}_{u|_{\cup_i E_i}}\|$.

(ii) If the sequence $\{E_i\}$ is pairwise disjoint, then $\sum_i T^{\varphi}_{u|_{E_i}} = T^{\varphi}_{u|_{\cup_i E_i}}$.

Proof. (i) Direct computation shows that

$$\begin{aligned}
\|(T^{\varphi}_{u|_{\cup_i E_i}})^*\|^2 &= \sup_{h\in H_1} \|(T^{\varphi}_{u|_{\cup_i E_i}})^*(h)\|^2 \\
&= \sup_{h\in H_1} \int_{\cup_i E_i} |h_0(x)E(\langle h,u\rangle)\circ\varphi^{-1}(x)|^2 d\mu \\
&= \sup_{h\in H_1} \lim_n \int_{\cup_{i=1}^n E_i} |h_0(x)E(\langle h,u\rangle)\circ\varphi^{-1}(x)|^2 d\mu \\
&= \lim_n \sup_{h\in H_1} \int_{\cup_{i=1}^n E_i} |h_0(x)E(\langle h,u\rangle)\circ\varphi^{-1}(x)|^2 d\mu \\
&= \lim_n \|(T^{\varphi}_{u|_{\cup_{i=1}^n E_i}})^*\|^2.
\end{aligned}$$

Thus $\lim_{n\to\infty} \|T^{\varphi}_{u|_{\cup_{i=1}^n E_i}}\| = \|T^{\varphi}_{u|_{\cup_i E_i}}\|$.

(ii) It is easily seen that

$$\begin{aligned}
\|T^{\varphi}_{u|_{\cup_i E_i}} - T^{\varphi}_{u|_{\cup_{i=1}^n E_i}}\|^2 &= \|T^{\varphi}_{u|_{\cup_{i=n}^\infty E_i}}\|^2 \\
&= \sup_{h\in H_1} \int_{\cup_{i=n}^\infty E_i} |h_0(x)E(\langle h,u\rangle)\circ\varphi^{-1}(x)|^2 d\mu \\
&= \sup_{h\in H_1} \int_X \sum_{i=n}^\infty \chi_{E_i} |h_0(x)E(\langle h,u\rangle)\circ\varphi^{-1}(x)|^2 d\mu \to 0
\end{aligned}$$

as $n\to\infty$. Therefor $T^{\varphi}_{u|_{\cup_i E_i}} = \lim_n \sum_{i=1}^n T^{\varphi}_{u|_{E_i}} = \sum_{i=1}^\infty T^{\varphi}_{u|_{E_i}}$.

□

Theorem 3.2. *Let K be a Hilbert space and $u : X \to H$ satisfies in the swbi-condition for H and $f : H \to K$ be a bounded linear mapping. Then*

(i) The mapping $fu : X \to K$ satisfies in the swbi-condition for K, and $fT^{\varphi}_u = T^{\varphi}_{fu}$.

(ii) T^{φ}_{fu} *is surjective if and only if* f *is surjective.*

Proof. (i) We have

$$\sup_{h\in H_1}\int_X |\langle h, f(u(x))\rangle|^2 d\mu \le \|f\|^2 \sup_{h\in H_1}\int_X |\langle h, u(x)\rangle|^2 d\mu.$$

Therefor if $f: H \to K$ be a bounded linear operator and $u: X \to H$ satisfies in the *swbi*-condition for H, then fu will be satisfies in the *swbi*-condition for K. Now if $g \in L^2(X)$, then for each $k \in K$ we have

$$\begin{aligned}\langle T^{\varphi}_{fu}(g), k\rangle &= \textstyle\int_X (g\circ\varphi)(x)\langle f(u(x)), k\rangle d\mu \quad (k \in K)\\ &= (g\circ\varphi, \langle fu, k\rangle)\\ &= (g\circ\varphi, \langle u, f^*k\rangle)\\ &= \langle T^{\varphi}_u(g), f^*k\rangle\\ &= \langle fT^{\varphi}_u(g), k\rangle.\end{aligned}$$

Hence $fT^{\varphi}_u = T^{\varphi}_{fu}$.

(ii) It is trivial. □

Theorem 3.3. *Let* u *satisfies in the wbi-condition for* H. *Then*

(i) $\sup_{h\in H_1} \| (T^{\varphi}_u)^*(h) \|^2 = \| S^{\varphi}_u \|$.

(ii) $\inf_{h\in H_1} \| (T^{\varphi}_u)^*(h) \|^2 = \| (S^{\varphi}_u)^{-1} \|^{-1}$.

Proof. (i) It is trivial.
(ii) Since $\inf_{h\in H_1} \|(T^{\varphi}_u)^*(h)\|^2 \le \|S^{\varphi}_u\| \le \sup_{h\in H_1} \|(T^{\varphi}_u)^*(h)\|^2$. So

$$(\sup_{h\in H_1} \|(T^{\varphi}_u)^*(h)\|^2)^{-1} \le \|S^{\varphi}_u\|^{-1} \le (\inf_{h\in H_1} \|(T^{\varphi}_u)^*(h)\|^2)^{-1}.$$

On the other hand, since $(S^{\varphi}_u)^{-1}u$ satisfies in the *swbi*-condition for H, then we obtain

$$S^{\varphi}_{(S^{\varphi}_u)^{-1}u} = T^{\varphi}_{(S^{\varphi}_u)^{-1}u}(T^{\varphi}_{((S^{\varphi}_u)^{-1}u)})^* = (S^{\varphi}_u)^{-1}T^{\varphi}_u((S^{\varphi}_u)^{-1}T^{\varphi}_u)^* = (S^{\varphi}_u)^{-1}.$$

It follows that

$$(\sup(\|(T^{\varphi}_{(S^{\varphi}_u)^{-1}u})^*(h)\|)^2)^{-1} \le \|S^{\varphi}_u\| \le (\inf(\|(T^{\varphi}_{(S^{\varphi}_u)^{-1}u})^*(h)\|)^2)^{-1}.$$

Hence

$$(\sup(\|(T^{\varphi}_{(S^{\varphi}_u)^{-1}u})^*(h)\|)^2)^{-1} \le \inf_{h\in H_1} \|(T^{\varphi}_u)^*(h)\|^2. \tag{1}$$

Similarly $\|S_u^\varphi\|^{-1} \leq (\inf_{h\in H_1} |(T_u^\varphi)^*(h)\|^2)^{-1}$ and also

$$\|S_u^\varphi\|^{-1} \leq \sup(\|(T^\varphi_{(S_u^\varphi)^{-1}u})^*(h)\|)^2.$$

Thus

$$\sup(\|(T^\varphi_{(S_u^\varphi)u^{-1}})^*(h)\|)^2 \leq (\inf_{h\in H_1} \|(T_u^\varphi)^*(h)\|^2)^{-1}. \tag{2}$$

Consequently by (1) and (2) we have

$$\begin{aligned}\inf_{h\in H_1} \|(T_u^\varphi)^*(h)\|^2 &= (\sup(\|(T^\varphi_{(S_u^\varphi)^{-1}u})^*(h)\|)^2)^{-1}\\ &= \|S^\varphi_{(S_u^\varphi)^{-1}u)}\|^{-1}\\ &= \|(S_u^\varphi)^{-1}\|^{-1}.\end{aligned}$$

□

Theorem 3.4. *Let f,g satisfies in the swbi-condition for H. Then the following assertion are equivalent:*

(i) For each $h \in H$, $h = T_f^\varphi(\langle h, g\circ\varphi^{-1}\rangle)$.

(ii) For each $h \in H$, $h = T_g^\varphi(\langle h, f\circ\varphi^{-1}\rangle)$.

(iii) For each $h,k \in H$, $\langle h,k\rangle = \int_X \langle h, f(x)\rangle\langle g(x),k\rangle d\mu$.

(iv) For each $h \in H$, $\langle h,h\rangle = \|h\|^2 = \int_X \langle h, f(x)\rangle\langle g(x),h\rangle d\mu$.

(v) For orthonormal bases $\{e_i\}_{i\in I}$ and $\{\gamma_j\}_{j\in J}$ for H

$$\langle e_i,\gamma_j\rangle = \int_X \langle e_i, f(x)\rangle\langle g(x),\gamma_j\rangle d\mu, \quad i\in I, j\in J.$$

(vi) For each orthonormal basis $\{e_i\}_{i\in I}$ for H

$$\langle e_i,e_j\rangle = \int_X \langle e_i, f(x)\rangle\langle g(x),e_j\rangle d\mu, \quad i,j\in I.$$

Proof. (i)⇒(ii) Let $h = T_f^{\varphi}(\langle h, g\circ\varphi^{-1}\rangle)$. For each $k \in H$ we have

$$\begin{aligned}\langle h,k\rangle &= \langle T_f^{\varphi}(\langle h, g\circ\varphi^{-1}\rangle), k\rangle\\ &= \textstyle\int_X \langle h, g\circ\varphi^{-1}\rangle\circ\varphi(x)\langle f(x),k\rangle d\mu\\ &= \textstyle\int_X \langle h, g(x)\rangle\langle f(x),k\rangle d\mu\\ &= \langle h, T_g^{\varphi}(\langle k, f\circ\varphi^{-1}\rangle)\rangle.\end{aligned}$$

Hence $k = T_g^{\varphi}(\langle k, f\circ\varphi^{-1}\rangle)$.

(ii)⇒(iii) Let $h = T_g^{\varphi}(\langle h, f\circ\varphi^{-1}\rangle)$ and $h,k \in H$, then

$$\langle h,k\rangle = \langle T_g^{\varphi}(\langle h, f\circ\varphi^{-1}\rangle), k) = \int_X \langle h, f(x)\rangle\langle g(x), h\rangle d\mu.$$

(iii)⇒(i) Let $h,k \in H$, then $\langle T_f^{\varphi}(\langle h, g\circ\varphi^{-1}\rangle), k\rangle = \int_X \langle f(x),k\rangle\langle h, g(x)\rangle d\mu = \langle h,k\rangle$. Thus $h = T_f^{\varphi}(\langle h, g\circ\varphi^{-1}\rangle)$.

(iv)⇒(i)

$$\begin{aligned}\langle h,h\rangle = \|h\|^2 &= \textstyle\int_X \langle h, f(x)\rangle\langle g(x), h\rangle d\mu\\ &= (h, T_f^{\varphi}(\langle h, g\circ\varphi^{-1}\rangle)),\end{aligned}$$

so $h = T_f^{\varphi}(\langle h, g\circ\varphi^{-1}\rangle)$.

The implications (iii)⇒(v), (iii)⇒(iv) and (v)⇒(vi) are evident.

(v)⇒(iii) For each $h,k \in H$, we get that

$$\begin{aligned}&\textstyle\int_X \langle h, f(x)\rangle\langle g(x),k\rangle d\mu\\ &\qquad = (\langle h,f\rangle, \langle k,g\rangle)\\ &\qquad = (\langle h, \sum_i \langle f, e_i\rangle e_i\rangle, \langle k, \sum_j \langle g,\gamma_j\rangle\gamma_j\rangle)\\ &\qquad = (\sum_i \langle e_i, f\rangle\langle h, e_i\rangle, \sum_j \langle k,\gamma_j\rangle\langle\gamma_j, g\rangle)\\ &\qquad = \sum_{i,j}\langle h, e_i\rangle\langle\gamma_j,k\rangle\langle\langle e_i,f\rangle, \langle\gamma_j,g\rangle\rangle\\ &\qquad = \sum_{i,j}\langle h, e_i\rangle\langle\gamma_j,k\rangle \int_X \langle e_i, f(x)\rangle\overline{\langle\gamma_j, g(x)\rangle}d\mu\\ &\qquad = \sum_{i,j}\langle h, e_i\rangle\langle\gamma_j,k\rangle\langle e_i,\gamma_j\rangle\\ &\qquad = \sum_j(\sum_i\langle\langle h,e_i\rangle e_i,\gamma_j\rangle\langle\gamma_j,k\rangle\\ &\qquad = \sum_j\langle h,\gamma_j\rangle\langle\gamma_j,k\rangle\\ &\qquad = \langle\sum_j\langle h,\gamma_j\rangle\gamma_j,k\rangle\\ &\qquad = \langle h,k\rangle.\end{aligned}$$

(vi)⇒(v) If $h = e_i$ and $k = \gamma_j$, by the similar method used in the proof of (v)⇒(iii), we obtain

$$\begin{aligned} \textstyle\int_X \langle e_i, f(x)\rangle\langle g(x), \gamma_j\rangle d\mu &= \sum_{k,l} \langle e_i, e_k\rangle\langle \gamma_l, \gamma_j\rangle\langle e_k, \gamma_k\rangle \\ &= \langle e_i, e_i\rangle\langle \gamma_j, \gamma_j\rangle\langle e_i, \gamma_j\rangle \\ &= \langle e_i, \gamma_j\rangle. \end{aligned}$$

□

Definition 3.5. Let f,g satisfy in the *swbi*-condition for H. We say that f,g are a dual pair, if one of the assertion of the Theorem 3.4 satisfies.

Remark 3.6. Let u satisfies in the *swbi*-condition for H. It is easily seen that for each $h \in H$
(i) $h = T^{\varphi}_{(S^{\varphi}_u)^{-1}u}(h_0 E(\langle h, u\rangle)\circ\varphi^{-1})$.
(ii) $h = T^{\varphi}_u(h_0 E(\langle (S^{\varphi}_u)^{-1}(h), u\rangle)\circ\varphi^{-1})$.

Lemma 3.7. *Let $u : X \to H$ satisfies in the swbi-condition for H. Then $(S^{\varphi}_u)^{-1}u\circ\varphi$ and u are a dual pair.*

Proof. We can write $h = T^{\varphi}_u(h_0 E(\langle (S^{\varphi}_u)^{-1}(h), u\rangle)\circ\varphi^{-1})$, on the other hand we have

$$\begin{aligned} \textstyle\int_X h_0 E(\langle (S^{\varphi}_u)^{-1}(h), u\rangle)\circ\varphi^{-1} d\mu &= \textstyle\int_X E(\langle (S^{\varphi}_u)^{-1}(h), u\rangle)\circ\varphi^{-1} d\mu\circ\varphi^{-1} \\ &= \textstyle\int_X E(\langle (S^{\varphi}_u)^{-1}(h), u\rangle) d\mu \\ &= \textstyle\int_X \langle (S^{\varphi}_u)^{-1}(h), u\rangle d\mu \\ &= \textstyle\int_X \langle h, (S^{\varphi}_u)^{-1}u\rangle d\mu. \end{aligned}$$

Then we get that $h = T^{\varphi}_u(\langle h, (S^{\varphi}_u)^{-1}u\rangle)$. So $(S^{\varphi}_u)^{-1}u\circ\varphi$ is a dual pair of u. □

Lemma 3.8. *Let f,g be a dual pair. Then f satisfies in the wbi-condition for H.*

Proof. Since f,g are a dual pair and also any function in $L^2(X)$ satisfies in *swbi*-condition, thus for each $h \in H$ we get that

$$\begin{aligned} \|h\|^2 &= \textstyle\int_X \langle h, f(x)\rangle\langle g(x), h\rangle d\mu \\ &\le \textstyle\int_X |\langle h, f(x)\rangle||\langle g(x), h\rangle| d\mu \\ &\le (\textstyle\int_X |\langle h, f(x)\rangle|^2 d\mu)^{\frac{1}{2}} (\textstyle\int_X |\langle h, g(x)\rangle|^2 d\mu)^{\frac{1}{2}} \\ &\le (\textstyle\int_X |\langle h, f(x)\rangle|^2 d\mu)^{\frac{1}{2}} B^{\frac{1}{2}}\|h\|, \quad B > 0. \end{aligned}$$

Thus

$$B^{-1}\|h\|^2 \le \int_X |\langle h, f(x)\rangle|^2 d\mu.$$

Consequently f satisfies in the *wbi*-condition for H. □

Definition 3.9. Let f,g satisfy in the *swbi*-condition for H. We say f and g are weakly equal if for each $h\in H$, $\langle h,f\rangle=\langle h,g\rangle$.

Theorem 3.10. *Let u satisfies in wbi-condition for H. Then*

(i) In the formula $h=T_u^{\varphi}(h_0E(\langle(S_u^{\varphi})^{-1}(h),u\rangle)\circ\varphi^{-1})$, $h_0E(\langle h,(S_u^{\varphi})^{-1}u\rangle)\circ\varphi^{-1}$ has the least norm among all of the retrieval formulas.

(ii) For each $h\in H$, $h=T_u^{\varphi}(\langle h,g\circ\varphi^{-1}\rangle)$ if and only if there exists a mapping ℓ such that satisfies in the swbi-condition for H and $\ell\circ\varphi=g\circ\varphi^{-1}-(S_u^{\varphi})^{-1}u$ and also for each $k\in H$, $\langle k,\ell\circ\varphi\rangle\in kerT_u^{\varphi}$.

(iii) u has unique dual if and only if $R((T_u^{\varphi})^*)=L^2(X)$.

Proof. (i) Let $M\in L^2(X)$ and $h=T_u^{\varphi}(M)$. Then for each $k\in H$, we have

$$\langle h,k\rangle=\langle T_u^{\varphi}(h_0E(\langle(S_u^{\varphi})^{-1}(h),u\rangle)\circ\varphi^{-1}),k\rangle$$

$$=\int_X\langle u(x),k\rangle((h_0\circ\varphi)(x)E(\langle(S_u^{\varphi})^{-1}(h),u(x)\rangle))d\mu.$$

On the other hand, $\langle h,k\rangle=\langle T_u^{\varphi}(M),k\rangle=\int_X\langle u(x),k\rangle M\circ\varphi(x)d\mu$. We have

$$\langle h,k\rangle-\langle h,k\rangle=\int_X\langle u(x),k\rangle((h_0\circ\varphi)(x)E(\langle(S_u^{\varphi})^{-1}(h),u(x)\rangle)-(M\circ\varphi)(x))d\mu.$$

Thus $h_0E(\langle(S_u^{\varphi})^{-1}(h),u\rangle)\circ\varphi^{-1}-M\in kerT_u^{\varphi}$. Since T_u^{φ} is surjective so $(T_u^{\varphi})^*$ has the closed range, it follows that

$$L^2(X)=kerT_u^{\varphi}\oplus R(T_u^{\varphi})^*.$$

On the other hand $h_0E(\langle(S_u^{\varphi})^{-1}(h),u\rangle)\circ\varphi^{-1}\in R(T_u^{\varphi})^*$. So we can write

$$\|M\|^2=\|h_0E(\langle(S_u^{\varphi})^{-1}(h),u\rangle)\circ\varphi^{-1}-M\|^2+\|h_0E(\langle(S_u^{\varphi})^{-1}(h),u\rangle)\circ\varphi^{-1}\|^2.$$

This implies that $h_0E(\langle h,(S_u^{\varphi})^{-1}u\rangle)\circ\varphi^{-1}$ has the least norm.

(ii) Let g satisfies in the *swbi*-condition for H and $h=T_u^{\varphi}(\langle h,g\circ\varphi^{-1}\rangle$. We set $g\circ\varphi^{-1}=\ell\circ\varphi+(S_u^{\varphi})^{-1}u$. By Theorem 2.10 for each $h,k\in H$ we have

$$\begin{aligned}\langle T_u^{\varphi}(\langle k,\ell\circ\varphi\rangle),h\rangle &=\langle T_u^{\varphi}(\langle k,g\circ\varphi^{-1}\rangle),h\rangle-\langle T_u^{\varphi}(\langle k,(S_u^{\varphi})^{-1}u\rangle,h\rangle\\ &=\textstyle\int_X\langle\langle k,g(x)\rangle\langle u(x),h\rangle d\mu\\ &-\textstyle\int_X\langle\langle u(x),h\rangle\langle k,(S_u^{\varphi})^{-1}u\circ\varphi(x)\rangle d\mu\\ &=\langle h,k\rangle-\langle h,k\rangle\\ &=0.\end{aligned}$$

Hence $\langle k, \ell\circ\phi\rangle \in kerT_u^{\varphi}$. Now, let $\ell\circ\varphi = g\circ\varphi^{-1} - (S_u^{\varphi})^{-1}u$ and for each $k \in H, \langle k, \ell\circ\phi\rangle \in kerT_u^{\varphi}$. Then

$$\begin{aligned}
\int_X \langle u(x),h\rangle\langle k,g(x)\rangle d\mu &= \int_X \langle u(x),h\rangle\langle k,(\ell\circ\varphi^2 + (S_u^{\varphi})^{-1}u\circ\varphi)(x)\rangle d\mu \\
&= \int_X \langle u(x),h\rangle\langle k,\ell\circ\varphi^2(x)\rangle d\mu \\
&+ \int_X \langle u(x),h\rangle\langle k,(S_u^{\varphi})^{-1}u\circ\varphi)(x)\rangle d\mu \\
&= \langle T_u^{\varphi}\langle(k,\ell\circ\varphi\rangle),h\rangle + \langle h,k\rangle \\
&= \langle h,k\rangle.
\end{aligned}$$

Thus g is a dual pair of u.

(iii) Let $R((T_u^{\varphi})^*) \neq L^2(X)$. Then there exists $\ell \in kerT_u^{\varphi}$ such that $\|\ell\|_2 = 1$. Now, Let $k : X \to L^2(X)$ be defined by $k(x) = \ell\circ\varphi(x)\ell$. The mapping

$$X \to C$$

$$x \mapsto \langle t, k(x)\rangle$$

for each $t \in L^2(X)$ is measurable, and

$$\int_X |\langle t,k(x)\rangle|^2 d\mu = \int_X |\langle t,\ell\rangle|^2|\ell\circ\varphi(x)|^2 d\mu \leq |\langle t,\ell\rangle|^2 \leq \|t\|^2.$$

Then k satisfies in the *swbi*-condition for $L^2(X)$. Also let $m : L^2(X) \to H$ be such that $m(\ell) \neq 0$. Thus mk satisfies in the *swbi*-condition for H, and so $(S_u^{\varphi})^{-1}u\circ\varphi + mk$ satisfies in the *swbi*-condition for H. For each $h \in H$ we have

$$\begin{aligned}
&\int_X |\langle h,(S_u^{\varphi})^{-1}u\circ\varphi(x) + mk(x)\rangle\langle u(x),h\rangle d\mu \\
&\qquad = \int_X |\langle h,(S_u^{\varphi})^{-1}u\circ\varphi(x)\rangle\langle u(x),h\rangle d\mu \\
&\qquad + \int_X |\langle h,mk(x)\rangle\langle u(x),h\rangle d\mu \\
&\qquad = \|h\|^2 + \langle m^*(h),\ell\rangle \int_X \overline{\ell\circ\varphi(x)}\langle u(x),h\rangle d\mu \\
&\qquad = \|h\|^2 + \langle T_u^{\varphi}(\ell),h\rangle \\
&\qquad = \|h\|^2.
\end{aligned}$$

So $(S_u^{\varphi})^{-1}u\circ\varphi + mk$ is a dual pair of u. On the other hand we have

$$\langle m(\ell),mk(x)\rangle = \langle m(\ell),\ell\circ\varphi(x)m(\ell)\rangle = \overline{\ell\circ\varphi(x)}\|m(\ell)\|^2 \neq 0.$$

It follows that $(S_u^{\varphi})^{-1}u\circ\varphi + mk$ is not weakly equal to $(S_u^{\varphi})^{-1}u\circ\varphi$.
Conversely, let $R((T_u^{\varphi})^*) = L^2(X)$, and also we suppose g is a dual of u such that $\ell\circ\varphi = g\circ\varphi^{-1} - (S_u^{\varphi})^{-1}u$ and for each $k \in H$ $\langle k,\ell\circ\varphi\rangle \in kerT_u^{\varphi}$. Since $kerT_u^{\varphi} = 0$, this implies that $\ell\circ\varphi = 0$ weakly. So $(S_u^{\varphi})^{-1}u\circ\varphi$ is only dual pair of u.

□

Example 3.11. Let $X = [0,2]$, Σ the Lebegue subsets of X and μ be a Lebegues measure on X. Also let $\varphi : X \to X$ is non-singular measurable transformation with $h_0 \in L^\infty$ and let $u : X \to L^2$ satisfies in *swbi*-condition for L^2. For $A \subseteq X$, put $h = \chi_A$. Then for $T_u^\varphi : L^2(X) \to L^2$ we obtain

$$\langle T_u^\varphi f, \chi_A \rangle = \int_0^2 \int_A u(x)(y) d\nu f \circ \varphi(x) d\mu,$$

where ν be Lebegues measure on A. Since $f \circ \varphi \in L^1(\mu)$ and $u(x) \in L^1(\nu)$, then by Fubini's Theorem $u(x)(y)(f \circ \varphi)(x) \in L^1(\mu \times \nu)$ and

$$\int_A T_u^\varphi f d\mu = \int_A \int_0^2 u(x)(y) f \circ \varphi(x) d\mu dv.$$

It follows that the expressive formula of bounded operator T_u^φ on $L^2(X)$ is $T_u^\varphi f = \int_0^2 u(x)(y) f \circ \varphi(x) d\mu$.

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