LE MATEMATICHE
Vol. LXXII (2017) – Fasc. I, pp. 23–32
doi: 10.4418/2017.72.1.3

ATOMIC AND NONATOMIC OPERATOR-VALUED MEASURES IN LOCALLY CONVEX CONES

DAVOOD AYASEH - ASGHAR RANJBARI

The operator valued measures and integrals for cone-valued functions have been investigated in [W. Roth, Operator-valued measures and integrals for cone-valued functions, Lecture Notes in Mathematics, vol. 1964, 2009, Springer Verlag, Heidelberg-Berlin-New York]. In this paper, we define atomic and nonatomic operator valued measures in locally convex cones and investigate their properties. We prove that every operator valued measure can be written as the sum of an atomic and a nonatomic measures.

1. Introduction

A *cone* is a set $\mathcal{P}$ endowed with an addition and a scalar multiplication for nonnegative real numbers. The addition is assumed to be associative and commutative, and there is a neutral element $0 \in \mathcal{P}$. For the scalar multiplication the usual associative and distributive properties hold, that is $\alpha(\beta a) = (\alpha\beta)a$, $(\alpha+\beta)a = \alpha a+\beta a$, $\alpha(a+b) = \alpha a+\alpha b$, $1a = a$ and $0a = 0$ for all $a,b \in \mathcal{P}$ and $\alpha,\beta \geq 0$.

The theory of locally convex cones as developed in [5] and [8] uses an order theoretical concept to introduce a topological structure on a cone. For recent researches see [1–4, 7].

Entrato in redazione: 4 gennaio 2014

AMS 2010 Subject Classification: 46A03, 46A12, 28A05
Keywords: Locally convex cones, atomic and nonatomic operator valued measures.

An *ordered cone* $\mathcal{P}$ carries a reflexive transitive relation $\leq$ such that $a \leq b$ implies $a+c \leq b+c$ and $\alpha a \leq \alpha b$ for all $a,b,c \in \mathcal{P}$ and $\alpha \geq 0$. The extended real numbers $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a natural example of an ordered cone with the usual order and algebraic operations in $\overline{\mathbb{R}}$, in particular $0 \cdot (+\infty) = 0$.

A subset $\mathcal{V}$ of the ordered cone $\mathcal{P}$ is called an *abstract neighborhood system,* if the following properties hold:
(1) $0 < v$ for all $v \in \mathcal{V}$;
(2) for all $u,v \in \mathcal{V}$ there is a $w \in \mathcal{V}$ with $w \leq u$ and $w \leq v$;
(3) $u+v \in \mathcal{V}$ and $\alpha v \in \mathcal{V}$ whenever $u,v \in \mathcal{V}$ and $\alpha > 0$.

For every $a \in \mathcal{P}$ and $v \in \mathcal{V}$ we define

$$v(a) = \{b \in \mathcal{P} | b \leq a+v\} \quad resp. \quad (a)v = \{b \in \mathcal{P} | a \leq b+v\},$$

to be a neighborhood of a in the upper, resp. lower topologies on $\mathcal{P}$. Their common refinement is called the symmetric topology generated by the neighborhoods $v^s(a) = v(a) \cap (a)v$. If we suppose that all elements of $\mathcal{P}$ are bounded below, that is for every $a \in \mathcal{P}$ and $v \in \mathcal{V}$, $0 \leq a + \lambda v$ for some $\lambda > 0$, then the pair $(\mathcal{P},\mathcal{V})$ is called a *full locally convex cone.* A *locally convex cone* $(\mathcal{P},\mathcal{V})$ is a subcone of a full locally convex cone, not necessarily containing the abstract neighborhood system $\mathcal{V}$. For example, the extended real number system $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ endowed with the usual order and algebraic operations and the neighborhood system $\mathcal{V} = \{\varepsilon \in \mathbb{R} | \varepsilon > 0\}$ is a full locally convex cone.

The locally convex cone $(\mathcal{P},\mathcal{V})$ is called a *uc*-cone whenever there is $v \in \mathcal{V}$ such that $\mathcal{V} = \{\alpha v : \alpha > 0\}$ (see [2]).

A subset B of the locally convex cone $(\mathcal{P},\mathcal{V})$ is called *bounded below* whenever for every $v \in \mathcal{V}$ there is $\lambda > 0$, such that $0 \leq b + \lambda v$ for all $b \in B$.

For cones $\mathcal{P}$ and $\mathcal{Q}$ a mapping $T : \mathcal{P} \to \mathcal{Q}$ is called a *linear operator* if $T(a+b) = T(a)+T(b)$ and $T(\alpha a) = \alpha T(a)$ hold for all $a,b \in \mathcal{P}$ and $\alpha \geq 0$. If both $\mathcal{P}$ and $\mathcal{Q}$ are ordered, then T is called *monotone,* if $a \leq b$ implies $T(a) \leq T(b)$. If both $(\mathcal{P},\mathcal{V})$ and $(\mathcal{Q},\mathcal{W})$ are locally convex cones, the operator T is called *(uniformly) continuous* if for every $w \in \mathcal{W}$ one can find $v \in \mathcal{V}$ such that $T(a) \leq T(b) + w$ whenever $a \leq b + v$ for $a,b \in \mathcal{P}$.

A *linear functional* on $\mathcal{P}$ is a linear operator $\mu : \mathcal{P} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. The *dual cone* $\mathcal{P}^*$ of a locally convex cone $(\mathcal{P},\mathcal{V})$ consists of all continuous linear functionals on $\mathcal{P}$ and is the union of all polars v° of neighborhoods $v \in \mathcal{V}$, where $\mu \in v^\circ$ means that $\mu(a) \leq \mu(b)+1$, whenever $a \leq b+v$ for $a,b \in \mathcal{P}$. In addition, to the given order $\leq$ on the locally convex cone $(\mathcal{P},\mathcal{V})$, the *weak pereorder* $\preccurlyeq$ is defined for $a,b \in \mathcal{P}$ by

$$a \preccurlyeq b \quad if \quad a \leq \gamma b + \varepsilon v$$

for all $v \in \mathcal{V}$ and $\varepsilon > 0$ with some $1 \leq \gamma \leq 1+\varepsilon$ (for details, see [8], I.3). It is obviously coarser than the given order, that is $a \leq b$ implies $a \preccurlyeq b$ for $a,b \in \mathcal{P}$.

Given a neighborhood $v \in \mathcal{V}$ and $\varepsilon > 0$, the corresponding upper and lower relative neighborhoods $v_\varepsilon(a)$ and $(a)v_\varepsilon$ for an element $a \in \mathcal{P}$ are defined by

$$v_\varepsilon(a) = \{b \in \mathcal{P} \,|\, b \le \gamma a + \varepsilon v \quad \text{for some} \quad 1 \le \gamma \le 1+\varepsilon\},$$

$$(a)v_\varepsilon = \{b \in \mathcal{P} \,|\, a \le \gamma b + \varepsilon v \quad \text{for some} \quad 1 \le \gamma \le 1+\varepsilon\}.$$

Their intersection $v^s_\varepsilon(a) = v_\varepsilon(a) \cap (a)v_\varepsilon$ is the corresponding symmetric relative neighborhood.
We shall say that a locally convex cone $(\mathcal{P},\mathcal{V})$ is a *locally convex* $\vee$*-semilattice cone* if its order is antisymmetric and if for any two elements $a,b \in \mathcal{P}$ their supremum $a \vee b$ exists in $\mathcal{P}$ and if
$(\vee 1)$ $(a+c)\vee(b+c) = a \vee b + c$ holds for all $a,b,c \in \mathcal{P}$,
$(\vee 2)$ $a \le c+v$ and $b \le c+w$ for $a,b,c \in \mathcal{P}$ and $v,w \in \mathcal{V}$ imply that $a \vee b \le c+(v+w)$.
Likewise, $(\mathcal{P},\mathcal{V})$ is a *locally convex* $\wedge$*-semilattice cone* if its order is antisymmetric and if for any two elements $a,b \in \mathcal{P}$ their infimum $a \wedge b$ exists in $\mathcal{P}$ and if
$(\wedge 1)$ $(a+c)\wedge(b+c) = a \wedge b + c$ holds for all $a,b,c \in \mathcal{P}$,
$(\wedge 2)$ $c \le a+v$ and $c \le b+w$ for $a,b,c \in \mathcal{P}$ and $v,w \in \mathcal{V}$ imply that $c \le a \wedge b + (v+w)$.
If both sets of the above conditions hold, then $(\mathcal{P},\mathcal{V})$ is called a *locally convex lattice cone* (cf. [8]).

We shall say that a locally convex cone $(\mathcal{P},\mathcal{V})$ is a locally convex $\vee^c$- *semilattice cone* if $\mathcal{P}$ carries the weak preorder (that is the given order coincides with the weak preorder for the elements and the neighborhoods in $\mathcal{P}$), this order is antisymmetric and if
$(\vee^c_1)$ every non-empty subset $A \subseteq \mathcal{P}$ has a supremum $\sup A \in \mathcal{P}$ and $\sup(A+b) = \sup A + b$ holds for all $b \in \mathcal{P}$,
$(\vee^c_2)$ let $\emptyset \ne A \subseteq \mathcal{P}$, $b \in \mathcal{P}$ and $v \in \mathcal{V}$. If $a \le b+v$ for all $a \in A$, then $\sup A \le b+v$.

Likewise, $(\mathcal{P},\mathcal{V})$ is said to be a locally convex $\wedge^c$*-semilattice cone* if $\mathcal{P}$ carries the weak preorder, this order is antisymmetric and if
$(\wedge^c_1)$ every bounded below subset $A \subset \mathcal{P}$ has an infimum $\inf A \in \mathcal{P}$ and $\inf(A+b) = \inf A + b$ holds for all $b \in \mathcal{P}$,
$(\wedge^c_2)$ let $A \subset \mathcal{P}$ be bounded below, $b \in \mathcal{P}$ and $v \in \mathcal{V}$. If $b \le a+v$ for all $a \in A$, then $b \le inf A + v$.
Combining both of the above notions, we shall say that a locally convex cone $(\mathcal{P},\mathcal{V})$ is a *locally convex complete lattice cone* if $\mathcal{P}$ is both a $\vee^c$-semilattice cone and a $\wedge^c$-semilattice cone.

As a simple example the locally convex cone $(\overline{\mathbb{R}},\mathcal{V})$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ and $\mathcal{V} = \{\varepsilon \in \mathbb{R} : \varepsilon > 0\}$, is a locally convex lattice cone and a locally convex complete lattice cone.

Suppose $(\mathcal{P},\mathcal{V})$ is a locally convex complete lattice cone. A net $(a_i)_{i\in\mathcal{I}}$ in $\mathcal{P}$ is called *bounded below* if there is $i_0\in\mathcal{I}$ such that the set $\{a_i \mid i\geq i_0\}$ is bounded below. We define the superior and the inferior limits of a bounded below net $(a_i)_{i\in\mathcal{I}}$ in $\mathcal{P}$ by

$$\liminf_{i\in\mathcal{I}} a_i = \sup_{i\in\mathcal{I}}(\inf_{k\geq i} a_k) \text{ and } \limsup_{i\in\mathcal{I}} a_i = \inf_{i\in\mathcal{I}}(\sup_{k\geq i} a_k).$$

If $\liminf_{i\in\mathcal{I}} a_i$ and $\limsup_{i\in\mathcal{I}} a_i$ coincide, then we denote their common value by $\lim_{i\in\mathcal{I}} a_i$ and say that the net $(a_i)_{i\in\mathcal{I}}$ is order convergent. A series $\sum_{i=1}^{\infty} a_i$ in $(\mathcal{P},\mathcal{V})$ is said to be order convergent to $s\in\mathcal{P}$ if the sequence $s_n=\sum_{i=1}^{n} a_i$ is order convergent to s.

2. Atomic and nonatomic operator-valued measures

We shall say that a locally convex cone $(\mathcal{P},\mathcal{V})$ is quasi-full if
$(QF1)$ $a\leq b+v$ for $a,b\in\mathcal{P}$ and $v\in\mathcal{V}$ if and only if $a\leq b+s$ for some $s\in\mathcal{P}$ such that $s\leq v$, and
$(QF2)$ $a\leq u+v$ for $a\in\mathcal{P}$ and $u,v\in\mathcal{V}$ if and only if $a\leq s+t$ for some $s,t\in\mathcal{P}$ such that $s\leq u$ and $t\leq v$.

The collection $\mathfrak{R}$ of subsets of X is called a (weak) σ-ring whenever:
$(R1)$ $\emptyset\in\mathfrak{R}$,
$(R2)$ If $E_1,E_2\in\mathfrak{R}$, then $E_1\cup E_2\in\mathfrak{R}$ and $E_1\backslash E_2\in\mathfrak{R}$,
$(R3)$ If $E_n\in\mathfrak{R}$ for $n\in\mathbb{N}$ and $E_n\subseteq E$ for some $E\in\mathfrak{R}$, then $\bigcup_{n\in\mathbb{N}} E_n\in\mathfrak{R}$.

Every σ-algebra is a σ-ring in this sense and a σ-ring is a σ-algebra if and only if $X\in\mathfrak{R}$. By any σ-ring $\mathfrak{R}$, we can associate the σ-algebra

$$\mathfrak{U}_{\mathfrak{R}}=\{A\subset X : \forall E\in\mathfrak{R}, A\cap E\in\mathfrak{R}\}.$$

The subset A of X is called measurable if $A\in\mathfrak{U}_{\mathfrak{R}}$. The operator–valued measures in locally convex cones have been defined in [8]. Let $(\mathcal{P},\mathcal{V})$ be a quasi-full locally convex cone and let $(\mathcal{Q},\mathcal{W})$ be a locally convex complete lattice cone (see Sections 5 and 6 in Chapter I from [8]). Let $\mathcal{L}(\mathcal{P},\mathcal{Q})$ denote the cone of all (uniformly) continuous linear operators from $\mathcal{P}$ to $\mathcal{Q}$. Recall from Section 3 in Chapter I from [8] that a continuous linear operator between locally convex cones is monotone with respect to the respective weak preorders. Because $\mathcal{Q}$ carries its weak preorder, this implies monotonicity with respect to the given orders of $\mathcal{P}$ and $\mathcal{Q}$ as well. Let X be a set, $\mathfrak{R}$ a (weak) σ-ring of subsets of X. An $\mathcal{L}(\mathcal{P},\mathcal{Q})$-valued measure θ on $\mathfrak{R}$ is a set function

$$E\to\theta_E : \mathfrak{R}\to\mathcal{L}(\mathcal{P},\mathcal{Q})$$

such that $\theta(\emptyset) = \theta_\emptyset = 0$ and

$$\theta(\bigcup_{i\in\mathbb{N}} E_i) = \theta_{(\bigcup_{i\in\mathbb{N}} E_i)} = \sum_{i\in\mathbb{N}} \theta_{E_i}$$

holds whenever the sets $E_i \in \mathfrak{R}$ are disjoint and $\bigcup_{i=1}^{\infty} E_i \in \mathfrak{R}$. Convergence for the series on the right-hand side is meant in the following way: for every $a \in \mathcal{P}$ the series $\sum_{i\in\mathbb{N}} \theta_{E_i}(a)$ is order convergent in $\mathcal{Q}$. We note that the order convergence is implied by convergence in the symmetric relative topology.

Let $(\mathcal{P},\mathcal{V})$ be a quasi-full locally convex cone, $(\mathcal{Q},\mathcal{W})$ be locally convex complete lattice cone and θ be a fixed $\mathcal{L}(\mathcal{P},\mathcal{Q})$-valued measure on $\mathfrak{R}$. For a neighborhood $v \in \mathcal{V}$ and a set $E \in \mathfrak{R}$, semivariation of θ is defined as follows:

$$|\theta|(E,v) = \sup\left\{ \sum_{i\in\mathbb{N}} \theta_{E_i}(s_i) : s_i \in \mathcal{P}, s_i \le v, E_i \in \mathfrak{R}\,\text{disjoint subsets of}\,E \right\}.$$

It is proved in ([8], II, Lemma 3.3), that if $v \in \mathcal{P}$, then $|\theta|(E,v) = \theta_E(v)$.

Proposition 2.1. *Let $(\mathcal{P},\mathcal{V})$ be a quasi-full locally convex cone, $(\mathcal{Q},\mathcal{W})$ be a locally convex complete lattice cone and θ be a fixed $\mathcal{L}(\mathcal{P},\mathcal{Q})$-valued measure on $\mathfrak{R}$.*
(a) If for $E \in \mathfrak{R}$, $\theta_E = 0$, then for every $v \in \mathcal{V}$, $|\theta|(E,v) = 0$,
(b) If for every $v \in \mathcal{V}$, $|\theta|(E,v) = 0$, then $\theta_E(a) = 0$ for every bounded element a of $\mathcal{P}$.

Proof. For (a), let $\theta_E = 0$ and $F_1,\cdots,F_n$, $n \in \mathbb{N}$ be a partition of E. Then for $0 \le s_i \le v$, $i = 1,\cdots,n$, we have $0 \le \theta_{F_i}(s_i) \le \theta_E(s_i) = 0$. Since the order of $\mathcal{Q}$ is antisymmetric, for every $i \in \{1,\cdots,n\}$, we have $\theta_{F_i}(s_i) = 0$. Then $|\theta|(E,v) = 0$.

For (b), let $a \in \mathcal{P}$ and for every $v \in \mathcal{V}$, $|\theta|(E,v) = 0$. Since a is bounded, for $v \in \mathcal{V}$, there is $\lambda > 0$ such that $0 \le a + \lambda v$ and $a \le \lambda v$. Now we have $0 \le \theta_E(a) + |\theta|(E,\lambda v)$ and $\theta_E(a) \le |\theta|(E,\lambda v)$ by Lemma II,3.4 of [8]. This shows that $0 \le \theta_E(a)$ and $\theta_E(a) \le 0$. Since the order of $\mathcal{Q}$ is antisymmetric, we have $\theta_E(a) = 0$. □

Corollary 2.2. *Let $(\mathcal{P},\mathcal{V})$ be a quasi-full locally convex cone, $(\mathcal{Q},\mathcal{W})$ be a locally convex complete lattice cone and θ be a fixed $\mathcal{L}(\mathcal{P},\mathcal{Q})$-valued measure on $\mathfrak{R}$. If all elements of $\mathcal{P}$ are bounded, then for $E \in \mathfrak{R}$, $\theta_E = 0$ if and only if $|\theta|(E,v) = 0$ for all $v \in \mathcal{V}$.*

In the following we shall define atomic and nonatomic operator valued measures. By considering the Corollary 2.2, we use the semivariation of θ for this aim.

Definition 2.3. Let $\mathfrak{R}$ be a σ-ring of subsets of X and $v \in \mathcal{V}$. For $v \in \mathcal{V}$, the set $E \in \mathfrak{R}$ is said to be of *positive v-semivariation* of the measure θ if $|\theta|(E,v) > 0$. Also, we say that the set E is of *bounded v-semivariation* of the measure θ, if $|\theta|(E,v)$ is bounded in $(\mathcal{Q},\mathcal{W})$.

Definition 2.4. Let $\mathfrak{R}$ be a σ-ring of subsets of X. We say that a set $E \in \mathfrak{R}$ of positive v-semivariation of the measure θ is a *θ_v-atom* if (1) $|\theta|(E,v) > 0$ and (2) for given $A \in \mathfrak{R}$ either $|\theta|(A \cap E, v) = 0$ or $|\theta|(E \setminus A, v) = 0$.

We note that if E is a θ_v-atom and $|\theta|(A \cap E, v) > 0$, then $E \cap A$ is a θ_v-atom for θ.

Definition 2.5. We say that the measure θ is *purely v-atomic* if each $E \in \mathfrak{R}$ of positive v-semivariation contains a θ_v-atom. We say that θ is *v-nonatomic* if there are no θ_v-atoms for θ. This means that every $E \in \mathfrak{R}$ of positive v-semivariation can be split into two disjoint elements of $\mathfrak{R}$, each having positive v-semivariation. The measure θ is called *purely atomic (or nonatomic)* if for every $v \in \mathcal{V}$, it is v-atomic (or v-nonatomic).

Clearly the zero measure is the only measure which is both purely atomic and nonatomic.

Lemma 2.6. If $(\mathcal{P},\mathcal{V})$ is a *uc*-cone, then θ is purely atomic (or nonatomic)if and only if it is purely v-atomic (or v-nonatomic) for some $v \in \mathcal{V}$.

Proof. Let $u \in \mathcal{V}$ be arbitrary. There is $\alpha > 0$ such that $u = \alpha v$. This shows that every θ_u-atom is a θ_v-atom and every set of positive u-semivariation has positive v-semivariation. Therefore the assertion holds. □

Example 2.7. Let $X = \mathbb{N} \cup \{+\infty\}$ and $\mathcal{P} = \mathcal{Q} = \overline{\mathbb{R}}$. We consider on $\overline{\mathbb{R}}$ the abstract neighborhood system $\mathcal{V} = \{\varepsilon \in \mathbb{R} : \varepsilon > 0\}$. Then $\mathcal{L}(\mathcal{P},\mathcal{Q})$ contains all nonnegative reals and the linear functional $\bar{0}$ acting as

$$\bar{0}(x) = \begin{cases} +\infty & x = +\infty \\ 0 & else. \end{cases}$$

We set $\mathfrak{R} = \{E \subset X : E \text{ is finite}\}$. Then $\mathfrak{R}$ is a σ-ring on X. We define the set function θ on $\mathfrak{R}$ as following: for $x \in \mathcal{P}$, $\theta_{\emptyset}(x) = 0$, $\theta_{\{n\}}(x) = nx$ for $n \in \mathbb{N}$ and $\theta_{\{+\infty\}}(x) = \bar{0}(x)$. For $E = \{a_1, \cdots, a_n\} \in \mathfrak{R}$, $n \in \mathbb{N}$, we define $\theta_E(x) = \sum_{i=1}^{n} \theta_{\{a_i\}}(x)$ for $x \in \mathcal{P}$. Then θ is clearly an operator valued measure on $\mathfrak{R}$. For $n \in \mathbb{N}$ and $\varepsilon > 0$, we have $|\theta|(\{n\},\varepsilon) = \theta_{\{n\}}(\varepsilon) = n\varepsilon$ and $|\theta|(\{+\infty\},\varepsilon) = \theta_{\{\infty\}}(\varepsilon) = \bar{0}(\varepsilon) = 0$. It is easy to see that for every $n \in \mathbb{N}$, $\{n\}$ is a θ_ε-atom but $\{+\infty\}$ is not. Now since every set in $\mathfrak{R}$ which has positive ε-semivariation contains a θ_ε-atom, θ is purely ε-atomic. Also since $(\overline{\mathbb{R}},\mathcal{V})$ is a *uc*-cone, then θ is purely atomic by Lemma 2.6.

Example 2.8. Let X be an uncountable set and $\mathcal{P} = \mathcal{Q} = \{0, +\infty\}$. We consider the cones $\mathcal{P}$ and $\mathcal{Q}$ endowed with the abstract neighborhood system $\mathfrak{U} = \{+\infty\}$. Then we have $\mathcal{L}(\mathcal{P}, \mathcal{Q}) = \{0, \overline{+\infty}\}$. We set $\mathfrak{R} = P(X)$. Then $\mathfrak{R}$ is a σ-ring. We define the operator valued measures θ as follows

$$\theta(E) = \theta_E = \begin{cases} 0 \text{ if E is finite or countable} \\ \\ \overline{+\infty} \ \ \textit{else.} \end{cases}$$

We have $|\theta|(E, +\infty) = 0$ if E is finite or countable and $|\theta|(E, +\infty) = +\infty$ else. Then every uncountable set has positive ∞-semivariation but they do not contain any $\theta_{+\infty}$-atom. Therefore θ is a $+\infty$-nonatomic operator valued measure. Since $(\mathcal{P}, \mathfrak{U})$ is a *uc*-cone, then θ is nonatomic.

Proposition 2.9. *Suppose θ and ϑ are purely atomic $\mathcal{L}(\mathcal{P}, \mathcal{Q})$-valued measures on $\mathfrak{R}$. Then so are $\theta + \vartheta$ and $\alpha\theta$ for $\alpha \geq 0$*

Proof. Let $\varphi = \theta + \vartheta$ and for $E \in \mathfrak{R}$, $|\varphi|(E, v) > 0$. Then $|\theta|(E, v) > 0$ or $|\vartheta|(E, v) > 0$. Suppose $|\theta|(E, v) > 0$. Since θ is purely v-atomic, there is a θ_v-atom F such that $F \subset E$. If $|\vartheta|(F, v) = 0$, then F is clearly a φ_v-atom. On the other hand, if $|\vartheta|(F, v) > 0$, then there is a ϑ_v-atom H such that $H \subset F$. Clearly we have $|\varphi|(H, v) > 0$. If H is a φ_v-atom, then the proof is complete. Otherwise, there is $G \in \mathfrak{R}$ such that $|\varphi|(H \cap G, v) > 0$ and $|\varphi|(H \setminus G, v) > 0$. We claim that $H \cap G$ is a φ_v-atom. Since $|\varphi|(H \cap G, v) > 0$, we have $|\vartheta|(H \cap G, v) > 0$ or $|\theta|(H \cap G, v) > 0$. If $|\vartheta|(H \cap G, v) > 0$, then $H \cap G$ is ϑ_v-atom, since H is ϑ_v-atom. Also, if $|\theta|(H \cap G, v) > 0$, then $H \cap G$ is θ_v-atom, since $H \cap G \subseteq F$ and F is a θ_v-atom. Therefore $H \cap G$ is a φ_v-atom.

It is obvious that if θ is purely v-atomic, then $\alpha\theta$ so is for $\alpha \geq 0$. □

Corollary 2.10. *Let $\mathcal{M}$ be the cone of all $\mathcal{L}(\mathcal{P}, \mathcal{Q})$-valued measures on $\mathfrak{R}$. The collection of all v-atomic $\mathcal{L}(\mathcal{P}, \mathcal{Q})$-valued measures on $\mathfrak{R}$ forms a subcone of $\mathcal{M}$.*

We can prove that the collection of all v-nonatomic operator valued measures forms a subcone of $\mathcal{M}$. We shall denote the cones of v-atomic and v-nonatomic operator valued measures on $\mathfrak{R}$ by $\mathcal{M}_A^v$ and $\mathcal{M}_{NA}^v$, respectively.

Definition 2.11. Let θ and ϑ be operator valued measures on $\mathfrak{R}$. We shall say that θ is absolutely v-continuous with respect to ϑ, denoted $\theta \ll_v \vartheta$, if for every $E \in \mathfrak{R}$, $|\vartheta|(E, v) = 0$ implies that $|\theta|(E, v) = 0$.

Proposition 2.12. *If E is a ϑ_v-atom, and $\theta \ll_v \vartheta$, then either $|\theta|(E, v) = 0$ or E is a θ_v-atom.*

Proof. Let E be a ϑ_v-atom. Then for $F \in \mathfrak{R}$, either $|\vartheta|(E \cap F, v) = 0$ or $|\vartheta|(E \setminus F, v) = 0$. Since $\theta \ll_v \vartheta$, we conclude that $|\theta|(E \cap F, v) = 0$ or $|\theta|(E \setminus F, v) = 0$. Therefore $|\theta|(E, v) = 0$ or E is a θ_v-atom. □

Suppose θ and ϑ are two operator measures on $\mathfrak{R}$. We shall say that θ is $\mathfrak{R}_v$-singular with respect to ϑ, denoted $\theta R_v \vartheta$, if for every $E \in \mathfrak{R}$ there is $F \in \mathfrak{R}$, $F \subset E$ such that $|\theta|(E, v) = |\theta|(F, v)$ and $|\vartheta|(F, v) = 0$.

Theorem 2.13. *Let θ, ϑ and φ be operator valued measures on $\mathfrak{R}$ such that $\varphi = \theta + \vartheta$. If $\theta R_v \vartheta$, then θ is v-atomic or v-nonatomic if φ has that property.*

Proof. Let for $E \in \mathfrak{R}$, $|\theta|(E, v) > 0$. Since $\theta R_v \vartheta$, there is $F \in \mathfrak{R}$, $F \subset E$ such that $|\theta|(F, v) > 0$ and $|\vartheta|(F, v) = 0$. Obviously $|\varphi|(F, v) > 0$. Now, if φ is v-atomic, then there is a φ_v-atom G such that $G \subset F$. Since $\theta \ll_v \varphi$, we conclude that either $|\theta|(G, v) = 0$ or G is a θ_v-atom. Since $|\theta|(G, v) = 0$ is impossible, then G is a θ_v-atom. On the other hand, if φ is v-nonatomic, then there is $G \in \mathfrak{R}$ such that $|\varphi|(F \cap G, v) > 0$ and $|\varphi|(F \setminus G, v) > 0$. Since $|\vartheta|(F, v) = 0$, thus $|\theta|(F \cap G, v) > 0$ and $|\theta|(F \setminus G, v) > 0$. Therefore θ is v-nonatomic. □

Suppose that θ is an operator valued measure on $\mathfrak{R}$ and $\mathcal{A}$ is a subfamily of $\mathfrak{R}$ which contains $\emptyset$ and it is closed under countable unions. For every $E \in \mathfrak{R}$ and $a \in \mathcal{P}$ we define $\theta_E^1(a) = \sup\{\theta_{E\cap A}(a) : A \in \mathcal{A}\}$ and $\theta_E^2(a) = \sup\{\theta_{E\cap B}(a) : \theta_B^1 = 0\}$. The mappings θ_E^1 and θ_E^2 are linear operators from $\mathcal{P}$ into $\mathcal{Q}$ by the Proposition 5.5, I from [8].

Lemma 2.14. The linear operators θ_E^1 and θ_E^2 are continuous.

Proof. Let $a, b \in \mathcal{P}$ and $w \in \mathcal{W}$. Since for $A \in \mathcal{A}$ and $E \in \mathfrak{R}$, $\theta_{E\cap A}$ is continuous, there is $v \in \mathcal{V}$ such that $a \le b + v$ implies $\theta_{E\cap A}(a) \le \theta_{E\cap A}(b) + w$. Then, we have $\theta_{E\cap A}(a) \le \theta_{E\cap A}(b) + w \le \sup_{A\in\mathcal{A}} \theta_{E\cap A}(b) + w$. Now $(\vee_2^c)$ shows that $\sup_{A\in\mathcal{A}} \theta_{E\cap A}(a) \le \sup_{A\in\mathcal{A}} \theta_{E\cap A}(b) + w$. Therefore $\theta_E^1(a) \le \theta_E^1(b) + w$. Thus θ_E^1 is continuous. Similarly, θ_E^2 is continuous. □

We define the set functions θ^1 and θ^2 as follows:

$$E \to \theta_E^1 : \mathfrak{R} \to \mathcal{L}(\mathcal{P}, \mathcal{Q}) \text{ and } E \to \theta_E^2 : \mathfrak{R} \to \mathcal{L}(\mathcal{P}, \mathcal{Q}).$$

Then θ^1 and θ^2 are $\mathcal{L}(\mathcal{P}, \mathcal{Q})$-valued measures on $\mathfrak{R}$. Obviously, we have $\theta = \theta^1 + \theta^2$.

Theorem 2.15. *Let θ be a $\mathcal{L}(\mathcal{P}, \mathcal{Q})$-valued measure on $\mathfrak{R}$ and $v \in \mathcal{V}$. Then there are v-atomic measure θ^1 and v-nonatomic measure θ^2 such that $\theta = \theta^1 + \theta^2$.*

Proof. Let $\mathcal{A}$ be the collection of all countable unions of θ_v-atoms and θ^1 and θ^2 be as the above. Then we have $\theta = \theta^1 + \theta^2$. We show that θ^1 is a v-atomic measure and θ^2 is a v-nonatomic measure. Let $E \in \mathfrak{R}$ and $|\theta^1|(E,v) > 0$. Then there is $A \in \mathcal{A}$ such that $|\theta^1|(E \cap A) > 0$. There are θ_v-atoms $A_1, A_2, \cdots$, such that $A = \cup_{n=1}^{\infty} A_n$. This implies that $|\theta^1|(E \cap A_n) > 0$ for some $n \in \mathbb{N}$. Since $E \cap A_n$ is a θ_v-atom and $\theta^1 \ll_v \theta$, then $E \cap A_n$ is a θ_v^1-atom by Proposition 2.12 which is contained in E.

Now we show that θ^2 is v-nonatomic. Suppose $|\theta^2|(E,v) > 0$. Then there is $B \in \mathfrak{R}$ such that $\theta_B^1 = 0$ and $|\theta^2|(E \cap B, v) > 0$. The set $E \cap B$ is not a θ_v-atom, since otherwise $|\theta^1|(E \cap B, v) > 0$, which is a contradiction. Now since $|\theta|(E \cap B, v) > 0$ and $E \cap B$ is not a θ_v-atom, there is $F \in \mathfrak{R}$ such that $|\theta|(E \cap B \cap F, v) > 0$ and $|\theta|(E \cap B \setminus F, v) > 0$. This shows that $|\theta^2|(E \cap F, v) > 0$ and $|\theta^2|(E \setminus F, v) > 0$. Then θ^2 is v-nonatomic. □

Corollary 2.16. *Every operator valued measure can be written as the sum of an atomic and a nonatomic operator valued measures.*

Proof. Since in the proof of Theorem 2.15, $v \in \mathcal{V}$ is arbitrary, we conclude that θ^1 is an atomic measure and θ^2 is a nonatomic measure. □

Theorem 2.17. *Let θ be an v-atomic operator valued measure on $\mathfrak{R}$ and $E \in \mathfrak{R}$. If $|\theta|(E,v) > 0$, then there are a countable disjoint collection of θ_v-atoms $A_k \subset E$, $k \in \mathbb{N}$ such that*

$$|\theta|(E,v) = |\theta|(\bigcup_{k=1}^{\infty} A_k, v) = \sum_{k=1}^{\infty} |\theta|(A_k, v).$$

Proof. The proof of Theorem 2.15 shows that there are θ_v-atoms E_k, $k \in \mathbb{N}$ such that

$$|\theta|(E,v) = |\theta|\big(E \cap \big(\bigcup_{k=1}^{\infty} E_k\big), v\big).$$

By setting $A_k = [E_k - (E_1 \cup \cdots \cup E_{k-1})] \cap E$ and disregarding those A_k, which have zero v-semivariations, we have

$$|\theta|(E,v) = |\theta|(\bigcup_{k=1}^{\infty} A_k, v) = \sum_{k=1}^{\infty} |\theta|(A_k, v).$$

Obviously, if $|\theta|(A_k, v) > 0$ for $k \in \mathbb{N}$, then A_k is an θ_v-atom. □

REFERENCES

[1] D. Ayaseh and A. Ranjbari, Bornological Convergence in Locally Convex Cones, Mediterr. J. Math., 13 (4), 1921-1931(2016).

[2] D. Ayaseh and A. Ranjbari, Bornological Locally Convex Cones, Le Matematiche, 69(2), 267-284(2014).

[3] D. Ayaseh and A. Ranjbari, Locally Convex Quotient Lattice Cones, Math. Nachr., 287(10), 1083-1092 (2014).

[4] D. Ayaseh and A. Ranjbari, Some notes on bornological and nonbornological locally convex cones, Le Matematiche, 70(2), 235-241(2015).

[5] K. Keimel and W. Roth, Ordered cones and approximation, Lecture Notes in Mathematics, vol. 1517, 1992, Springer Verlag, Heidelberg-Berlin-New York.

[6] P. R. Halmos, Measure theory, Van Nostrand, Princeton, N. J., 1950. MR 11, 504.

[7] A. Ranjbari, Strict inductive limits in locally convex cones, Positivity, 15(3), 465-471(2o11).

[8] W. Roth, Operator-valued measures and integrals for cone-valued functions, Lecture Notes in Mathematics, vol. 1964, 2009, Springer Verlag, Heidelberg-Berlin-New York.

DAVOOD AYASEH
Department of Pure Mathematics,
Faculty of Mathematical Sciences,
University of Tabriz,
Tabriz, Iran.
e-mail: d_ayaseh@tabrizu.ac.ir

ASGHAR RANJBARI
Department of Pure Mathematics,
Faculty of Mathematical Sciences,
University of Tabriz,
Tabriz, Iran.
e-mail: ranjbari@tabrizu.ac.ir