LE MATEMATICHE
Vol. LXXI (2016) – Fasc. II, pp. 81–97
doi: 10.4418/2016.71.2.7

q-WEIERSTRASS TRANSFORM ASSOCIATED WITH THE q-FOURIER BESSEL OPERATOR

SOUMAYA CHEFAI - KAMEL BRAHIM

This work deals kernels in Hilbert spaces. Namely, we study the Weierstrass transform in quantum calculus related to the Fourier Bessel transform and we give the appropriate best approximations of the inversion.

1. Introduction

The present paper develops a general theory of integral transforms in the framework of Hilbert spaces. We consider the following integral transform in quantum calculus called q-Weierstrass transform connected with the q-Bessel Fourier transform :

$$\mathcal{W}^{\nu}_{q,t}(f) = \vartheta_t *_q f; \quad \nu > \frac{-1}{2},$$

where ϑ_t is the Gauss kernel [1] and $*_q$ is the q-convolution product operator related to the q-Bessel Fourier transform.

This paper is devoted first to study the q-Weierstrass transform related to the q-Bessel Fourier transform and second to give its properties on some Hilbert

Entrato in redazione: 2 settembre 2015

AMS 2010 Subject Classification: 33B15, 33D05.
Keywords: Weierstrass transform, q-Harmonic analysis, q-Fourier Bessel transform, reproducing kernels.

space **H** and finally to use the theory of reproducing kernels to obtain the best approximation of its inversion.

On the other hand, if the space of departure is a Sobolev space included in **H**, we lose the surjectivity of this q-transform, which leads us to study approximation of its inversion via the theory of Saitoh ([7], [8]) to characterize the extremals functions in the approximations.

This paper is organized as follows.
In section 2, we recall the main results about the q-harmonic analysis.
Section 3 is devoted to define the q-Weierstrass transformation associated with the q-Fourier Bessel transform and we establish its properties. We deal with a bounded linear operator and some of its properties acting in the Hilbert spaces

$$\mathbf{H}_\beta = \{f \in \mathbf{H} \mid \left(1+\lambda^2\right)^{\frac{\beta}{2}} \mathcal{F}_{\nu,q}(f)(\lambda) \in \mathbf{H}\},\ \beta \in \mathbb{R}, \beta > \nu+1,$$

equipped with the inner product

$$\langle f \mid g \rangle_{\beta,q} = \int_0^{+\infty} \left(1+\lambda^2\right)^{\beta} \mathcal{F}_{q,\nu}(f)(\lambda)\overline{\mathcal{F}_{q,\nu}(g)(\lambda)}\lambda^{2\nu+1} d_q\lambda,$$

and $\mathbf{H}_{\beta,\xi}$ the space $\mathbf{H}_\beta$ equipped with the inner product

$$< f \mid g >_{\beta,\xi,q} = \xi < f \mid g >_{\beta,q} + < \mathcal{W}_{q,t}(f) \mid \mathcal{W}_{q,t}(g) >_{\nu,q},$$

where $< \mid >_{\nu,q}$ is the inner product on the Hilbert space **H** defined by

$$< f \mid g >_{\nu,q} = \frac{1}{(1-q)} \frac{\left(q^{2\nu+2}, q^2\right)_\infty}{(q^2,q^2)_\infty} \int_0^{+\infty} f(x)\overline{g(x)} x^{2\nu+1} d_q x.$$

Then applying Saitoh's Theorem [7], we estimate the extremal functions and their properties.

2. Preliminaries

Throughout this paper, we will assume that $0 < q < 1$.

We refer to [4] for the definitions, notations and properties of the q-shifted factorials, the Jackson's q-derivative and the Jackson's q-integrals.

2.1. Basic symbols

Let $a \in \mathbb{C}$, the q-shifted factorial is defined by

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

We also denote for $x \in \mathbb{C}$

$$[x]_q = \frac{1-q^x}{1-q} \quad \text{and} \quad [n]_q! = \frac{(q;q)_n}{(1-q)^n}.$$

The q-derivative of a function f is given by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x} \quad \text{if} \quad x \neq 0,$$

and

$$D_q^n f(x) = D_q^{n-1}(D_q f)(x) \;\; \forall n > 0.$$

The q-Jackson integrals from 0 to a and from 0 to ∞ are defined by

$$\int_0^a f(x) d_q x = (1-q) a \sum_{n=0}^{\infty} f(aq^n) q^n,$$

$$\int_0^\infty f(x) d_q x = (1-q) \sum_{n=-\infty}^{\infty} f(q^n) q^n,$$

provided the sums converge absolutely.

We denote by

- $\mathbb{R}_q^+ = \left\{ q^k, k \in \mathbb{Z} \right\}$.
- $d\mu_{\nu,q}$ the measure defined on $\mathbb{R}_q^+$ by

$$d\mu_{\nu,q}(x) = c\left(\nu, q^2\right) x^{2\nu+1} d_q x,$$

where

$$c\left(\nu, q^2\right) = \frac{1}{(1-q)} \frac{\left(q^{2\nu+2}, q^2\right)_\infty}{\left(q^2, q^2\right)_\infty}.$$

- $L^p_{\nu,q}(\mathbb{R}_q^+)$ is the space of measurable functions f on $\mathbb{R}_q^+$, $p \geq 1$, satisfying

$$\|f\|_{\nu,q,p} = \left(\int_{\mathbb{R}_q^+} |f(x)|^p \, d\mu_{\nu,q}(x) \right)^{\frac{1}{p}} < +\infty. \qquad (1)$$

- $\mathcal{C}_{0,q}\left(\mathbb{R}_q^+\right)$ is the space of restriction on $\mathbb{R}_q^+$ of even continuous functions f such that

$$\|f\|_{\infty,q} = \sup_{x \in \mathbb{R}_q^+} |f(x)|$$

and

$$\lim_{|x|\to+\infty} f(x) = 0.$$

The q-Bessel operator $\Delta_{q,\nu}$ is defined on $\mathbb{R}_q^+$ by

$$\Delta_{q,\nu} f(x) = q^{2\nu+1} D_q^2 f\left(q^{-1}x\right) + \frac{1-q^{2\nu+1}}{(1-q)\,q^{-1}x} D_q f\left(q^{-1}x\right). \tag{2}$$

The q-modified Bessel function j_ν of the first kind is defined by

$$j_\nu\left(x;q^2\right) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)}}{\left(q^{2\nu+2};q^2\right)_n \left(q^2;q^2\right)_n} x^{2n}. \tag{3}$$

For λ complex, the function $x \mapsto j_\nu\left(\lambda x;q^2\right)$ is the unique even solution of the following problem:

$$\Delta_{q,\nu} y(x) = -\lambda^2 y(x), \;\; y(0) = 1.$$

Definition 2.1. The q-Bessel Fourier transform $\mathcal{F}_{q,\nu}$ is defined by $(see[1--3,$ $6])$

$$\mathcal{F}_{\nu,q}(f)(\lambda) = \int_0^{+\infty} f(x)\, j_\nu\left(\lambda x;q^2\right) d_{\mu_{\nu,q}}(x), \qquad \nu \geq -\frac{1}{2}. \tag{4}$$

Definition 2.2.
1) Let $f \in L^p_{\nu,q}(\mathbb{R}_q^+)$, the q-Bessel translation operator is defined by [1–3]:

$$T^\nu_{q,x} f(y) = \int_0^{\infty} \mathcal{F}_{\nu,q}(f)(t)\, j_\nu(yt,q^2)\, j_\nu(xt,q^2)\, d_{\mu_{\nu,q}}(t).$$

2) Let f and g in $L^1_{\nu,q}(\mathbb{R}_q^+)$, the q-convolution product is defined by:

$$f *_q g(x) = \int_0^{+\infty} f(t)\, T^\nu_{q,x}(g)(t)\, d_{\mu_{\nu,q}}(t). \tag{5}$$

The q-Bessel Fourier Transform satisfies the following properties:

- For $f \in L^1_{\nu,q}(\mathbb{R}_q^+)$,

$$\mathcal{F}_{\nu,q}\left(T^\nu_{q,x} f\right)(\lambda) = j_\nu\left(\lambda x;q^2\right) \mathcal{F}_{\nu,q}(f)(\lambda). \tag{6}$$

- For f, g in $L^1_{\nu,q}(\mathbb{R}_q^+)$,

$$\mathcal{F}_{\nu,q}(f *_q g)(\lambda) = \mathcal{F}_{\nu,q}(f)(\lambda)\, \mathcal{F}_{\nu,q}(g)(\lambda). \tag{7}$$

- For $f \in L^p_{\nu,q}(\mathbb{R}^+_q), p \geq 1$, the function $T^\nu_{q,x}f$ belongs to the space $L^p_{\nu,q}(\mathbb{R}^+_q)$ and
$$\left\|T^\nu_{q,x}f\right\|_{\nu,q,p} \leq \|f\|_{\nu,q,p}. \tag{8}$$

Theorem 2.3. *(q-inversion formula) Let $f \in L^1_{\nu,q}(\mathbb{R}^+_q)$ such that $\mathcal{F}_{\nu,q}f \in L^1_{\nu,q}(\mathbb{R}^+_q)$, then for all $x \in \mathbb{R}^+_q$ and $\nu > -\frac{1}{2}$, we have:*

$$f(x) = \int_0^{+\infty} \mathcal{F}_{\nu,q}(f)(\lambda)\, j_\nu\left(\lambda x; q^2\right) d_{\mu_{\nu,q}}(\lambda). \tag{9}$$

Theorem 2.4. *(q-Plancherel theorem)The operator $\mathcal{F}_{\nu,q}$ can be extended to an isometric isomorphism from $L^2_{\nu,q}(\mathbb{R}^+_q)$ onto itself. In particular for all f,g in $L^2_{\nu,q}(\mathbb{R}^+_q)$, we have*

$$\int_0^{+\infty} f(x)\overline{g(x)} d_{\mu_{\nu,q}}(x) = \int_0^{+\infty} \mathcal{F}_{\nu,q}(f)(x)\overline{\mathcal{F}_{\nu,q}(g)(x)} d_{\mu_{\nu,q}}(x). \tag{10}$$

Theorem 2.5. *The q-Bessel Fourier transform satisfies:*

1. *For $f \in L^p_{\nu,q}(\mathbb{R}^+_q)$, $\quad \mathcal{F}^2_{q,\nu}f(x) = f(x), \quad \forall x \in \mathbb{R}^+_q$.*
2. *For $f \in L^2_{\nu,q}(\mathbb{R}^+_q)$, $\quad \|\mathcal{F}_{q,\nu}f\|_{q,2,\nu} = \|f\|_{q,2,\nu}$.*
3. *Let $f \in L^p_{\nu,q}(\mathbb{R}^+_q)$ and $g \in L^r_{\nu,q}(\mathbb{R}^+_q)$ then $f *_q g \in L^s_{\nu,q}(\mathbb{R}^+_q)$ and*

$$\mathcal{F}_{q,\nu}(f *_q g)(x) = \mathcal{F}_{q,\nu}(f)(x) \times \mathcal{F}_{q,\nu}(g)(x), \quad \forall x \in \mathbb{R}^+_q,$$

where $\dfrac{1}{p} + \dfrac{1}{r} - 1 = \dfrac{1}{s}$.

In the remainder, we denote by $\mathbf{H}$ the space $L^2_{\nu,q}(\mathbb{R}^+_q)$ which is a Hilbert space with the inner product

$$< f \mid g >_{\nu,q} = \int_0^{+\infty} f(\lambda)\overline{g(\lambda)} d\mu_{\nu,q}(\lambda).$$

3. On q-Weierstrass transform associated with the Fourier Bessel operator

In the literature we find many q-analogue of the exponential function, here we just recall that we needed namely the q-exponential function defined by

$$e(z,q) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_{q!}} = \frac{1}{((1-q)z;q)_\infty}, \quad |z| < \frac{1}{1-q}.$$

We introduce the following Hilbert space

$$\mathbf{H}_\beta = \left\{ f \in \mathbf{H} \mid \left(1+\lambda^2\right)^{\frac{\beta}{2}} \mathcal{F}_{q,\nu}(f)(\lambda) \in \mathbf{H} \right\}, \quad \beta > \nu + 1,$$

equipped With the inner product

$$\langle f \mid g \rangle_{\beta,q} = \int_0^{+\infty} \left(1+\lambda^2\right)^\beta \mathcal{F}_{q,\nu}(f)(\lambda) \overline{\mathcal{F}_{q,\nu}(g)(\lambda)} \lambda^{2\nu+1} d_q\lambda,$$

and the norm

$$\|f\|_{\beta,q} = \sqrt{\langle f, f \rangle_{\beta,q}}.$$

Proposition 3.1. *For $x \in \mathbb{R}_q^+$, the space $\mathbf{H}_\beta$ admits the reproducing kernel $\mathcal{K}_x^\beta$ given by*

$$\mathcal{K}_x^\beta(y) = \int_0^{+\infty} \frac{j_\nu(\lambda x, q^2) j_\nu(\lambda y, q^2)}{(1+\lambda^2)^\beta} \lambda^{2\nu+1} d_q\lambda.$$

Proof. we shall prove

$$f \in \mathbf{H}_\beta \Rightarrow < f \mid \mathcal{K}_x^\beta >_{\beta,q} = f(x), \forall x \in \mathbb{R}_q^+.$$

(It's called the reproducing property)

Indeed, a simple computation shows that

$$\mathcal{F}_{\nu,q}\left(\mathcal{K}_x^\beta\right)(\lambda) = \frac{j_\nu(\lambda x, q^2)}{(1+\lambda^2)^\beta} \Rightarrow \mathcal{K}_x^\beta \in \mathbf{H}_\beta. \tag{11}$$

Furthermore,

$$\begin{aligned} f \in \mathbf{H}_\beta &\Rightarrow < f \mid \mathcal{K}_x^\beta >_{\beta,q} \\ &= \int_0^{+\infty} \left(1+\lambda^2\right)^\beta \mathcal{F}_{\nu,q}(f)(\lambda) \overline{\mathcal{F}_{\nu,q}\left(\mathcal{K}_x^\beta\right)(\lambda)} d\mu_{\nu,q}(\lambda) \\ &= \int_0^{+\infty} \left(1+\lambda^2\right)^\beta \mathcal{F}_{\nu,q}(f)(\lambda) \frac{j_\nu(\lambda x, q^2)}{(1+\lambda^2)^\beta} d\mu_{\nu,q}(\lambda) \\ &= \int_0^{+\infty} \mathcal{F}_{\nu,q}(f)(\lambda) j_\nu(\lambda x, q^2) d\mu_{\nu,q}(\lambda) \\ &= f(x). \end{aligned}$$

□

3.1. q-Heat equation

Lemma 3.2. *The unique solution of the following q-difference equation*

$$D_{q^2,t}\psi(x,t) = -x^2\psi(x,t); \quad \psi(x,0) = c(x)$$

is given in the following form

$$\psi(x,t) = c(x)e(-tx^2, q^2).$$

Lemma 3.3. *Let $f \in L^2_{\nu,q}(\mathbb{R}_q^+)$ such that $\Delta_{q,\nu} f \in L^2_{\nu,q}(\mathbb{R}_q^+)$ then*

$$\mathcal{F}_{q,\nu}\left[\Delta_{q,\nu} f\right](x) = -x^2 \mathcal{F}_{q,\nu} f(x).$$

Proof. Let $g = \mathcal{F}_{q,\nu} f$. From the inversion formula we obtain

$$f = \mathcal{F}^2_{q,\nu} f = \mathcal{F}_{q,\nu} g.$$

Then

$$\Delta_{q,\nu} f(x) = \mathcal{F}_{q,\nu}\left[y \mapsto -y^2 g(y)\right](x)$$

Again, by using the inversion formula we obtain the result. □

Definition 3.4. The q-Weierstrass transform is defined as follows:

$$\mathcal{W}^\nu_{q,t} f(x) = (G^\nu(.,t,q^2) *_q f)(x), \quad t > 0,$$

where $G^\nu(.,t,q^2)$ is the q-Gauss kernel [1]

$$G^\nu(x,t,q^2) = \mathcal{F}_{q,\nu}\left[y \mapsto e(-ty^2,q^2)\right](x) = A(t)\ e\left(-\frac{q^{-2\nu}}{t}x^2, q^2\right),$$

and

$$A(t) = \frac{(-q^{2\nu+2}t, -q^{-2\nu}/t; q^2)_\infty}{(-t, -q^2/t; q^2)_\infty}.$$

Proposition 3.5. *For $\lambda \in \mathbb{R}_q^+$ and f on $L^2_{\nu,q}(\mathbb{R}_q^+)$, we denote*

$$\delta_\lambda f(x) = \frac{1}{\lambda^{(\nu+1)^2}} f\left(\frac{x}{\lambda^{\nu+1}}\right),$$

we have immediately the following properties:
1- $\|\delta_\lambda f\|_{\nu,q} = \|f\|_{\nu,q}$,
2- $\mathcal{F}_{q,\nu}\delta_\lambda = \delta_{\frac{1}{\lambda}}\mathcal{F}_{q,\nu}$,
3- $T^\nu_{q,x}\delta_\lambda = \delta_\lambda T^\nu_{q,\frac{x}{\lambda^{\nu+1}}}$,
therefore, for $f \in L^2_{\nu,q}(\mathbb{R}_q^+)$ and $x \in \mathbb{R}_q^+$, we get

$$\mathcal{W}^\nu_{q,t}(\delta_\lambda f)(x) = \delta_\lambda\left(\mathcal{W}^\nu_{q,\frac{t}{\lambda^{2\nu+2}}} f\right)(x).$$

Proof. For $t > 0$, we have

$$\begin{aligned}\mathcal{F}_{q,\nu}\left[e(-ty^2,q^2)\right](y) &= \frac{1}{\lambda^{(\nu+1)^2}}\mathcal{F}_{q,\nu}\left(\delta_{\frac{1}{\lambda}}\left[e(-\frac{t}{\lambda^{2\nu+2}}y^2,q^2)\right]\right)\\ &= \frac{1}{\lambda^{(\nu+1)^2}}\delta_\lambda\left(\mathcal{F}_{q,\nu}\left[e(-\frac{t}{\lambda^{2\nu+2}}y^2,q^2)\right]\right).\end{aligned}$$

Then for all $x \in \mathbb{R}_q^+$, we have

$$\begin{aligned}
&\mathcal{W}_{q,t}^{\nu}(\delta_\lambda f)(x)\\
&= G^{\nu}(.,t,q^2) *_q \delta_\lambda f(x)\\
&= \int_0^\infty T_{q,x}^{\nu}(\delta_\lambda f)(y) G^{\nu}(y,t,q^2) d\mu_{\nu,q}(y)\\
&= \frac{1}{\lambda^{(\nu+1)^2}} \int_0^\infty T_{q,x}^{\nu}(\delta_\lambda f)\delta_\lambda \Big(\mathcal{F}_{q,\nu}\big[e(-\frac{t}{\lambda^{2\nu+2}}y^2,q^2)\big]\Big) d\mu_{\nu,q}(y)\\
&= \frac{1}{\lambda^{(\nu+1)^2}} \int_0^\infty \delta_\lambda(T_{q,\frac{x}{\lambda^{\nu+1}}}^{\nu} f)(y)\delta_\lambda \Big(\mathcal{F}_{q,\nu}\big[e(-\frac{t}{\lambda^{2\nu+2}}y^2,q^2)\big]\Big) d\mu_{\nu,q}(y)\\
&= \frac{1}{\lambda^{3(\nu+1)^2}} \int_0^\infty T_{q,\frac{x}{\lambda^{\nu+1}}}^{\nu} f(\frac{y}{\lambda^{\nu+1}}) \Big(\mathcal{F}_{q,\nu}\big[e(-\frac{t}{\lambda^{4\nu+4}}y^2,q^2)\big]\Big) d\mu_{\nu,q}(y),
\end{aligned}$$

by the change of variable $y = z\lambda^{\nu+1}$, we get the desired result. □

In the following we denote by ϑ_t the q-Gauss kernel

$$\vartheta_t(x) = G^{\nu}(x,t,q^2),$$

which satisfies

$$\mathcal{F}(\vartheta_t)(x) = e(-tx^2,q^2).$$

Proposition 3.6. *For all $f \in L^2_{\nu,q}(\mathbb{R}_q^+)$ and $t > 0$, the q-Weierstrass transform associated with the Fourier Bessel operator is satisfying*

$$\lim_{t\to 0^+} \mathcal{W}_{q,t}^{\nu}(f) = f, in \quad L^2_{\nu,q}(\mathbb{R}_q^+).$$

For all $t > 0$, $\beta > \nu+1$, and for all $f \in \mathbf{H}_\beta^\nu$, the bounded linear operator $\mathcal{W}_{q,t}$ satisfies

$$\left\|\mathcal{W}_{q,t}(f)\right\|_{\nu,q} \leq \|f\|_{\beta,q}.$$

Proof. According the Ramanujan identity [1]

$$\sum_{s\in\mathbb{Z}} \frac{z^s}{(bq^{2s};q^2)_\infty} = \frac{\left(bz, \frac{q^2}{bz}, q^2; q^2\right)_\infty}{\left(b, z, \frac{q^2}{b}; q^2\right)_\infty},$$

we obtain

$$\begin{aligned}
&c_{q,\nu}\int_0^\infty e\left(-\frac{q^{-2\nu}}{t}x^2,q^2\right)x^{2\nu+1}d_qx\\
&=(1-q)c_{q,\nu}\sum_{n\in\mathbb{Z}}q^{(2\nu+2)n}e\left(-\frac{q^{-2\nu}}{t}q^{2n},q^2\right)\\
&=(1-q)c_{q,\nu}\sum_{n\in\mathbb{Z}}\frac{q^{(2\nu+2)n}}{\left(-\frac{q^{-2\nu}}{t}q^{2n},q^2\right)_\infty}\\
&=(1-q)c_{q,\nu}\frac{(-q^2/t,-t,q^2;q^2)_\infty}{(-q^{-2\nu}/t,q^{2\nu+2},-q^{2\nu+2}t;q^2)_\infty}\\
&=\frac{(-q^2/t,-t;q^2)_\infty}{(-q^{-2\nu}/t,-q^{2\nu+2}t;q^2)_\infty}\\
&=\frac{1}{A(t)},
\end{aligned}$$

therefore

$$\left\|G^\nu(.,t,q^2)\right\|_{\nu,q,1}=1,$$

which implies that for $t>0$, the kernel $\left(G^\nu(x,t,q^2)\right)_{t>0}$ is an approximate identity, in particular for $f\in L^2_{\nu,q}(\mathbb{R}_q^+)$, we get

$$\lim_{t\to0^+}G^\nu(.,t,q^2)*_qf=f, in \quad L^2_{\nu,q}(\mathbb{R}_q^+).$$

By using the inequality of Young, the properties of the convolution product and q-Plancherel theorem, we deduce for $f\in\mathbf{H}_\beta$

$$\begin{aligned}
\left\|\mathcal{W}_{q,t}(f)\right\|_{\nu,q,2}&=\left\|\vartheta_t\star_qf\right\|_{\nu,q,2}\\
&=\left\|\mathcal{F}_{\nu,q}(\vartheta_t\star_qf)\right\|_{\nu,q,2}\\
&=\left\|\left(e(-t\lambda^2,q^2)\mathcal{F}_{\nu,q}f\right)\right\|_{\nu,q,2}\\
&\leqslant\left\|e(-t\lambda^2,q^2)\right\|_{\nu,q,1}\|f\|_{\nu,q,2}\leqslant\|f\|_{\beta,q}.
\end{aligned}$$

□

Proposition 3.7. *The unique solution of the q-heat equation*

$$\begin{cases}\Delta_{q,\nu}u(x,t)=D_{q^2}u(x,t), & \forall x\in\mathbb{R}_q^+\\ \lim\limits_{t\to0}u(x,t)=f(x), & f\in L^2_{\nu,q}(\mathbb{R}_q^+),\\ x\mapsto u(x,t)\in L^2_{\nu,q}(\mathbb{R}_q^+), & \forall t>0\end{cases}$$

is

$$u(x,t)=\mathcal{W}^\nu_{q,t}f(x)=\vartheta_t*_qf(x).$$

Proof. Let

$$\psi(x,t)=\mathcal{F}_{q,v}\Big[u(.,t)\Big](x).$$

From the following equation

$$\Delta_{q,v}u(x,t)=D_{q^2}u(x,t),$$

we have

$$\Delta_{q,v}u(.,t)\in L^2_{v,q}(\mathbb{R}_q^+).$$

Lemma 3.3 implies that

$$\mathcal{F}_{q,v}\Big[\Delta_{q,v}u(.,t)\Big](x)=-x^2\mathcal{F}_{q,v}\Big[u(.,t)\Big](x)$$

and

$$\mathcal{F}_{q,v}\Big[D_{q^2}u(.,t)\Big](x)=D_{q^2}\mathcal{F}_{q,v}\Big[u(.,t)\Big](x).$$

The function ψ satisfies

$$D_{q^2}\psi(x,t)=-x^2\psi(x,t),\quad \lim_{t\to 0}\psi(x,t)=\mathcal{F}_{q,v}f(x).$$

From Lemma 3.2 and Theorem 2.5 we see that

$$\psi(x,t)=e(-tx^2,q^2)\mathcal{F}_{q,v}f(x)=\mathcal{F}_{q,v}\Big[\vartheta_t\Big](x)\times\mathcal{F}_{q,v}f(x)=\mathcal{F}_{q,v}\Big[\vartheta_t*_q f\Big](x).$$

Therefore

$$u(x,t)=\vartheta_t*_q f(x),$$

which completes the proof. □

3.2. Practical real inversion formulas for $\mathcal{W}_{q,t}$

For $\beta\in\mathbb{R},\beta>v+1$, we denote $\mathbf{H}_{\beta,\xi}$ the space $\mathbf{H}_\beta$ equipped with the inner product

$$<f\,|\,g>_{\beta,\xi,q}=\xi<f\,|\,g>_{\beta,q}+<\mathcal{W}_{q,t}(f)\,|\,\mathcal{W}_{q,t}(g)>_{v,q},$$

and the norm

$$\|f\|^2_{\beta,\xi,q}=\xi\,\|f\|^2_{\beta,q}+\left\|\mathcal{W}_{q,t}(f)\right\|^2_{v,q}.$$

Proposition 3.8. *For $x\in\mathbb{R}_q^+$, the space $\mathbf{H}_{\beta,\xi}$ admits the reproducing kernel $\mathcal{K}_x^{\beta,\xi}$ given by*

$$\mathcal{K}_x^{\beta,\xi}(y)=\int_0^{+\infty}\frac{j_v(\lambda x,q^2)j_v(\lambda y,q^2)}{\xi\,(1+\lambda^2)^\beta+e\left(-t\lambda^2;q^2\right)^2}d\mu_{v,q}(\lambda).$$

Proof. We shall prove

$$f \in \mathbf{H}_{\beta,\xi} \Rightarrow < f \mid \mathcal{K}_x^{\beta,\xi} >_{\beta,\xi} = f(x), \forall x \in \mathbb{R}_q^+.$$

(It's called the reproducing property)

The following computation proves that

$$\mathcal{F}_{\nu,q}\left(\mathcal{K}_x^{\beta,\xi}\right)(\lambda) = \frac{j_\nu(\lambda x, q^2)}{\xi\,(1+\lambda^2)^\beta + e\,(-t\lambda^2;q^2)^2} \Rightarrow \mathcal{K}_x^{\beta,\xi} \in \mathbf{H}_{\beta,\xi}. \qquad (12)$$

Let f be in $\mathbf{H}_{\beta,\xi}^\nu$, then we have

$$< f \mid \mathcal{K}_x^{\beta,\xi} >_{\beta,\xi,q} = \xi < f \mid \mathcal{K}_x^{\beta,\xi} >_{\beta,q} + < \mathcal{W}_{q,t}(f) \mid \mathcal{W}_{q,t}\left(\mathcal{K}_x^{\beta,\xi}\right) >_{\nu,q},$$

Indeed, we see that

$$< f \mid \mathcal{K}_x^{\beta,\xi} >_{\beta,q} = \int_0^{+\infty} \frac{\left(1+\lambda^2\right)^\beta j_\nu(\lambda x, q^2)}{\xi\,(1+\lambda^2)^\beta + e\,(-t\lambda^2;q^2)^2} \mathcal{F}_{\nu,q}(f)(\lambda)\,d\mu_{\nu,q}(\lambda).$$

$$\begin{aligned}\mathcal{W}_{q,t}\left(\mathcal{K}_x^{\beta,\xi}\right)(\lambda) &= \vartheta_t \star_q \mathcal{K}_x^{\beta,\xi}(\lambda)\\ &= \mathcal{F}_{\nu,q}\left(\frac{j_\nu(xy,q^2)e\left(-ty^2;q^2\right)}{\xi\,(1+y^2)^\beta + e\,(-ty^2;q^2)^2}\right)(\lambda).\end{aligned}$$

Similarly, we have

$$\mathcal{W}_{q,t}(f)(x) = \mathcal{F}_{\nu,q}\left(e\left(-t\lambda^2;q^2\right)\mathcal{F}_{\nu,q}(f)\right)(x).$$

Therefore according to the Parseval theorem we get

$$\begin{aligned}&< \mathcal{W}_{q,t}(f) \mid \mathcal{W}_{q,t}\left(\mathcal{K}_x^{\beta,\xi}\right) >_{\nu,q}\\ &=< \mathcal{F}_{\nu,q}\mathcal{W}_{q,t}(f) \mid \mathcal{F}_{\nu,q}\mathcal{W}_{q,t}\left(\mathcal{K}_x^{\beta,\xi}\right) >_{\nu,q}\\ &=< \mathcal{F}_{\nu,q}(\vartheta_t)\mathcal{F}_{\nu,q}(f)) \mid \left(\frac{j_\nu(\lambda x,q^2)e\left(-t\lambda^2;q^2\right)}{\xi\,(1+\lambda^2)^\beta + e\,(-t\lambda^2;q^2)^2}\right) >_{\nu,q}\\ &= \int_0^{+\infty} \frac{e\left(-t\lambda^2;q^2\right)^2 j_\nu(\lambda x,q^2)}{\xi\,(1+\lambda^2)^\beta + e\,(-t\lambda^2;q^2)^2}\mathcal{F}_{\nu,q}(f)(\lambda)\,d\mu_{\nu,q}(\lambda).\end{aligned}$$

By the inversion formula for $\mathcal{F}_{\nu,q}$, we get the desired result:

$$\begin{aligned}< f \mid \mathcal{K}_x^{\beta,\xi}(x,y) >_{\beta,\xi,q} &= \int_0^{+\infty} j_\nu(\lambda x,q^2)\mathcal{F}_{\nu,q}(f)(\lambda)\,d\mu_{\nu,q}(\lambda)\\ &= f(x).\end{aligned}$$

□

In the following section we are interested by extremal functions to solve the problem of inverse approximation. We use reproducing kernel Hilbert spaces to give the best approximation for the bounded linear operator $\mathcal{W}_{q,t}$. Using the Saitoh's Theorem [7] we obtain the following results :

3.3. Extremal function for $\mathcal{W}_{q,t}$

Theorem 3.9. *Let $\xi > 0$, $\beta > \nu + 1$ and $g \in L^2_{\nu,q}(\mathbb{R}^+_q)$. Then the approximation problem*

$$\inf_{f \in \mathbf{H}_\beta} \left(\xi \, \|f\|^2_{\beta,q} + \left\| g - \mathcal{W}_{q,t}(f) \right\|^2_{\nu,q} \right) \tag{13}$$

is solvable and

$$f^*_{\xi,g}(x) = < g \,|\, \mathcal{W}_{q,t}\left(\mathcal{K}^{\beta,\xi}_x \right) >_{\nu,q}$$

is the element of $\mathbf{H}_\beta$ with the smallest $\mathbf{H}_\beta$-norm where the infimum (13) is attained.

For $g \in L^2_{\nu,q}(\mathbb{R}^+_q)$, the function

$$f(x) = \lim_{\xi \to 0} f^*_{\xi,g}(x)$$

solves the problem of approximation

$$\inf_{f \in \mathbf{H}} \left\| g - \mathcal{W}_{q,t}(f) \right\|^2_{\nu,q}.$$

Remark 3.10. Applying Parseval's equality, the q-analogue of extremal function $f^*_{\xi,g}$ is also written as

$$\begin{aligned}
f^*_{\xi,g}(x) &= < g \,|\, \mathcal{W}_{q,t}\left(\mathcal{K}^{\beta,\xi}_x \right) >_{\nu,q} \\
&= \int_0^{+\infty} g(\lambda) \, \mathcal{W}_{q,t}\left(\mathcal{K}^{\beta,\xi}_x \right)(\lambda) \, d\mu_{\nu,q}(\lambda) \\
&= \int_0^{+\infty} g(\lambda) \, \mathcal{F}_{\nu,q}\left(\frac{j_\nu(xy,q^2) e\left(-ty^2;q^2\right)}{\xi\,(1+y^2)^\beta + e\left(-ty^2;q^2\right)^2} \right)(\lambda) \, d\mu_{\nu,q}(\lambda) \\
&= \int_0^{+\infty} g(y) \, \mathcal{F}_{\nu,q}\left(\frac{j_\nu(\lambda x,q^2) e\left(-t\lambda^2;q^2\right)}{\xi\,(1+\lambda^2)^\beta + e\left(-t\lambda^2;q^2\right)^2} \right)(y) \, d\mu_{\nu,q}(y) \\
&= \int_0^{+\infty} \mathcal{F}_{\nu,q}(g)(\lambda) \, \frac{j_\nu(\lambda x,q^2) e\left(-t\lambda^2;q^2\right)}{\xi\,(1+\lambda^2)^\beta + e\left(-t\lambda^2;q^2\right)^2} d\mu_{\nu,q}(\lambda) \\
&= \mathcal{F}_{\nu,q}\left(\mathcal{F}_{\nu,q}(g)(\lambda) \, \frac{e\left(-t\lambda^2;q^2\right)}{\xi\,(1+\lambda^2)^\beta + e\left(-t\lambda^2;q^2\right)^2} \right)(x),
\end{aligned}$$

which implies that

$$\mathcal{F}_{\nu,q}\left(f^*_{\xi,g}\right)(\lambda) = \frac{e\left(-t\lambda^2;q^2\right)}{\xi\,(1+\lambda^2)^\beta + e\,(-2t\lambda^2;q^2)}\mathcal{F}_{\nu,q}\,(g)\,(\lambda)\,. \tag{14}$$

Corollary 3.11. *Let* $\xi > 0$, $\beta > \nu + 1$ *and* $g \in L^2_{\nu,q}(\mathbb{R}^+_q)$. *The q-extremal function* $f^*_{\xi,g}$ *satisfies*

$$\left\|f^*_{\xi,g}\right\|^2_{\nu,q,2} \leqslant c\,(\xi,\beta,\nu)\,\|g\|^2_{\nu,q,2}\,,$$

where $c\,(\xi,\beta,\nu)$ *is constant.*

Proof. Hölder's inequality and integrating over x lead to

$$\left\|f^*_{\xi,g}\right\|^2_{\nu,q,2} \leqslant \|g\|^2_{\nu,q,2}\left\|\mathcal{W}_{q,t}(\mathcal{K}^{\beta,\xi}_x)\right\|^2_{\nu,q,2}.$$

Since $a^2+b^2 \geqslant 2ab$ and by the Plancherel theorem and Remark 3.10 we get

$$\begin{aligned}
&\left\|\mathcal{W}_{q,t}\left(\mathcal{K}^{\beta,\xi}_x\right)\right\|^2_{\nu,q,2} \\
&\leqslant \int_0^{+\infty}\left|\mathcal{F}_{\nu,q}\left(\frac{j^2_\nu(\lambda x,q^2)e\left(-t\lambda^2;q^2\right)^2}{\left(\xi\,(1+\lambda^2)^\beta + e\,(-t\lambda^2;q^2)^2\right)^2}\right)(y)\right| d\mu_{\nu,q}\,(\lambda) \\
&\leqslant \frac{1}{(q;q^2)^2_\infty\,\xi}\int_0^{+\infty}\frac{1}{(1+\lambda^2)^\beta}d\mu_{\nu,q}\,(\lambda) \\
&= c\,(\xi,\beta,\nu)\,.
\end{aligned}$$

□

Corollary 3.12. *Let* $\xi > 0$, $\beta > \nu + 1$.*For all* $g_1, g_2 \in L^2_{\nu,q}(\mathbb{R}^+_q)$, *we have*

$$\left\|f^*_{\xi,g_1} - f^*_{\xi,g_2}\right\|^2_{\beta,q} \leqslant \frac{\|g_1-g_2\|^2_{\nu,q,2}}{4\xi}.$$

Proof. From Remark 3.10, we get

$$\begin{aligned}
&(f^*_{\xi,g_1} - f^*_{\xi,g_2})(x) \\
&= \int_0^{+\infty}(g_1-g_2)\,(y)\,\mathcal{W}_{q,t}\left(\mathcal{K}^{\beta,\xi}_x\right)(y)\,d\mu_{\nu,q}\,(y) \\
&= \int_0^{+\infty}(g_1-g_2)\,(y)\,\mathcal{F}_{\nu,q}\left(\frac{j_\nu(\lambda x,q^2)e\left(-t\lambda^2;q^2\right)}{\xi\,(1+\lambda^2)^\beta + e\,(-t\lambda^2;q^2)^2}\right)(y)\,d\mu_{\nu,q}\,(y) \\
&= \int_0^{+\infty}\mathcal{F}_{\nu,q}\,(g_1-g_2)\,(\lambda)\,\frac{j_\nu(\lambda x,q^2)e\left(-t\lambda^2;q^2\right)}{\xi\,(1+\lambda^2)^\beta + e\,(-t\lambda^2;q^2)^2}d\mu_{\nu,q}\,(\lambda)\,,
\end{aligned}$$

and

$$\mathcal{F}_{\nu,q}\left(f^*_{\xi,g_1}-f^*_{\xi,g_2}\right)(\lambda)=\frac{e\left(-t\lambda^2;q^2\right)}{\xi\,(1+\lambda^2)^{\beta}+e\left(-t\lambda^2;q^2\right)^2}\mathcal{F}_{\nu,q}(g_1-g_2)(\lambda).$$

Since $2ab\le a^2+b^2$, we deduce

$$\begin{aligned}
&\left\|f^*_{\xi,g_1}-f^*_{\xi,g_2}\right\|^2_{\beta,q}\\
&=\int_0^{+\infty}\left(1+\lambda^2\right)^{\beta}\left|\mathcal{F}_{\nu,q}\left(f^*_{\xi,g_1}-f^*_{\xi,g_2}\right)(\lambda)\right|^2 d\mu_{\nu,q}(\lambda)\\
&\le\int_0^{+\infty}\frac{\left(1+\lambda^2\right)^{\beta}e\left(-t\lambda^2;q^2\right)^2}{4\xi\,(1+\lambda^2)^{\beta}e\left(-t\lambda^2;q^2\right)^2}\left|\mathcal{F}_{\nu,q}(g_1-g_2)(\lambda)\right|^2 d\mu_{\nu,q}(\lambda)\\
&\leqslant\frac{1}{4\xi}\int_0^{+\infty}\left|\mathcal{F}_{\nu,q}(g_1-g_2)(\lambda)\right|^2 d\mu_{\nu,q}(\lambda)\\
&=\frac{1}{4\xi}\left\|g_1-g_2\right\|^2_{\nu,q,2}.
\end{aligned}$$

□

Corollary 3.13. *Let $\xi>0$, $\beta>\nu+1$. For all $g_1,g_2\in L^2_{\nu,q}(\mathbb{R}^+_q)$, we have*

$$\left\|f^*_{\xi,g_1}-f^*_{\xi,g_2}\right\|^2_{\beta,\xi,q}\leqslant\frac{\xi+1}{4\xi}\left\|g_1-g_2\right\|^2_{\nu,q,2}.$$

Proof. We have

$$\begin{aligned}
\left\|f^*_{\xi,g_1}-f^*_{\xi,g_2}\right\|^2_{\beta,\xi,q}&=\xi\left\|f^*_{\xi,g_1}-f^*_{\xi,g_2}\right\|^2_{\beta,q}+\left\|\mathcal{W}_t\left(f^*_{\xi,g_1}-f^*_{\xi,g_2}\right)\right\|^2_{\nu,q,2}\\
&\leqslant\xi\left\|f^*_{\xi,g_1}-f^*_{\xi,g_2}\right\|^2_{\beta,q}+\left\|f^*_{\xi,g_1}-f^*_{\xi,g_2}\right\|^2_{\beta,q}\\
&=(\xi+1)\left\|f^*_{\xi,g_1}-f^*_{\xi,g_2}\right\|^2_{\beta,q}.
\end{aligned}$$

Apply again the fact that $a^2+b^2\geqslant 2ab$ and by the relation (14) we obtain

$$\begin{aligned}
&\left\|f^*_{\xi,g_1}-f^*_{\xi,g_2}\right\|^2_{\beta,q}=\int_0^{+\infty}\left(1+\lambda^2\right)^{\beta}\left|\mathcal{F}_{\nu,q}\left(f^*_{\xi,g_1}-f^*_{\xi,g_2}\right)(\lambda)\right|^2 d\mu_{\nu,q}(\lambda)=\\
&\int_0^{+\infty}\left(1+\lambda^2\right)^{\beta}\frac{e\left(-t\lambda^2;q^2\right)^2}{\left(\xi\,(1+\lambda^2)^{\beta}+e\left(-t\lambda^2;q^2\right)^2\right)^2}\left|\mathcal{F}_{\nu,q}(g_1-g_2)(\lambda)\right|^2 d\mu_{\nu,q}(\lambda)\\
&\leqslant\frac{1}{4\xi}\int_0^{+\infty}\left|\mathcal{F}_{\nu,q}(g_1-g_2)(\lambda)\right|^2 d\mu_{\nu,q}(\lambda)\\
&=\frac{1}{4\xi}\left\|g_1-g_2\right\|^2_{\nu,q,2},
\end{aligned}$$

and we deduce that

$$\left\| f^*_{\xi,g_1} - f^*_{\xi,g_2} \right\|^2_{\beta,\xi,q} \leqslant \frac{\xi+1}{4\xi} \left\| g_1 - g_2 \right\|^2_{\nu,q,2}.$$

□

Corollary 3.14. *Let $\xi > 0$, $\beta > \nu + 1$. For all $f \in \mathbf{H}_\beta$, and for $g = \mathcal{W}_{q,t}(f)$ we have*

$$\lim_{\xi \to 0^+} \left\| f^*_{\xi,g} - f \right\|^2_{\beta,q} = 0.$$

Moreover, $\left(f^*_{\xi,g} \right)_{\xi>0}$ converges uniformly to f as $\xi \to 0^+$.

Proof. If $f \in \mathbf{H}_\beta$, then we have $g = \mathcal{W}_{q,t}(f) \in L^2_{\nu,q}(\mathbb{R}^+_q)$. On the other hand, simple computation and Remark 3.10 show that

$$\mathcal{F}_{\nu,q}(\mathcal{W}_t(f))(\lambda) = e\left(-t\lambda^2; q^2\right) \mathcal{F}_{\nu,q}(f)(\lambda), \tag{15}$$

and

$$\begin{aligned}
\mathcal{F}_{\nu,q}\left(f^*_{\xi,g_1} - f \right)(\lambda) &= \frac{e\left(-t\lambda^2; q^2\right) \mathcal{F}_{\nu,q}(g)(\lambda)}{\xi (1+\lambda^2)^\beta + e(-t\lambda^2; q^2)^2} - \mathcal{F}_{\nu,q}(f)(\lambda) \\
&= \frac{e\left(-t\lambda^2; q^2\right) \mathcal{F}_{\nu,q}(\mathcal{W}_t(f))(\lambda)}{\xi (1+\lambda^2)^\beta + e(-t\lambda^2; q^2)^2} - \mathcal{F}_{\nu,q}(f)(\lambda) \\
&= \frac{-\xi \left(1+\lambda^2\right)^\beta}{\xi (1+\lambda^2)^\beta + e(-t\lambda^2; q^2)^2} \mathcal{F}_{\nu,q}(f)(\lambda), \qquad (16)
\end{aligned}$$

then

$$\begin{aligned}
&\left\| f^*_{\xi,g_1} - f \right\|^2_{\beta,q} \\
&= \int_0^{+\infty} \left(1+\lambda^2\right)^\beta \left| \mathcal{F}_{\nu,q}\left(f^*_{\xi,g_1} - f \right)(\lambda) \right|^2 d\mu_{\nu,q}(\lambda) \\
&= \int_0^{+\infty} \left(1+\lambda^2\right)^\beta \left| \frac{-\xi \left(1+\lambda^2\right)^\beta}{\xi (1+\lambda^2)^\beta + e(-t\lambda^2; q^2)^2} \mathcal{F}_{\nu,q}(f)(\lambda) \right|^2 d\mu_{\nu,q}(\lambda) \\
&= \int_0^{+\infty} \frac{\xi^2 \left(1+\lambda^2\right)^{3\beta}}{\left(\xi (1+\lambda^2)^\beta + e(-t\lambda^2; q^2)^2 \right)^2} \left| \mathcal{F}_{\nu,q}(f)(\lambda) \right|^2 d\mu_{\nu,q}(\lambda).
\end{aligned}$$

For $\lambda \in \mathbb{R}^+_q$ we have,

$$\lim_{\xi \to 0^+} \frac{\xi^2 \left(1+\lambda^2\right)^{3\beta}}{\left(\xi (1+\lambda^2)^\beta + e(-t\lambda^2; q^2)^2 \right)^2} \left| \mathcal{F}_{\nu,q}(f)(\lambda) \right|^2 = 0,$$

and

$$\frac{\xi^2\left(1+\lambda^2\right)^{3\beta}}{\left(\xi\left(1+\lambda^2\right)^{\beta}+e\left(-t\lambda^2;q^2\right)^2\right)^2}\left|\mathcal{F}_{\nu,q}(f)(\lambda)\right|^2\leqslant\left(1+\lambda^2\right)^{\beta}\left|\mathcal{F}_{\nu,q}(f)(\lambda)\right|^2.$$

Since $f\in\mathbf{H}_\beta$, then the function

$$\left(1+\lambda^2\right)^{\beta}\left|\mathcal{F}_{\nu,q}(f)(\lambda)\right|^2$$

is integrable with respect to the measure $d\mu_{\nu,q}$. So, according the dominated convergence theorem, we deduce that

$$\lim_{\xi\to0^+}\left\|f^*_{\xi,g}-f\right\|^2_{\beta,q}=0.$$

Remark 3.10 and the relations (14), (15) and (34) clearly show that the function $\mathcal{F}_{\nu,q}\left(f^*_{\xi,g}-f\right)$ belongs to $L^1_{\nu,q}\left(\mathbb{R}^+_q\right)$, therefore it satisfies the inversion formula, namely for all $x\in\mathbb{R}^+_q$, we have

$$\begin{aligned}\left|f^*_{\xi,g}(x)-f(x)\right|&=\left|\int_0^{+\infty}\mathcal{F}_{\nu,q}\left(f^*_{\xi,g}-f\right)(\lambda)\,j_\nu\left(\lambda x;q^2\right)d_{\mu_{\nu,q}}(\lambda)\right|\\&=\left|\int_0^{+\infty}\frac{-\xi\left(1+\lambda^2\right)^{\beta}\mathcal{F}_{\nu,q}(f)(\lambda)}{\xi\left(1+\lambda^2\right)^{\beta}+e\left(-t\lambda^2;q^2\right)^2}j_\nu\left(\lambda x;q^2\right)d_{\mu_{\nu,q}}(\lambda)\right|\\&\leqslant\int_0^{+\infty}\frac{\xi\left(1+\lambda^2\right)^{\beta}}{\xi\left(1+\lambda^2\right)^{\beta}+e\left(-t\lambda^2;q^2\right)^2}\left|\mathcal{F}_{\nu,q}(f)(\lambda)\right|d_{\mu_{\nu,q}}(\lambda),\end{aligned}$$

and

$$\frac{\xi\left(1+\lambda^2\right)^{\beta}}{\xi\left(1+\lambda^2\right)^{\beta}+e\left(-t\lambda^2;q^2\right)^2}\left|\mathcal{F}_{\nu,q}(f)(\lambda)\right|\leqslant\left|\mathcal{F}_{\nu,q}(f)(\lambda)\right|$$

which is integrable with respect to the measure $d\mu_{\nu,q}$. Then, the result follows from dominated convergence theorem. □

REFERENCES

[1] L. Dhaouadi - A. Fitouhi - J. El Kamel, *Inequalities in q -Fourier Analysis*, Journal of Inequalities in Pure and Applied Mathematics 7 (5) (2006), Article 171.

[2] L. Dhaouadi, *On the q-Bessel Fourier transform*, Bulletin of Mathematical Analysis and Applications 5 (2) (2013), 42–60.

[3] A. Fitouhi - M. Hamza - F. Bouzeffour, *The q-j_α Bessel function*, J. Appr. Theory 115 (2002), 144–166.

[4] G. Gasper - R. Rahman, *Basic Hypergeometric Series*, 2nd Edition, Encyclopedia of Mathematics and Its Applications, 96, Cambridge University Press, Cambridge, 2004.

[5] F. H. Jackson, *On a q-Definite Integrals*, Quarterly Journal of Pure and Applied Mathematics 41 (1910), 193–203.

[6] T. H. Koornwinder - R. F. Swarttouw, *On q-analogues of the Hankel and Fourier Transforms*, Trans. A. M. S. 333 (1992), 445–461.

[7] S. Saitoh, *Theory of reproducing kernels: applications to approximate solutions of bounded linear operator equations on Hilbert spaces*, Amer. Math. Soc. Transl. Ser. 2 230, Amer. Math. Soc., Providence, RI, 2010.

[8] S. Saitoh, *Approximate real inversion formulas of the gaussian convolution*, Appl. Anal. 83 (7) (2004), 727–733.

[9] R. F. Swarttouw, *The Hahn-Exton q-Bessel functions*, Ph. D. Thesis, Delft Technical University (1992).

SOUMAYA CHEFAI
Department of Mathematic
Faculty of Science of Tunis, University of Tunis el Manar, 1060 Tunis, Tunisia.
e-mail: klembi.soumaya56@hotmail.fr

KAMEL BRAHIM
Department of Mathematic
Faculty of Science of Tunis, University of Tunis el Manar, 1060 Tunis, Tunisia.
e-mail: Kamel.Brahim@ipeit.rnu.tn