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EXPANSION FORMULAS FOR APOSTOL TYPE Q-APPELL POLYNOMIALS, AND THEIR SPECIAL CASES

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We present identities of various kinds for generalized q–Apostol-Bernoulli and Apostol-Euler polynomials and power sums, which resemble q–analogues of formulas from the 2009 paper by Liu and Wang. These formulas are divided into two types: formulas with only q–Apostol-Bernoulli, and only q–Apostol-Euler polynomials, or so-called mixed formulas, which contain polynomials of both kinds. This can be seen as a logical consequence of the fact that the q–Appell polynomials form a commutative ring. The functional equations for Ward numbers operating on the q–exponential function, as well as symmetry arguments, are essential for many of the proofs. We conclude by finding multiplication formulas for two q–Appell polynomials of general form. This brings us to the q–H polynomials, which were discussed in a previous paper.

1. Introduction

In the second article on q–analogues of two Appell polynomials [4], the Apostol-Bernoulli and Apostol-Euler polynomials, focus was on multiplication formulas and on formulas including (multiple) λ power sums. In this article we will find a corresponding multiplication formula for a more general q–Appell polynomial, which is a generalization of both q–Apostol-Euler and q–Apostol-H polynomials.

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There are many new formulas on this subject, both Apostol-Appell and similar Appell, which have recently been published; in all cases the limit $\lambda \to 1$ is straightforward. Sometimes we write q-analogue of etc., not bothering about the above dichotomy.

This paper is organized as follows: In section 1 we give a general introduction och the definitions. In section 2 we present formulas with only q–Apostol-Bernoulli, and only q–Apostol-Euler polynomials. In section 3 we present mixed formulas for these polynomials. In section 4, two general polynomials are defined, which generalize the q–Apostol-Bernoulli and q–Apostol-Euler polynomials. Then multiplication formulas for these polynomials are proved, which specialize to the q–Apostol-H polynomials.

We now start with the definitions. Some of the notation is well-known and can be found in the book [1]. The variables i, j, k, l, m, n, ν vill denote positive integers, and λ, μ will denote complex numbers when nothing else is stated.

Definition 1.1. The Gauss q–binomial coefficient are defined by

$$\binom{n}{k}_q \equiv \frac{\{n\}_q!}{\{k\}_q!\{n-k\}_q!}, k = 0, 1, \ldots, n. \tag{1}$$

Let a and b be any elements with commutative multiplication. Then the NWA q–addition is given by

$$(a \oplus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q a^k b^{n-k},\ n = 0, 1, 2, \ldots. \tag{2}$$

If $0 < |q| < 1$ and $|z| < |1-q|^{-1}$, the q–exponential function is defined by

$$\mathrm{E}_q(z) \equiv \sum_{k=0}^{\infty} \frac{1}{\{k\}_q!} z^k. \tag{3}$$

The following theorem shows how Ward numbers usually appear in applications.

Theorem 1.1. *Assume that $n, k \in \mathbb{N}$. Then*

$$(\overline{n}_q)^k = \sum_{m_1+\ldots+m_n=k} \binom{k}{m_1, \ldots, m_n}_q, \tag{4}$$

where each partition of k is multiplied with its number of permutations.

Theorem 1.2. *Functional equations for Ward numbers operating on the q–exponential function. First assume that the letters $\overline{m}_q$ and $\overline{n}_q$ are independent,* *i.e. come from two different functions, when operating with the functional. Furthermore, $mnt < \frac{1}{1-q}$. Then we have*

$$\mathrm{E}_q(\overline{m}_q\overline{n}_q t) = \mathrm{E}_q(\overline{mn}_q t). \tag{5}$$

Furthermore,

$$\mathrm{E}_q(\overline{jm}_q) = \mathrm{E}_q(\overline{j}_q)^m = \mathrm{E}_q(\overline{m}_q)^j. \tag{6}$$

Compare with the semiring of Ward numbers [1, p. 167].

Proof. Formula (5) is proved as follows:

$$\mathrm{E}_q(\overline{m}_q\overline{n}_q t) = \mathrm{E}_q((1 \oplus_q 1 \oplus_q \cdots \oplus_q 1)\overline{n}_q t), \tag{7}$$

where the number of 1s to the left is m. But this means exactly $\mathrm{E}_q(\overline{n}_q t)^m$, and the result follows. □

Definition 1.2. The generalized NWA q–Apostol-Bernoulli polynomials $\mathscr{B}^{(n)}_{\mathrm{NWA},\lambda,\nu,q}(x)$ are defined by

$$\frac{t^n}{(\lambda \mathrm{E}_q(t)-1)^n}\mathrm{E}_q(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu \mathscr{B}^{(n)}_{\mathrm{NWA},\lambda,\nu,q}(x)}{\{\nu\}_q!}, \; |t+\log\lambda| < 2\pi. \tag{8}$$

Definition 1.3. The generalized NWA q–Apostol-Euler polynomials $\mathscr{F}^{(n)}_{\mathrm{NWA},\lambda,\nu,q}(x)$ are defined by

$$\frac{2^n}{(\lambda \mathrm{E}_q(t)+1)^n}\mathrm{E}_q(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu \mathscr{F}^{(n)}_{\mathrm{NWA},\lambda,\nu,q}(x)}{\{\nu\}_q!}, \; |t+\log\lambda| < \pi. \tag{9}$$

Definition 1.4. The generalized NWA q–$\mathscr{H}$ polynomials
are defined by

$$\frac{(2t)^n}{(\lambda \mathrm{E}_q(t)+1)^n}\mathrm{E}_q(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu \mathscr{H}^{(n)}_{\mathrm{NWA},\lambda,\nu,q}(x)}{\{\nu\}_q!}, \; |t+\log\lambda| < \pi. \tag{10}$$

Definition 1.5. The generalized JHC q–$\mathscr{H}$ polynomials
are defined by

$$\frac{(2t)^n}{(\lambda \mathrm{E}_{\frac{1}{q}}(t)+1)^n}\mathrm{E}_q(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu \mathscr{H}^{(n)}_{\mathrm{JHC},\lambda,\nu,q}(x)}{\{\nu\}_q!}, \; |t+\log\lambda| < \pi. \tag{11}$$

Definition 1.6. The generating function for $\mathrm{H}_{\mathrm{NWA},\nu,q}^{(n)}(x)$ is given by

$$\frac{(2t)^n}{(\mathrm{E}_q(t)+1)^n}\mathrm{E}_q(xt)=\sum_{\nu=0}^{\infty}\frac{t^\nu \mathrm{H}_{\mathrm{NWA},\nu,q}^{(n)}(x)}{\{\nu\}_q!},\ |t|<2\pi. \tag{12}$$

Definition 1.7. The generating function for $\mathrm{H}_{\mathrm{JHC},\nu,q}^{(n)}(x)$ is given by

$$\frac{(2t)^n}{(\mathrm{E}_{\frac{1}{q}}(t)+1)^n}\mathrm{E}_q(xt)=\sum_{\nu=0}^{\infty}\frac{t^\nu \mathrm{H}_{\mathrm{JHC},\nu,,q}^{(n)}(x)}{\{\nu\}_q!},\ |t|<2\pi. \tag{13}$$

The polynomials in (12) and (13) are q–analogues of the generalized H polynomials.

Definition 1.8. The polynomials $\mathrm{b}_{\lambda,\nu,q}^{(n)}(x)$ are defined by

$$\frac{t^n g(t)}{(\lambda\mathrm{E}_q(t)-1)^n}\mathrm{E}_q(xt)=\sum_{\nu=0}^{\infty}\frac{t^\nu \mathrm{b}_{\lambda,\nu,q}^{(n)}(x)}{\{\nu\}_q!}. \tag{14}$$

Definition 1.9. The e polynomials are defined by

$$\frac{2^n g(t)}{(\lambda\mathrm{E}_q(t)+1)^n}\mathrm{E}_q(xt)=\sum_{\nu=0}^{\infty}\frac{t^\nu \mathrm{e}_{\lambda,\nu,q}^{(n)}(x)}{\{\nu\}_q!}. \tag{15}$$

The f polynomials are more general forms of the JHC q–$\mathscr{H}$ polynomials.

Definition 1.10. The f polynomials $\mathrm{f}_{\lambda,\nu,q}^{(n)}(x)$ are defined by

$$\frac{2^n g(t)}{(\lambda\mathrm{E}_{\frac{1}{q}}(t)+1)^n}\mathrm{E}_q(xt)=\sum_{\nu=0}^{\infty}\frac{t^\nu \mathrm{f}_{\lambda,\nu,q}^{(n)}(x)}{\{\nu\}_q!}. \tag{16}$$

Definition 1.11. A q–analogue of [7, (20) p. 381], the multiple q–power sum is defined by

$$s_{\mathrm{NWA},\lambda,m,q}^{(l)}(n)\equiv\sum_{|\vec{j}|=l}\binom{l}{\vec{j}}\lambda^k\left(\overline{k_q}\right)^m, \tag{17}$$

where $k\equiv j_1+2j_2+\cdots+(n-1)j_{n-1},\ \forall j_i\geq 0$.

Definition 1.12. A q–analogue of [7, (46) p. 386], the multiple alternating q–power sum is defined by

$$\sigma_{\mathrm{NWA},\lambda,m,q}^{(l)}(n)\equiv(-1)^l\sum_{|\vec{j}|=l}\binom{l}{\vec{j}}(-\lambda)^k\left(\overline{k_q}\right)^m, \tag{18}$$

where $k\equiv j_1+2j_2+\cdots+(n-1)j_{n-1},\ \forall j_i\geq 0$.

Theorem 1.3. *A symmetry relation for the generalized q–$\mathscr{H}$ numbers.*

$$(-1)^{\nu}\mathscr{H}_{\mathrm{JHC},\lambda^{-1},\nu,q}=\mathscr{H}_{\mathrm{NWA},\lambda,\nu,q}. \tag{19}$$

Proof. A simple computation with generating functions shows the way:

$$\begin{aligned}&\sum_{\nu=0}^{\infty}\frac{(-t)^{\nu}\mathscr{H}_{\mathrm{JHC},\lambda^{-1},\nu,q}}{\{\nu\}_q!}=\frac{-2t}{\lambda^{-1}\mathrm{E}_{\frac{1}{q}}(-t)+1}=\frac{-2t\lambda\mathrm{E}_q(t)}{\lambda\mathrm{E}_q(t)+1}\\&=-\lambda\sum_{\nu=0}^{\infty}\frac{t^{\nu}\mathscr{H}_{\mathrm{NWA},\lambda,\nu,q}(1)}{\{\nu\}_q!}.\end{aligned} \tag{20}$$

Equating the coefficients of t^{ν} gives (19). □

Theorem 1.4. *Assume that $g(t)$ in (15) and (16) are equal and even functions. Then*

$$\mathrm{f}^{(n)}_{\lambda^{-1},\nu,q}(x)=(-1)^{\nu}\lambda^{n}\mathrm{e}^{(n)}_{\lambda,\nu,q}(\overline{n}_q\ominus_q x). \tag{21}$$

This implies a complementary argument theorem for the generalized q–$\mathscr{H}$ polynomials.

Theorem 1.5.

$$\mathscr{H}^{(n)}_{\mathrm{JHC},\lambda^{-1},\nu,q}(x)=(-1)^{\nu}\lambda^{n}\mathscr{H}^{(n)}_{\mathrm{NWA},\lambda,\nu,q}(\overline{n}_q\ominus_q x),n\text{ even}. \tag{22}$$

$$\mathscr{H}^{(n)}_{\mathrm{JHC},\lambda^{-1},\nu,q}(x)=(-1)^{\nu+1}\lambda^{n}\mathscr{H}^{(n)}_{\mathrm{NWA},\lambda,\nu,q}(\overline{n}_q\ominus_q x),n\text{ odd}. \tag{23}$$

Definition 1.13. The following functions named the q–power sum, and the alternate q–power sum (with respect to λ), were introduced in [4].

$$s_{\mathrm{NWA},\lambda,m,q}(n)\equiv\sum_{k=0}^{n-1}\lambda^{k}(\overline{k}_q)^{m}\text{ and }\sigma_{\mathrm{NWA},\lambda,m,q}(n)\equiv\sum_{k=0}^{n-1}(-1)^{k}\lambda^{k}(\overline{k}_q)^{m}. \tag{24}$$

Their respective generating functions are

$$\sum_{m=0}^{\infty}s_{\mathrm{NWA},\lambda,m,q}(n)\frac{t^m}{\{m\}_q!}=\frac{\lambda^{n}\mathrm{E}_q(\overline{n}_q t)-1}{\lambda\mathrm{E}_q(t)-1} \tag{25}$$

and

$$\sum_{m=0}^{\infty}\sigma_{\mathrm{NWA},\lambda,m,q}(n)\frac{t^m}{\{m\}_q!}=\frac{(-1)^{n+1}\lambda^{n}\mathrm{E}_q(\overline{n}_q t)+1}{\lambda\mathrm{E}_q(t)+1}. \tag{26}$$

2. The first expansion formulas

Theorem 2.1. *A triple sum of NWA q–Apostol-Euler polynomials is equal to another triple sum of NWA q–Apostol-Euler polynomials.*

$$\sum_{|\nu|=n} \binom{n}{\vec{\nu}}_q (\bar{i}_q)^{\nu_1} (\bar{j}_q)^{\nu_2} \mathscr{F}^{(k)}_{\mathrm{NWA},\lambda^i,\nu_1,q}\left(\bar{j}_q x\right) \mathscr{F}^{(k-1)}_{\mathrm{NWA},\lambda^j,\nu_2,q}\left(\bar{i}_q y\right) \sigma_{\mathrm{NWA},\lambda^j,\nu_3,q}(i) (\bar{j}_q)^{\nu_3}$$
$$= \sum_{\nu=0}^{n} \binom{n}{\nu}_q (\bar{i}_q)^{\nu} (\bar{j}_q)^{n-\nu} \mathscr{F}^{(k-1)}_{\mathrm{NWA},\lambda^j,n-\nu,q}\left(\bar{i}_q y\right) \sum_{m=0}^{i-1} \lambda^{jm}(-1)^m$$
$$\mathscr{F}^{(k)}_{\mathrm{NWA},\lambda^i,\nu,q}\left(\bar{j}_q x \oplus_q \frac{\overline{jm}_q}{\bar{i}_q}\right). \tag{27}$$

Proof. Define the following function, note that $f_q(t)$ is symmetric when i, j have the same parity.

$$f_q(t) \equiv \frac{\mathrm{E}_q(\overline{ij}_q(x\oplus y)t)((-1)^{i+1}\lambda^{ij}\mathrm{E}_q(\overline{ij}_q t)+1)}{(\lambda^i \mathrm{E}_q(\bar{i}_q t)+1)^k(\lambda^j \mathrm{E}_q(\bar{j}_q t)+1)^k} = 2^{1-2k}\mathrm{E}_q(\overline{ij}_q(x\oplus y)t)$$
$$\left(\frac{2}{\lambda^i \mathrm{E}_q(\bar{i}_q t)+1}\right)^k \left(\frac{2}{\lambda^j \mathrm{E}_q(\bar{j}_q t)+1}\right)^{k-1} \left(\frac{(-1)^{i+1}\lambda^{ij}\mathrm{E}_q(\overline{ij}_q t)+1}{\lambda^j \mathrm{E}_q(\bar{j}_q t)+1}\right). \tag{28}$$

By using the formula for a geometric sequence, we can expand $f_q(t)$ in two ways:

$$f_q(t) \overset{\text{by}(26,9)}{=} 2^{1-2k}\left(\sum_{\nu=0}^{\infty} \mathscr{F}^{(k)}_{\mathrm{NWA},\lambda^i,\nu,q}\left(\bar{j}_q x\right) \frac{(\bar{i}_q t)^{\nu}}{\{\nu\}_q!}\right) \left(\sum_{m=0}^{\infty} \sigma_{\mathrm{NWA},\lambda^j,m,q}(i) \frac{(\bar{j}_q t)^m}{\{m\}_q!}\right)$$
$$\left(\sum_{l=0}^{\infty} \mathscr{F}^{(k-1)}_{\mathrm{NWA},\lambda^j,l,q}\left(\bar{i}_q y\right) \frac{(\bar{j}_q t)^l}{\{l\}_q!}\right) = 2^{1-2k} \frac{2^k}{\left(\lambda^i \mathrm{E}_q(\bar{i}_q t)+1\right)^k}$$
$$\frac{2^{k-1}}{\left(\lambda^j \mathrm{E}_q(\bar{j}_q t)+1\right)^{k-1}} \sum_{m=0}^{i-1}(-1)^m \lambda^{jm} \mathrm{E}_q\left(\left(\bar{j}_q x \oplus_q \bar{j}_q y \oplus_q \frac{\overline{jm}_q}{\bar{i}_q}\right)\bar{i}_q t\right)$$
$$= 2^{1-2k}\sum_{m=0}^{i-1}(-1)^m \lambda^{jm} \sum_{l=0}^{\infty} \frac{(\bar{i}_q)^l t^l}{\{l\}_q!} \mathscr{F}^{(k)}_{\mathrm{NWA},\lambda^i,l,q}\left(\bar{j}_q x \oplus_q \frac{\overline{jm}_q}{\bar{i}_q}\right)$$
$$\sum_{n=0}^{\infty} \frac{(\bar{j}_q)^n t^n}{\{n\}_q!} \mathscr{F}^{(k-1)}_{\mathrm{NWA},\lambda^j,n,q}(\bar{i}_q y). \tag{29}$$

The theorem follows by equating the coefficients of $\frac{t^n}{\{n\}_q!}$. □

Theorem 2.2. *Almost a q–analogue of [5, p. 3351]. Assume that i and j are either both odd, or both even. Then we have*

$$\sum_{\nu=0}^{n}\binom{n}{\nu}_q(\bar{j}_q)^{\nu}(\bar{i}_q)^{n-\nu}\mathscr{F}^{(k-1)}_{\mathrm{NWA},\lambda^i,n-\nu,q}\left(\bar{j}_q y\right)\sum_{m=0}^{j-1}\lambda^{im}(-1)^m\mathscr{F}^{(k)}_{\mathrm{NWA},\lambda^j,\nu,q}\left(\bar{i}_q x\oplus_q\frac{\overline{im}_q}{\bar{j}_q}\right)$$
$$=\sum_{\nu=0}^{n}\binom{n}{\nu}_q(\bar{i}_q)^{\nu}(\bar{j}_q)^{n-\nu}\mathscr{F}^{(k-1)}_{\mathrm{NWA},\lambda^j,n-\nu,q}\left(\bar{i}_q y\right)$$
$$\sum_{m=0}^{i-1}\lambda^{jm}(-1)^m\mathscr{F}^{(k)}_{\mathrm{NWA},\lambda^i,\nu,q}\left(\bar{j}_q x\oplus_q\frac{\overline{jm}_q}{\bar{i}_q}\right)\tag{30}$$

Proof. This follows from the previous proof, and then using the symmetry for i and j. □

Theorem 2.3. *A triple sum of NWA q–Apostol-Bernoulli polynomials is equal to a double sum of NWA q–Apostol-Bernoulli polynomials.*

$$\sum_{|\nu|=n}\binom{n}{\nu}_q(\bar{i}_q)^{\nu_1}(\bar{j}_q)^{\nu_2}(\bar{j}_q)^{\nu_3}\mathscr{B}^{(k)}_{\mathrm{NWA},\lambda^i,\nu_1,q}\left(\bar{j}_q x\right)\mathscr{B}^{(k-1)}_{\mathrm{NWA},\lambda^j,\nu_2,q}\left(\bar{i}_q y\right)s_{\mathrm{NWA},\lambda^j,\nu_3,q}(i)$$
$$=\sum_{\nu=0}^{n}\binom{n}{\nu}_q(\bar{i}_q)^{\nu}(\bar{j}_q)^{n-\nu}\mathscr{B}^{(k-1)}_{\mathrm{NWA},\lambda^j,n-\nu,q}\left(\bar{i}_q y\right)\sum_{m=0}^{i-1}\lambda^{jm}\mathscr{B}^{(k)}_{\mathrm{NWA},\lambda^i,\nu,q}\left(\bar{j}_q x\oplus_q\frac{\overline{jm}_q}{\bar{i}_q}\right)\tag{31}$$

Proof. Define the following symmetric function

$$\phi_q(t)\equiv\frac{\mathrm{E}_q(\overline{ij}_q(x\oplus y)t)(\lambda^{ij}\mathrm{E}_q(\overline{ij}_q t)-1)}{(\lambda^i\mathrm{E}_q(\bar{i}_q t)-1)^k(\lambda^j\mathrm{E}_q(\bar{j}_q t)-1)^k}t^k=\mathrm{E}_q(\overline{ij}_q(x\oplus y)t)$$
$$\left(\frac{\bar{i}_q t}{\lambda^i\mathrm{E}_q(\bar{i}_q t)-1}\right)^k\left(\frac{\bar{j}_q t}{\lambda^j\mathrm{E}_q(\bar{j}_q t)-1}\right)^{k-1}\left(\frac{\lambda^{ij}\mathrm{E}_q(\overline{ij}_q t)-1}{\lambda^j\mathrm{E}_q(\bar{j}_q t)-1}\right)\frac{t^{1-2k}}{(\bar{i}_q)^k(\bar{j}_q)^{k-1}}.\tag{32}$$

By using the formula for a geometric sequence, we can expand $\phi_q(t)$ in two ways:

$$\begin{aligned}\phi_q(t) &\overset{\text{by}(25)}{=} \left(\sum_{\nu=0}^{\infty}\mathscr{B}^{(k)}_{\text{NWA},\lambda^i,\nu,q}\left(\bar{j}_q x\right)\frac{(\bar{i}_q t)^\nu}{\{\nu\}_q!}\right)\left(\sum_{m=0}^{\infty}s_{\text{NWA},\lambda^j,m,q}(i)\frac{(\bar{j}_q t)^m}{\{m\}_q!}\right)\\
&\left(\sum_{l=0}^{\infty}\mathscr{B}^{(k-1)}_{\text{NWA},\lambda^j,l,q}\left(\bar{i}_q y\right)\frac{(\bar{j}_q t)^l}{\{l\}_q!}\right)\frac{t^{1-2k}}{(\bar{i}_q)^k(\bar{j}_q)^{k-1}} = \frac{(\bar{i}_q t)^k}{\left(\lambda^i \mathrm{E}_q(\bar{i}_q t)-1\right)^k}\\
&\frac{(\bar{j}_q t)^{k-1}}{\left(\lambda^j \mathrm{E}_q(\bar{j}_q t)-1\right)^{k-1}}\sum_{m=0}^{i-1}\lambda^{jm}\mathrm{E}_q\left(\left(\bar{j}_q x\oplus_q \bar{j}_q y\oplus_q \frac{\overline{jm}_q}{\bar{i}_q}\right)\bar{i}_q t\right)\frac{t^{1-2k}}{(\bar{i}_q)^k(\bar{j}_q)^{k-1}}\\
&= \frac{t^{1-2k}}{(\bar{i}_q)^k(\bar{j}_q)^{k-1}}\sum_{m=0}^{i-1}\lambda^{jm}\sum_{l=0}^{\infty}\frac{(\bar{i}_q)^l t^l}{\{l\}_q!}\mathscr{B}^{(k)}_{\text{NWA},\lambda^i,l,q}\left(\bar{j}_q x\oplus_q\frac{\overline{jm}_q}{\bar{i}_q}\right)\\
&\sum_{n=0}^{\infty}\frac{(\bar{j}_q)^n t^n}{\{n\}_q!}\mathscr{B}^{(k-1)}_{\text{NWA},\lambda^j,n,q}(\bar{i}_q y).
\end{aligned}\tag{33}$$

The theorem follows by equating the coefficients of $\frac{t^n}{\{n\}_q!}$. □

Theorem 2.4. *A q–analogue of [12, p. 2994], [11, p. 551].*

$$\begin{aligned}&\sum_{|\nu|=n}\binom{n}{\nu}_q(\bar{i}_q)^{\nu_1}(\bar{j}_q)^{\nu_2}(\bar{j}_q)^{\nu_3}\mathscr{B}^{(k)}_{\text{NWA},\lambda^i,\nu_1,q}\left(\bar{j}_q x\right)\mathscr{B}^{(k-1)}_{\text{NWA},\lambda^j,\nu_2,q}\left(\bar{i}_q y\right)s_{\text{NWA},\lambda^j,\nu_3,q}(i)\\
&=\sum_{|\nu|=n}\binom{n}{\nu}_q(\bar{j}_q)^{\nu_1}(\bar{i}_q)^{\nu_2}(\bar{i}_q)^{\nu_3}\mathscr{B}^{(k)}_{\text{NWA},\lambda^j,\nu_1,q}\left(\bar{i}_q x\right)\mathscr{B}^{(k-1)}_{\text{NWA},\lambda^i,\nu_2,q}\left(\bar{j}_q y\right)s_{\text{NWA},\lambda^i,\nu_3,q}(j)
\end{aligned}\tag{34}$$

Proof. Use the symmetry in $\phi_q(t)$. □

Theorem 2.5. *A q–analogue of [12, p. 2996]. We have*

$$\begin{aligned}&\sum_{\nu=0}^{n}\binom{n}{\nu}_q\sum_{l=0}^{i-1}\sum_{m=0}^{j-1}\lambda^{l+m}(\bar{i}_q)^\nu(\bar{j}_q)^{n-\nu}\mathscr{B}^{(k)}_{\text{NWA},\lambda,\nu,q}\left(\bar{j}_q x\oplus_q\frac{\overline{jl}_q}{\bar{i}_q}\right)\\
&\quad\mathscr{B}^{(k)}_{\text{NWA},\lambda,n-\nu,q}\left(\bar{i}_q y\oplus_q\frac{\overline{im}_q}{\bar{j}_q}\right)\\
&=\sum_{\nu=0}^{n}\binom{n}{\nu}_q\sum_{l=0}^{j-1}\sum_{m=0}^{i-1}\lambda^{l+m}(\bar{j}_q)^\nu(\bar{i}_q)^{n-\nu}\mathscr{B}^{(k)}_{\text{NWA},\lambda,\nu,q}\left(\bar{i}_q x\oplus_q\frac{\overline{il}_q}{\bar{j}_q}\right)\\
&\quad\mathscr{B}^{(k)}_{\text{NWA},\lambda,n-\nu,q}\left(\bar{j}_q y\oplus_q\frac{\overline{jm}_q}{\bar{i}_q}\right)
\end{aligned}\tag{35}$$

Proof. We can expand the following symmetric function $\phi'_q(t)$ by using the formula for a geometric sequence:

$$\phi'_q(t) \equiv \frac{\mathrm{E}_q(\overline{ij}_q(x\oplus y)t)(\lambda^i\mathrm{E}_q(\overline{ij}_q t)-1)(\lambda^j\mathrm{E}_q(\overline{ij}_q t)-1)}{(\lambda\mathrm{E}_q(\bar{i}_q t)-1)^k(\lambda\mathrm{E}_q(\bar{j}_q t)-1)^k}t^{2k-2}$$
$$=\mathrm{E}_q(\overline{ij}_q(x\oplus y)t)\frac{1}{(\bar{i}_q)^{k-1}(\bar{j}_q)^{k-1}}$$
$$\left(\frac{\bar{i}_q t}{\lambda\mathrm{E}_q(\bar{i}_q t)-1}\right)^{k-1}\left(\frac{\bar{j}_q t}{\lambda\mathrm{E}_q(\bar{j}_q t)-1}\right)^{k-1}\left(\frac{\lambda^i\mathrm{E}_q(\overline{ij}_q t)-1}{\lambda\mathrm{E}_q(\bar{j}_q t)-1}\right)\left(\frac{\lambda^j\mathrm{E}_q(\overline{ij}_q t)-1}{\lambda\mathrm{E}_q(\bar{i}_q t)-1}\right)$$
$$=\frac{1}{(\bar{i}_q)^{k-1}(\bar{j}_q)^{k-1}}\sum_{l=0}^{i-1}\sum_{m=0}^{j-1}\lambda^{l+m}\left(\frac{\bar{i}_q t}{\lambda\mathrm{E}_q(\bar{i}_q t)-1}\right)^{k-1}\left(\frac{\bar{j}_q t}{\lambda\mathrm{E}_q(\bar{j}_q t)-1}\right)^{k-1}$$
$$\mathrm{E}_q\left(\left(\bar{j}_q x\oplus_q\frac{\overline{jl}_q}{\bar{i}_q}\right)\bar{i}_q t\right)\mathrm{E}_q\left(\left(\bar{i}_q y\oplus_q\frac{\overline{im}_q}{\bar{j}_q}\right)\bar{j}_q t\right)$$
$$=\frac{1}{(\bar{i}_q)^{k-1}(\bar{j}_q)^{k-1}}\left(\sum_{l=0}^{i-1}\lambda^l\sum_{\nu_1=0}^{\infty}\frac{(\bar{i}_q)^{\nu_1}t^{\nu_1}}{\{\nu_1\}_q!}\mathscr{B}^{(k-1)}_{\mathrm{NWA},\lambda,\nu_1,q}\left(\bar{j}_q x\oplus_q\frac{\overline{jl}_q}{\bar{i}_q}\right)\right)$$
$$\left(\sum_{m=0}^{j-1}\lambda^m\sum_{\nu_2=0}^{\infty}\frac{(\bar{j}_q)^{\nu_2}t^{\nu_2}}{\{\nu_2\}_q!}\mathscr{B}^{(k-1)}_{\mathrm{NWA},\lambda,\nu_2,q}\left(\bar{i}_q y\oplus_q\frac{\overline{im}_q}{\bar{j}_q}\right)\right). \tag{36}$$

The theorem follows by using the symmetry in $\phi'_q(t)$ and changing $k-1$ to k. □

Theorem 2.6. *A q–analogue of [12, p. 2997]. We have*

$$\begin{aligned}&\sum_{\nu=0}^{n}\binom{n}{\nu}_q\sum_{l=0}^{i-1}(\bar{i}_q)^{\nu}(\bar{j}_q)^{n-\nu}\mathscr{B}^{(k)}_{\mathrm{NWA},\lambda,n-\nu,q}(\bar{i}_q y)\\&\sum_{m=0}^{j-1}\lambda^{l+m}\mathscr{B}^{(k)}_{\mathrm{NWA},\lambda,\nu,q}\left(\bar{j}_q x\oplus_q\frac{\overline{jl}_q}{\bar{i}_q}\oplus_q\overline{m}_q\right)\\&=\sum_{\nu=0}^{n}\binom{n}{\nu}_q\sum_{l=0}^{j-1}(\bar{j}_q)^{\nu}(\bar{i}_q)^{n-\nu}\mathscr{B}^{(k)}_{\mathrm{NWA},\lambda,n-\nu,q}(\bar{j}_q y)\\&\sum_{m=0}^{i-1}\lambda^{l+m}\mathscr{B}^{(k)}_{\mathrm{NWA},\lambda,\nu,q}\left(\bar{i}_q x\oplus_q\frac{\overline{il}_q}{\bar{j}_q}\oplus_q\overline{m}_q\right).\end{aligned} \tag{37}$$

Proof. Similar to above. □

Theorem 2.7. *A q–analogue of [11, p. 552]. We have*

$$\begin{aligned}&\frac{1}{(\bar{i}_q)^k(\bar{j}_q)^{k-1}}\sum_{m=0}^{n}\binom{n}{m}_q(\bar{i}_q)^m(\bar{j}_q)^{n-m}\\&\mathscr{B}^{(k-1)}_{\mathrm{NWA},\lambda^j,n-m,q}\left(\bar{i}_q y\right)\sum_{l=0}^{i-1}\lambda^{jl}\mathscr{B}^{(k)}_{\mathrm{NWA},\lambda^i,m,q}\left(\bar{j}_q x\oplus_q\frac{\overline{jl}_q}{\bar{i}_q}\right)\\&=\frac{1}{(\bar{j}_q)^k(\bar{i}_q)^{k-1}}\sum_{m=0}^{n}\binom{n}{m}_q(\bar{j}_q)^m(\bar{i}_q)^{n-m}\\&\mathscr{B}^{(k-1)}_{\mathrm{NWA},\lambda^i,n-m,q}\left(\bar{j}_q y\right)\sum_{l=0}^{j-1}\lambda^{il}\mathscr{B}^{(k)}_{\mathrm{NWA},\lambda^j,m,q}\left(\bar{i}_q x\oplus_q\frac{\overline{il}_q}{\bar{j}_q}\right).\end{aligned}\tag{38}$$

Proof. We can expand the following symmetric function $\psi_q(t)$ by using the formula for a geometric sequence:

$$\begin{aligned}&\psi_q(t)\equiv\frac{\mathrm{E}_q(\overline{ij}_q(x\oplus y)t)(\lambda^{ij}\mathrm{E}_q(\overline{ij}_q t)-1)}{(\lambda^i\mathrm{E}_q(\bar{i}_q t)-1)^k(\lambda^j\mathrm{E}_q(\bar{j}_q t)-1)^k}t^{2k-1}=\mathrm{E}_q(\overline{ij}_q(x\oplus y)t)\frac{1}{(\bar{i}_q)^k(\bar{j}_q)^{k-1}}\\&\left(\frac{\bar{i}_q t}{\lambda^i\mathrm{E}_q(\bar{i}_q t)-1}\right)^k\left(\frac{\bar{j}_q t}{\lambda^j\mathrm{E}_q(\bar{j}_q t)-1}\right)^{k-1}\left(\frac{\lambda^{ij}\mathrm{E}_q(\overline{ij}_q t)-1}{\lambda^j\mathrm{E}_q(\bar{j}_q t)-1}\right)\\&=\frac{1}{(\bar{i}_q)^k(\bar{j}_q)^{k-1}}\left(\frac{\bar{i}_q t}{\lambda^i\mathrm{E}_q(\bar{i}_q t)-1}\right)^k\left(\frac{\bar{j}_q t}{\lambda^j\mathrm{E}_q(\bar{j}_q t)-1}\right)^{k-1}\sum_{l=0}^{i-1}\lambda^{lj}\\&\mathrm{E}_q\left(\left(\bar{j}_q x\oplus_q\frac{\overline{jl}_q}{\bar{i}_q}\right)\bar{i}_q t\right)\mathrm{E}_q\left(\left(\bar{i}_q y\right)\bar{j}_q t\right)\\&=\frac{1}{(\bar{i}_q)^k(\bar{j}_q)^{k-1}}\left(\sum_{l=0}^{i-1}\lambda^{jl}\sum_{\nu_1=0}^{\infty}\frac{(\bar{i}_q)^{\nu_1}t^{\nu_1}}{\{\nu_1\}_q!}\mathscr{B}^{(k)}_{\mathrm{NWA},\lambda^i,\nu_1,q}\left(\bar{j}_q x\oplus_q\frac{\overline{jl}_q}{\bar{i}_q}\right)\right)\\&\sum_{\nu_2=0}^{\infty}\frac{(\bar{j}_q)^{\nu_2}t^{\nu_2}}{\{\nu_2\}_q!}\mathscr{B}^{(k-1)}_{\mathrm{NWA},\lambda^j,\nu_2,q}\left(\bar{i}_q y\right)=\frac{1}{(\bar{i}_q)^k(\bar{j}_q)^{k-1}}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}_q\sum_{l=0}^{i-1}\right.\\&\left.\lambda^{jl}(\bar{i}_q)^m(\bar{j}_q)^{n-m}\mathscr{B}^{(k)}_{\mathrm{NWA},\lambda^i,m,q}\left(\bar{j}_q x\oplus_q\frac{\overline{jl}_q}{\bar{i}_q}\right)\mathscr{B}^{(k-1)}_{\mathrm{NWA},\lambda^j,n-m,q}\left(\bar{i}_q y\right)\right)\frac{t^n}{\{n\}_q!}.\end{aligned}\tag{39}$$

The theorem follows by using the symmetry in $\psi_q(t)$. □

3. Mixed formulas

This is a continuation of the very similar computations in [4], to which we will refer.

Corollary 3.1. *A q–analogue of [10, (31) p. 314]. If i is even then*

$$\begin{aligned}
&\sum_{m=0}^{1}\lambda^{im}\mathscr{F}_{\mathrm{NWA},\lambda^2,n-1,q}\left(\bar{i}_q x\oplus_q\frac{\overline{im}_q}{\overline{2}_q}\right)\\
&=-\frac{2}{\{n\}_q(\overline{2}_q)^{n-1}}\sum_{k=0}^{n}\binom{n}{k}_q\frac{(\bar{i}_q)^k}{i}(\overline{2}_q)^{n-k}\mathscr{B}_{\mathrm{NWA},\lambda^i,k,q}\left(\overline{2}_q x\right)\sigma_{\mathrm{NWA},\lambda^2,n-k,q}(i)\\
&=\frac{1}{(\overline{2}_q)^{n-1}}\sum_{k=0}^{n-1}\binom{n-1}{k}_q(\overline{2}_q)^k(\bar{i}_q)^{n-k-1}\mathscr{F}_{\mathrm{NWA},\lambda^2,k,q}\left(\bar{i}_q x\right)s_{\mathrm{NWA},\lambda^i,n-k-1,q}(2)\\
&=-\frac{2}{\{n\}_q(\overline{2}_q)^{n-1}}\frac{(\bar{i}_q)^n}{i}\sum_{m=0}^{i-1}(-1)^m\lambda^{2m}\mathscr{B}_{\mathrm{NWA},\lambda^i,n,q}\left(\overline{2}_q x\oplus_q\frac{\overline{2m}_q}{\bar{i}_q}\right).
\end{aligned}\tag{40}$$

Proof. Put $j=2$ in formula (56) [4], and multiply by $-\frac{2}{\{n\}_q(\overline{2}_q)^{n-1}}$. □

Corollary 3.2. *A q–analogue of [10, (32) p. 314].*

$$\begin{aligned}
&\sum_{m=0}^{1}(-1)^{m+1}\lambda^m\mathscr{B}_{\mathrm{NWA},\lambda,n,q}\left(x\oplus_q\frac{\overline{2m}_q}{\overline{2}_q}\right)\\
&=\frac{\{n\}_q(\overline{2}_q)^{n-1}}{(\overline{2}_q)^n}\sum_{m=0}^{1}\lambda^m\mathscr{F}_{\mathrm{NWA},\lambda,n-1,q}\left(x\oplus_q\frac{\overline{2m}_q}{\overline{2}_q}\right).
\end{aligned}\tag{41}$$

Proof. Put $i=2$ in formula (40), replace x and λ^2 by $\frac{x}{\overline{2}_q}$ and λ, and multiply by $\frac{\{n\}_q(\overline{2}_q)^{n-1}}{(\overline{2}_q)^n}$. □

Corollary 3.3. *A q–analogue of [10, (33) p. 314].*

$$\begin{aligned}
&\sum_{m=0}^{1}(-1)^m\lambda^{jm}\mathscr{B}_{\mathrm{NWA},\lambda^2,n,q}\left(\bar{j}_q x\oplus_q\frac{\overline{jm}_q}{\overline{2}_q}\right)=-\frac{\{n\}_q}{(\overline{2}_q)^n}\sum_{k=0}^{n-1}\binom{n-1}{k}_q\\
&(\bar{j}_q)^k(\overline{2}_q)^{n-k-1}\mathscr{F}_{\mathrm{NWA},\lambda^j,k,q}\left(\overline{2}_q x\right)s_{\mathrm{NWA},\lambda^2,n-k-1,q}(j)\\
&=-\frac{\{n\}_q}{(\overline{2}_q)^n}(\bar{j}_q)^{n-1}\sum_{m=0}^{j-1}\lambda^{2m}\mathscr{F}_{\mathrm{NWA},\lambda^j,n-1,q}\left(\overline{2}_q x\oplus_q\frac{\overline{2m}_q}{\bar{j}_q}\right).
\end{aligned}\tag{42}$$

Proof. Put $i=2$ in formula (56) [4], and multiply by $\frac{2}{(\overline{2}_q)^n}$. □

The following formula is a generalization of [4, (57)].

Theorem 3.1. *A q–analogue of [5, (3.9) p. 3356].*

$$\sum_{|v|=n}\binom{n}{\vec{v}}_q(\bar{i}_q)^{v_1}(\bar{j}_q)^{v_2}\mathscr{B}^{(k)}_{\mathrm{NWA},\lambda^i,v_1,q}\left(\bar{j}_q x\right)\mathscr{F}^{(k-1)}_{\mathrm{NWA},\lambda^j,v_2,q}\left(\bar{i}_q y\right)\sigma_{\mathrm{NWA},\lambda^j,v_3,q}(i)(\bar{j}_q)^{v_3}$$
$$=\sum_{v=0}^{n}\binom{n}{v}_q(\bar{i}_q)^{v}(\bar{j}_q)^{n-v}\mathscr{F}^{(k-1)}_{\mathrm{NWA},\lambda^j,n-v,q}\left(\bar{i}_q y\right)\sum_{m=0}^{i-1}\lambda^{jm}(-1)^m$$
$$\mathscr{B}^{(k)}_{\mathrm{NWA},\lambda^i,v,q}\left(\bar{j}_q x\oplus_q\frac{\overline{jm}_q}{\bar{i}_q}\right). \tag{43}$$

Proof. Define the following function

$$g_q(t)\equiv\frac{\mathrm{E}_q(\overline{ij}_q(x\oplus y)t)((-1)^{i+1}\lambda^{ij}\mathrm{E}_q(\overline{ij}_q t)+1)}{(\lambda^i\mathrm{E}_q(\bar{i}_q t)-1)^k(\lambda^j\mathrm{E}_q(\bar{j}_q t)+1)^k}=\frac{2^{1-k}}{(\bar{i}_q t)^k}\mathrm{E}_q(\overline{ij}_q(x\oplus y)t)$$
$$\left(\frac{\bar{i}_q t}{\lambda^i\mathrm{E}_q(\bar{i}_q t)-1}\right)^k\left(\frac{2}{\lambda^j\mathrm{E}_q(\bar{j}_q t)+1}\right)^{k-1}\left(\frac{(-1)^{i+1}\lambda^{ij}\mathrm{E}_q(\overline{ij}_q t)+1}{\lambda^j\mathrm{E}_q(\bar{j}_q t)+1}\right). \tag{44}$$

By using the formula for a geometric sequence, we can expand $g_q(t)$ in two ways:

$$g_q(t)\overset{\text{by}(26)}{=}\frac{2^{1-k}}{(\bar{i}_q t)^k}\left(\sum_{v=0}^{\infty}\mathscr{B}^{(k)}_{\mathrm{NWA},\lambda^i,v,q}\left(\bar{j}_q x\right)\frac{(\bar{i}_q t)^v}{\{v\}_q!}\right)\left(\sum_{m=0}^{\infty}\sigma_{\mathrm{NWA},\lambda^j,m,q}(i)\frac{(\bar{j}_q t)^m}{\{m\}_q!}\right)$$
$$\left(\sum_{l=0}^{\infty}\mathscr{F}^{(k-1)}_{\mathrm{NWA},\lambda^j,l,q}\left(\bar{i}_q y\right)\frac{(\bar{j}_q t)^l}{\{l\}_q!}\right)=\frac{2^{1-k}}{(\bar{i}_q t)^k}\left(\frac{\bar{i}_q t}{\lambda^i\mathrm{E}_q(\bar{i}_q t)-1}\right)^k$$
$$\frac{2^{k-1}}{\left(\lambda^j\mathrm{E}_q(\bar{j}_q t)+1\right)^{k-1}}\sum_{m=0}^{i-1}(-1)^m\lambda^{jm}\mathrm{E}_q\left(\left(\bar{j}_q x\oplus_q\bar{j}_q y\oplus_q\frac{\overline{jm}_q}{\bar{i}_q}\right)\bar{i}_q t\right)$$
$$=\frac{2^{1-k}}{(\bar{i}_q t)^k}\sum_{m=0}^{i-1}(-1)^m\lambda^{jm}\sum_{l=0}^{\infty}\frac{(\bar{i}_q)^l t^l}{\{l\}_q!}\mathscr{B}^{(k)}_{\mathrm{NWA},\lambda^i,l,q}\left(\bar{j}_q x\oplus_q\frac{\overline{jm}_q}{\bar{i}_q}\right)$$
$$\sum_{n=0}^{\infty}\frac{(\bar{j}_q)^n t^n}{\{n\}_q!}\mathscr{F}^{(k-1)}_{\mathrm{NWA},\lambda^j,n,q}(\bar{i}_q y). \tag{45}$$

The theorem follows by equating the coefficients of $\frac{t^n}{\{n\}_q!}$. □

Theorem 3.2. *A q–analogue of [5, p. 3353]. Under the assumption that i is even, we have*

$$\begin{aligned}&\sum_{|\nu|=n}\binom{n}{\vec{\nu}}_q(\bar{i}_q)^{\nu_1}(\bar{j}_q)^{\nu_2}\mathscr{B}^{(k)}_{\mathrm{NWA},\lambda^i,\nu_1,q}\left(\bar{j}_qx\right)\mathscr{F}^{(k-1)}_{\mathrm{NWA},\lambda^j,\nu_2,q}\left(\bar{i}_qy\right)s_{\mathrm{NWA},\lambda^j,\nu_3,q}(i)(\bar{j}_q)^{\nu_3}\\&=-\frac{\{n\}_q(\bar{i}_q)^k}{2(\bar{i}_q)^{k-1}}\sum_{|\nu|=n-1}\binom{n-1}{\vec{\nu}}_q(\bar{i}_q)^{\nu_1}(\bar{j}_q)^{\nu_2}(\bar{j}_q)^{\nu_3}\mathscr{B}^{(k-1)}_{\mathrm{NWA},\lambda^i,\nu_1,q}\left(\bar{j}_qy\right)\\&\mathscr{F}^{(k)}_{\mathrm{NWA},\lambda^j,\nu_2,q}\left(\bar{i}_qx\right)s_{\mathrm{NWA},\lambda^i,\nu_3,q}(j).\end{aligned} \tag{46}$$

Proof. We can write $g_q(t)$ as follows:

$$\begin{aligned}&g_q(t)\overset{\text{by}(25),(44)}{=}\frac{2^{1-k}}{(\bar{i}_qt)^k}\left(\sum_{\nu=0}^{\infty}\mathscr{B}^{(k)}_{\mathrm{NWA},\lambda^i,\nu,q}\left(\bar{j}_qx\right)\frac{(\bar{i}_qt)^\nu}{\{\nu\}_q!}\right)\left(\sum_{m=0}^{\infty}s_{\mathrm{NWA},\lambda^j,m,q}(i)\frac{(\bar{j}_qt)^m}{\{m\}_q!}\right)\\&\left(\sum_{l=0}^{\infty}\mathscr{F}^{(k-1)}_{\mathrm{NWA},\lambda^j,l,q}\left(\bar{i}_qy\right)\frac{(\bar{j}_qt)^l}{\{l\}_q!}\right)=-\frac{2^{-k}}{(\bar{i}_qt)^{k-1}}\mathrm{E}_q(\overline{ij}_q(x\oplus y)t)\left(\frac{\bar{i}_qt}{\lambda^i\mathrm{E}_q(\bar{i}_qt)-1}\right)^{k-1}\\&\left(\frac{2}{\lambda^j\mathrm{E}_q(\bar{j}_qt)+1}\right)^k\left(\frac{\lambda^{ij}\mathrm{E}_q(\overline{ij}_qt)-1}{\lambda^i\mathrm{E}_q(\bar{i}_qt)-1}\right)\overset{\text{by}(25)}{=}\\&\quad-\frac{2^{-k}}{(\bar{i}_qt)^{k-1}}\left(\sum_{\nu=0}^{\infty}\mathscr{F}^{(k)}_{\mathrm{NWA},\lambda^j,\nu,q}\left(\bar{i}_qx\right)\frac{(\bar{j}_qt)^\nu}{\{\nu\}_q!}\right)\\&\quad\left(\sum_{m=0}^{\infty}s_{\mathrm{NWA},\lambda^i,m,q}(j)\frac{(\bar{j}_qt)^m}{\{m\}_q!}\right)\left(\sum_{l=0}^{\infty}\mathscr{B}^{(k-1)}_{\mathrm{NWA},\lambda^i,l,q}\left(\bar{j}_qy\right)\frac{(\bar{i}_qt)^l}{\{l\}_q!}\right).\end{aligned} \tag{47}$$

The theorem follows by equating the coefficients of $\frac{t^n}{\{n\}_q!}$. □

Theorem 3.3. *A q–analogue of [5, p. 3353]. Under the assumption that i is* *even,*

$$\sum_{\nu=0}^{n}\binom{n}{\nu}_q(\bar{i}_q)^{\nu}(\bar{j}_q)^{n-\nu}\mathscr{F}^{(k-1)}_{\mathrm{NWA},\lambda^j,n-\nu,q}\left(\bar{i}_q y\right)\sum_{m=0}^{i-1}\lambda^{jm}(-1)^m$$
$$\mathscr{B}^{(k)}_{\mathrm{NWA},\lambda^i,\nu,q}\left(\bar{j}_q x\oplus_q\frac{\overline{jm}_q}{\bar{i}_q}\right)$$
$$=-\frac{\{n\}_q(\bar{i}_q)^k}{2(\bar{i}_q)^{k-1}}\sum_{k=0}^{n-1}\binom{n-1}{k}_q(\bar{i}_q)^{n-k-1}(\bar{j}_q)^k\mathscr{B}^{(k-1)}_{\mathrm{NWA},\lambda^i,n-k-1,q}\left(\bar{j}_q y\right)\sum_{m=0}^{j-1}\lambda^{im}$$
$$\mathscr{F}^{(k)}_{\mathrm{NWA},\lambda^j,k,q}\left(\bar{i}_q x\oplus_q\frac{\overline{im}_q}{\bar{j}_q})\right). \tag{48}$$

Proof. We can expand $g_q(t)$ as follows:

$$g_q(t)\overset{\mathrm{by}(44)}{=}-\frac{2^{-k}}{(\bar{i}_q t)^{k-1}}\mathrm{E}_q(\overline{ij}_q(x\oplus y)t)$$
$$\left(\frac{\bar{i}_q t}{\lambda^i\mathrm{E}_q(\bar{i}_q t)-1}\right)^{k-1}\left(\frac{2}{\lambda^j\mathrm{E}_q(\bar{j}_q t)+1}\right)^k\left(\frac{\lambda^{ij}\mathrm{E}_q(\overline{ij}_q t)-1}{\lambda^i\mathrm{E}_q(\bar{i}_q t)-1}\right).$$

$$=-\frac{2^{-k}}{(\bar{i}_q t)^{k-1}}\left(\frac{\bar{i}_q t}{\lambda^i\mathrm{E}_q(\bar{i}_q t)-1}\right)^{k-1} \tag{49}$$
$$\frac{2^k}{\left(\lambda^j\mathrm{E}_q(\bar{j}_q t)+1\right)^k}\sum_{m=0}^{j-1}\lambda^{im}\mathrm{E}_q\left(\left(\bar{i}_q x\oplus_q\frac{\overline{im}_q}{\bar{j}_q}\right)\bar{j}_q t\right)E_q(\overline{ij}_q yt)$$
$$=-\frac{2^{-k}}{(\bar{i}_q t)^{k-1}}\sum_{m=0}^{j-1}\lambda^{im}\sum_{l=0}^{\infty}\frac{(\bar{i}_q)^l t^l}{\{l\}_q!}\mathscr{B}^{(k-1)}_{\mathrm{NWA},\lambda^i,l,q}\left(\bar{j}_q y\right)$$
$$\sum_{n=0}^{\infty}\frac{(\bar{j}_q)^n t^n}{\{n\}_q!}\mathscr{F}^{(k)}_{\mathrm{NWA},\lambda^j,n,q}(\bar{i}_q x\oplus_q\frac{\overline{im}_q}{\bar{j}_q}).$$

The theorem follows by equating the coefficients of $\frac{t^n}{\{n\}_q!}$. □

Theorem 3.4.

$$\sum_{|\nu|=n}\binom{n}{\nu}_q(\bar{i}_q)^{\nu_1}(\bar{j}_q)^{\nu_2}(\bar{j}_q)^{\nu_3}\mathscr{F}^{(k)}_{\mathrm{NWA},\lambda^i,\nu_1,q}\left(\bar{j}_q x\right)\mathscr{B}^{(k-1)}_{\mathrm{NWA},\lambda^j,\nu_2,q}\left(\bar{i}_q y\right)s_{\mathrm{NWA},\lambda^j,\nu_3,q}(i)$$
$$=\sum_{\nu=0}^{n}\binom{n}{\nu}_q(\bar{i}_q)^{\nu}(\bar{j}_q)^{n-\nu}\mathscr{B}^{(k-1)}_{\mathrm{NWA},\lambda^j,n-\nu,q}\left(\bar{i}_q y\right)\sum_{m=0}^{i-1}\lambda^{jm}\mathscr{F}^{(k)}_{\mathrm{NWA},\lambda^i,\nu,q}\left(\bar{j}_q x\oplus_q\frac{\overline{jm}_q}{\bar{i}_q}\right) \tag{50}$$

Proof. Define the following function

$$\Psi_q(t) \equiv \frac{\mathrm{E}_q(\overline{ij}_q(x\oplus y)t)(\lambda^{ij}\mathrm{E}_q(\overline{ij}_q t)-1)}{(\lambda^i\mathrm{E}_q(\bar{i}_q t)+1)^k(\lambda^j\mathrm{E}_q(\bar{j}_q t)-1)^k}t^{k-1} = \mathrm{E}_q(\overline{ij}_q(x\oplus y)t) \left(\frac{2}{\lambda^i\mathrm{E}_q(\bar{i}_q t)+1}\right)^k \left(\frac{\bar{j}_q t}{\lambda^j\mathrm{E}_q(\bar{j}_q t)-1}\right)^{k-1} \left(\frac{\lambda^{ij}\mathrm{E}_q(\overline{ij}_q t)-1}{\lambda^j\mathrm{E}_q(\bar{j}_q t)-1}\right)\frac{2^{-k}}{(\bar{j}_q)^{k-1}}. \tag{51}$$

By using the formula for a geometric sequence, we can expand $\Psi_q(t)$ in two ways:

$$\begin{aligned}\Psi_q(t) &\overset{\text{by}(25)}{=} \left(\sum_{\nu=0}^{\infty}\mathscr{F}^{(k)}_{\mathrm{NWA},\lambda^i,\nu,q}\left(\bar{j}_q x\right)\frac{(\bar{i}_q t)^\nu}{\{\nu\}_q!}\right)\left(\sum_{m=0}^{\infty}s_{\mathrm{NWA},\lambda^j,m,q}(i)\frac{(\bar{j}_q t)^m}{\{m\}_q!}\right)\\ &\left(\sum_{l=0}^{\infty}\mathscr{B}^{(k-1)}_{\mathrm{NWA},\lambda^j,l,q}\left(\bar{i}_q y\right)\frac{(\bar{j}_q t)^l}{\{l\}_q!}\right)\frac{2^{-k}}{(\bar{j}_q)^{k-1}} = \frac{2^k}{\left(\lambda^i\mathrm{E}_q(\bar{i}_q t)-1\right)^k}\\ &\frac{(\bar{j}_q t)^{k-1}}{\left(\lambda^j\mathrm{E}_q(\bar{j}_q t)-1\right)^{k-1}}\sum_{m=0}^{i-1}\lambda^{jm}\mathrm{E}_q\left(\left(\bar{j}_q x\oplus_q\bar{j}_q y\oplus_q\frac{\overline{jm}_q}{\bar{i}_q}\right)\bar{i}_q t\right)\frac{2^{-k}}{(\bar{j}_q)^{k-1}}\\ &=\sum_{m=0}^{i-1}\frac{2^{-k}\lambda^{jm}}{(\bar{j}_q)^{k-1}}\sum_{l=0}^{\infty}\frac{(\bar{i}_q)^l t^l}{\{l\}_q!}\mathscr{F}^{(k)}_{\mathrm{NWA},\lambda^i,l,q}\left(\bar{j}_q x\oplus_q\frac{\overline{jm}_q}{\bar{i}_q}\right)\\ &\sum_{n=0}^{\infty}\frac{(\bar{j}_q)^n t^n}{\{n\}_q!}\mathscr{B}^{(k-1)}_{\mathrm{NWA},\lambda^j,n,q}(\bar{i}_q y).\end{aligned} \tag{52}$$

The theorem follows by equating the coefficients of $\frac{t^n}{\{n\}_q!}$. □

The following example illustrates that similar formulas with $\mathscr{H}$ polynomials can easily be constructed.

Theorem 3.5.

$$\begin{aligned}&\sum_{|\nu|=n}\binom{n}{\nu}_q(\bar{i}_q)^{\nu_1}(\bar{j}_q)^{\nu_2}(\bar{j}_q)^{\nu_3}\mathscr{H}^{(k)}_{\mathrm{NWA},\lambda^i,\nu_1,q}\left(\bar{j}_q x\right)\mathscr{B}^{(k-1)}_{\mathrm{NWA},\lambda^j,\nu_2,q}\left(\bar{i}_q y\right)s_{\mathrm{NWA},\lambda^j,\nu_3,q}(i)\\ &=\sum_{\nu=0}^{n}\binom{n}{\nu}_q(\bar{i}_q)^{\nu}(\bar{j}_q)^{n-\nu}\mathscr{B}^{(k-1)}_{\mathrm{NWA},\lambda^j,n-\nu,q}\left(\bar{i}_q y\right)\sum_{m=0}^{i-1}\lambda^{jm}\mathscr{H}^{(k)}_{\mathrm{NWA},\lambda^i,\nu,q}\left(\bar{j}_q x\oplus_q\frac{\overline{jm}_q}{\bar{i}_q}\right)\end{aligned} \tag{53}$$

Proof. Use $\Psi_q(t)$ again. □

Theorem 3.6. *A q–analogue of [5, (3.11) p. 3356].*

$$\sum_{|\nu|=n}\binom{n}{\nu}_q(\bar{i}_q)^{\nu_1}(\bar{j}_q)^{\nu_2}(\bar{j}_q)^{\nu_3}\mathscr{F}^{(k)}_{\mathrm{NWA},\lambda^i,\nu_1,q}\left(\bar{j}_q x\right)\mathscr{B}^{(k-1)}_{\mathrm{NWA},\lambda^j,\nu_2,q}\left(\bar{i}_q y\right)n_{\mathrm{NWA},\lambda^j,\nu_3,q}(i)$$
$$=\sum_{\nu=0}^{n}\binom{n}{\nu}_q(\bar{i}_q)^{\nu}(\bar{j}_q)^{n-\nu}\mathscr{B}^{(k-1)}_{\mathrm{NWA},\lambda^j,n-\nu,q}\left(\bar{i}_q y\right)\sum_{m=0}^{i-1}\lambda^{jm}\mathscr{F}^{(k)}_{\mathrm{NWA},\lambda^i,\nu,q}\left(\bar{j}_q x\oplus_q\frac{\overline{jm}_q}{\bar{i}_q}\right) \tag{54}$$

Proof. Define the following function

$$f_q(t)\equiv\frac{\mathrm{E}_q(\overline{ij}_q(x\oplus y)t)(\lambda^{ij}\mathrm{E}_q(\overline{ij}_q t)-1)}{(\lambda^i\mathrm{E}_q(\bar{i}_q t)+1)^k(\lambda^j\mathrm{E}_q(\bar{j}_q t)-1)^k}t^k=\mathrm{E}_q(\overline{ij}_q(x\oplus y)t)$$
$$\left(\frac{2}{\lambda^i\mathrm{E}_q(\bar{i}_q t)+1}\right)^k\left(\frac{\bar{j}_q t}{\lambda^j\mathrm{E}_q(\bar{j}_q t)-1}\right)^{k-1}\left(\frac{\lambda^{ij}\mathrm{E}_q(\overline{ij}_q t)-1}{\lambda^j\mathrm{E}_q(\bar{j}_q t)-1}\right)\frac{1}{2^k(\bar{j}_q)^{k-1}}. \tag{55}$$

By using the formula for a geometric sequence, we can expand $f_q(t)$ in two ways:

$$f_q(t)\overset{\text{by(25)}}{=}\left(\sum_{\nu=0}^{\infty}\mathscr{B}^{(k)}_{\mathrm{NWA},\lambda^i,\nu,q}\left(\bar{j}_q x\right)\frac{(\bar{i}_q t)^{\nu}}{\{\nu\}_q!}\right)\left(\sum_{m=0}^{\infty}s_{\mathrm{NWA},\lambda^j,m,q}(i)\frac{(\bar{j}_q t)^m}{\{m\}_q!}\right)$$
$$\left(\sum_{l=0}^{\infty}\mathscr{B}^{(k-1)}_{\mathrm{NWA},\lambda^j,l,q}\left(\bar{i}_q y\right)\frac{(\bar{j}_q t)^l}{\{l\}_q!}\right)\frac{1}{2^k(\bar{j}_q)^{k-1}}=\frac{2^k}{\left(\lambda^i\mathrm{E}_q(\bar{i}_q t)+1\right)^k}$$
$$\frac{(\bar{j}_q t)^{k-1}}{\left(\lambda^j\mathrm{E}_q(\bar{j}_q t)-1\right)^{k-1}}\sum_{m=0}^{i-1}\lambda^{jm}\mathrm{E}_q\left(\left(\bar{j}_q x\oplus_q\bar{j}_q y\oplus_q\frac{\overline{jm}_q}{\bar{i}_q}\right)\bar{i}_q t\right)\frac{1}{2^k(\bar{j}_q)^{k-1}} \tag{56}$$
$$=\sum_{m=0}^{i-1}\frac{\lambda^{jm}}{2^k(\bar{j}_q)^{k-1}}\sum_{l=0}^{\infty}\frac{(\bar{i}_q)^l t^l}{\{l\}_q!}\mathscr{F}^{(k)}_{\mathrm{NWA},\lambda^i,l,q}\left(\bar{j}_q x\oplus_q\frac{\overline{jm}_q}{\bar{i}_q}\right)$$
$$\sum_{n=0}^{\infty}\frac{(\bar{j}_q)^n t^n}{\{n\}_q!}\mathscr{B}^{(k-1)}_{\mathrm{NWA},\lambda^j,n,q}(\bar{i}_q y).$$

The theorem follows by equating the coefficients of $\frac{t^n}{\{n\}_q!}$. □

4. Multiplication formulas

We will now define two quite general q–Appell polynomials, which have some similarities with the Appell polynomials in [9]. The names are chosen to resemble the Euler and Bernoulli polynomials.

Definition 4.1. A q–analogue of Lu, Luo [6, p. 4]. The generating function for the generalized NWA q–Apostol $\mathscr{E}$ polynomials of degree ν and order n, $\mathscr{E}^{(n)}_{\mathrm{NWA},\lambda,\mu,\theta;\nu,q}(x)$, is given by

$$\left(\frac{2^{\mu}t^{\theta}}{\lambda\mathrm{E}_q(t)+1}\right)^n\mathrm{E}_q(xt)=\sum_{\nu=0}^{\infty}\frac{t^{\nu}}{\{\nu\}_q!}\mathscr{E}^{(n)}_{\mathrm{NWA},\lambda,\mu,\theta;\nu,q}(x),\theta\in\mathbb{N}. \tag{57}$$

Several q–Appell polynomials in this article are special cases of these polynomials, e.g. the q–Euler polynomial is the case $\theta=0,\ \mu=1$.

Theorem 4.1. *A q–analogue of [6, (2.3) p. 5], first multiplication formula for q–Apostol-$\mathscr{E}$ polynomials*

$$\mathscr{E}^{(n)}_{\mathrm{NWA},\lambda,\mu,\theta;\nu,q}(\overline{m}_qx)=\frac{(\overline{m}_q)^{\nu}}{((\overline{m}_q)^{\theta})^n}\sum_{|\vec{j}|=n}(-\lambda)^k\binom{n}{\vec{j}}\mathscr{E}^{(n)}_{\mathrm{NWA},\lambda^m,\mu,\theta;\nu,q}\left(x\oplus_q\frac{\overline{k}_q}{\overline{m}_q}\right), \tag{58}$$

where $k=j_1+2j_2+\cdots+(m-1)j_{m-1}$, m odd.

Proof.

$$\begin{aligned}&\sum_{\nu=0}^{\infty}\mathscr{E}^{(n)}_{\mathrm{NWA},\lambda,\mu,\theta;\nu,q}(\overline{m}_qx)\frac{t^{\nu}}{\{\nu\}_q!}=\frac{(2^{\mu}t^{\theta})^n}{(\lambda\mathrm{E}_q(t)+1)^n}\mathrm{E}_q(\overline{m}_qxt)\\&=\frac{(2^{\mu}t^{\theta})^n}{(\lambda^m\mathrm{E}_q(\overline{m}_qt)+1)^n}\left(\sum_{i=0}^{m-1}(-\lambda)^i\mathrm{E}_q(\overline{i}_qt)\right)^n\mathrm{E}_q(\overline{m}_qxt)\\&\stackrel{\text{by}(6)}{=}\left(\frac{2^{\mu}t^{\theta}\overline{m}_q^{\theta}}{(\lambda^m\mathrm{E}_q(\overline{m}_qt)+1)}\right)^n\sum_{|\vec{j}|=n}\binom{n}{\vec{j}}(-\lambda)^k\mathrm{E}_q\left((x\oplus_q\frac{\overline{k}_q}{\overline{m}_q})\overline{m}_qt\right)\frac{1}{((\overline{m}_q)^{\theta})^n}\\&=\sum_{\nu=0}^{\infty}\left(\frac{(\overline{m}_q)^{\nu}}{((\overline{m}_q)^{\theta})^n}\sum_{|\vec{j}|=n}\binom{n}{\vec{j}}(-\lambda)^k\mathscr{E}^{(n)}_{\mathrm{NWA},\lambda^m,\mu,\theta;\nu,q}\left(x\oplus_q\frac{\overline{k}_q}{\overline{m}_q}\right)\right)\frac{t^{\nu}}{\{\nu\}_q!}.\end{aligned} \tag{59}$$

The theorem follows by equating the coefficients of $\frac{t^{\nu}}{\{\nu\}_q!}$. ∎

The following formula only applies for special values of the integers.

Theorem 4.2. *A q–analogue of [6, (2.4) p. 5], second multiplication formula for q–Apostol-$\mathscr{E}$ polynomials.*

$$\begin{aligned}&\mathscr{E}^{(n)}_{\mathrm{NWA},\lambda,\mu,\theta;\nu,q}(\overline{m}_qx)\\&=\frac{(-1)^n2^{\mu n}(\overline{m}_q)^{\nu+(1-\theta)n}}{\{\nu+1\}_{(1-\theta)n,q}(\overline{m}_q)^n}\sum_{|\vec{j}|=n}(-\lambda^k)\binom{n}{\vec{j}}\mathscr{B}^{(n)}_{\mathrm{NWA},\lambda^m,\nu+(1-\theta)n,q}\left(x\oplus_q\frac{\overline{k}_q}{\overline{m}_q}\right),\end{aligned} \tag{60}$$

where $k=j_1+2j_2+\cdots+(m-1)j_{m-1}$, *m even,* $\nu\geq(\theta-1)n$.

Proof.

$$\begin{aligned}&\sum_{\nu=0}^{\infty}\mathscr{E}_{\mathrm{NWA},\lambda,\mu,\theta;\nu,q}^{(n)}(\overline{m}_q x)\frac{t^\nu}{\{\nu\}_q!}=\frac{(2^\mu t^\theta)^n}{(\lambda\mathrm{E}_q(t)+1)^n}\mathrm{E}_q(\overline{m}_q xt)\\&=\frac{(2^\mu t^\theta)^n}{(1-\lambda^m\mathrm{E}_q(\overline{m}_q t))^n}\left(\sum_{i=0}^{m-1}(-\lambda)^i\mathrm{E}_q(\overline{i}_q t)\right)^n\mathrm{E}_q(\overline{m}_q xt)\\&=\left(\frac{2^\mu t\overline{m}_q}{1-\lambda^m\mathrm{E}_q(\overline{m}_q t)}\right)^n\sum_{|\vec{j}|=n}\binom{n}{\vec{j}}(-\lambda)^k\mathrm{E}_q\left((x\oplus_q\frac{\overline{k}_q}{\overline{m}_q})\overline{m}_q t\right)\frac{t^{(\theta-1)n}}{(\overline{m}_q)^n}\\&=(-1)^n t^{(\theta-1)n}2^{\mu n}\sum_{\nu=0}^{\infty}\left(\frac{(\overline{m}_q)^\nu}{(\overline{m}_q)^n}\sum_{|\vec{j}|=n}\binom{n}{\vec{j}}(-\lambda)^k\mathscr{B}_{\mathrm{NWA},\lambda^m,\nu;q}^{(n)}\left(x\oplus_q\frac{\overline{k}_q}{\overline{m}_q}\right)\right)\frac{t^\nu}{\{\nu\}_q!}\end{aligned}\tag{61}$$

The theorem follows by equating the coefficients of $\frac{t^\nu}{\{\nu\}_q!}$. □

Corollary 4.2. *A q–analogue of [8, (2.1) p. 49], [6, p. 7], first multiplication formula for generalized q–$\mathscr{H}$ polynomials.*

$$\mathscr{H}_{\mathrm{NWA},\lambda,\nu,q}^{(n)}(\overline{m}_q x)=\frac{(\overline{m}_q)^\nu}{(\overline{m}_q)^n}\sum_{|\vec{j}|=n}(-\lambda)^k\binom{n}{\vec{j}}\mathscr{H}_{\mathrm{NWA},\lambda^m,\nu,q}^{(n)}\left(x\oplus_q\frac{\overline{k}_q}{\overline{m}_q}\right),\tag{62}$$

where $k=j_1+2j_2+\cdots+(m-1)j_{m-1}$, *m odd.*

Corollary 4.3. *A q–analogue of [8, (2.2) p. 49], [6, p. 7], second multiplication formula for generalized q–$\mathscr{H}$ polynomials.*

$$\mathscr{H}_{\mathrm{NWA},\lambda,\nu,q}^{(n)}(\overline{m}_q x)=\frac{(-2)^n(\overline{m}_q)^\nu}{(\overline{m}_q)^n}\sum_{|\vec{j}|=n}(-\lambda^k)\binom{n}{\vec{j}}\mathscr{B}_{\mathrm{NWA},\lambda^m,\nu,q}^{(n)}\left(x\oplus_q\frac{\overline{k}_q}{\overline{m}_q}\right),\tag{63}$$

where $k=j_1+2j_2+\cdots+(m-1)j_{m-1}$, *m even.*

Theorem 4.3. *A q–analogue of [8, p. 51], an explicit formula for the multiple alternating q–power sum:*

$$\begin{aligned}&\sigma_{\mathrm{NWA},\lambda,\nu,q}^{(l)}(n)=2^{-l}\sum_{j=0}^{l}\binom{l}{j}\frac{(-1)^{jn}\lambda^{(n-1)j+l}}{\{\nu+1\}_{l,q}}\\&\sum_{m=0}^{\nu+l}\binom{\nu+l}{m}_q\mathscr{H}_{\mathrm{NWA},\lambda,m,q}^{(j)}\left(\overline{(n-1)j+l}_q\right)\mathscr{H}_{\mathrm{NWA},\lambda,\nu+l-m,m,q}^{(l-j)},\end{aligned}\tag{64}$$

Proof. We use the generating function technique. Put $k = j_1 + 2j_2 + \cdots + (n-1)j_{n-1}$. It is assumed that $j_i \geq 0, 1 \leq i \leq n-1$. All zeros are neglected.

$$\sum_{\nu=0}^{\infty} \sigma^{(l)}_{\mathrm{NWA},\lambda,\nu,q}(n)\frac{t^\nu}{\{\nu\}_q!} \overset{\text{by}(17)}{=} (-1)^l \sum_{\nu=0}^{\infty}\left(\sum_{|\vec{j}|=l}\binom{l}{\vec{j}}(-\lambda)^k\left(\overline{k}_q\right)^\nu\right)\frac{t^\nu}{\{\nu\}_q!}$$
$$= \left(\lambda \mathrm{E}_q(t) - \lambda^2\mathrm{E}_q(\overline{2}_q t) + \cdots + (-1)^n\lambda^{n-1}\mathrm{E}_q(\overline{n-1}_q t)\right)^l$$
$$= \left(\frac{(-\lambda)^n \mathrm{E}_q(\overline{n}_q t)}{\lambda \mathrm{E}_q(t)+1} + \frac{\lambda \mathrm{E}_q(t)}{\lambda \mathrm{E}_q(t)+1}\right)^l$$
$$= \sum_{j=0}^{l}\binom{l}{j}\left(\frac{(-\lambda)^n \mathrm{E}_q(\overline{n}_q t)}{\lambda \mathrm{E}_q(t)+1}\right)^j\left(\frac{\lambda \mathrm{E}_q(t)}{\lambda \mathrm{E}_q(t)+1}\right)^{l-j}$$
$$\overset{\text{by}(6)}{=} (2t)^{-l}\sum_{j=0}^{l}\binom{l}{j}(-1)^{jn}\lambda^{(n-1)j+l}\sum_{m=0}^{\infty}\mathscr{H}^{(j)}_{\mathrm{NWA},\lambda,m,q}\left(\overline{(n-1)j+l}_q\right)\frac{t^m}{\{m\}_q!}$$
$$\sum_{i=0}^{\infty}\mathscr{H}^{(l-j)}_{\mathrm{NWA},\lambda,i,q}\frac{t^i}{\{i\}_q!} = \sum_{\nu=0}^{\infty}\left[2^{-l}\sum_{j=0}^{l}\binom{l}{j}\frac{(-1)^{jn}\lambda^{(n-1)j+l}}{\{\nu+1\}_{l,q}}\right.$$
$$\left.\sum_{m=0}^{\nu+l}\binom{\nu+l}{m}_q \mathscr{H}^{(j)}_{\mathrm{NWA},\lambda,m,q}\left(\overline{(n-1)j+l}_q\right)\mathscr{H}^{(l-j)}_{\mathrm{NWA},\lambda,\nu+l-m,m,q}\right]\frac{t^\nu}{\{\nu\}_q!}. \tag{65}$$

The theorem follows by equating the coefficients of $\frac{t^\nu}{\{\nu\}_q!}$. □

Theorem 4.4. *For m odd, we have the following recurrence relation for q–Apostol-$\mathscr{E}$-numbers.*

$$\mathscr{E}^{(l)}_{\mathrm{NWA},\lambda,\mu,\theta;n,q} = (-1)^l\sum_{j=0}^{n}\binom{n}{j}_q\frac{(\overline{m}_q)^n}{((\overline{m}_q)^\theta)^l(\overline{m}_q)^{n-j}}\mathscr{E}^{(l)}_{\mathrm{NWA},\lambda^m,\mu,\theta;j,q}\sigma^{(l)}_{\mathrm{NWA},\lambda,n-j,q}(m), \tag{66}$$

where $k = j_1 + 2j_2 + \cdots + (m-1)j_{m-1}$ *in* $\sigma^{(l)}_{\mathrm{NWA},\lambda,n-j,q}(m)$.

Proof.

$$\mathscr{E}^{(l)}_{\mathrm{NWA},\lambda,\mu,\theta;n,q} \overset{\text{by}(58)}{=} \frac{(\overline{m}_q)^n}{((\overline{m}_q)^\theta)^l}\sum_{|\vec{\nu}|=l}(-\lambda)^k\binom{l}{\vec{\nu}}\mathscr{E}^{(l)}_{\mathrm{NWA},\lambda^m,\mu,\theta;n,q}\left(\frac{\overline{k}_q}{\overline{m}_q}\right)$$
$$= \frac{(\overline{m}_q)^n}{((\overline{m}_q)^\theta)^l}\sum_{|\vec{\nu}|=l}(-\lambda)^k\binom{l}{\vec{\nu}}\sum_{j=0}^{n}\binom{n}{j}_q\mathscr{E}^{(l)}_{\mathrm{NWA},\lambda^m,\mu,\theta;j,q}\left(\frac{\overline{k}_q}{\overline{m}_q}\right)^{n-j}$$
$$= \sum_{j=0}^{n}\binom{n}{j}_q\frac{(\overline{m}_q)^n}{(\overline{m}_q)^{n-j}((\overline{m}_q)^\theta)^l}\mathscr{E}^{(l)}_{\mathrm{NWA},\lambda^m,\mu,\theta;j,q}\sum_{|\vec{\nu}|=l}(-\lambda)^k\binom{l}{\vec{\nu}}(\overline{k}_q)^{n-j}\overset{\text{by}(17)}{=}\text{LHS}. \tag{67}$$

□

Definition 4.4. The generating function for the generalized NWA q–Apostol $\mathscr{C}$ polynomials of degree ν and order n, $\mathscr{C}^{(n)}_{\mathrm{NWA},\lambda,\theta;\nu,q}(x)$, is given by

$$\left(\frac{t^{\theta}}{\lambda \mathrm{E}_q(t)-1}\right)^n \mathrm{E}_q(xt)=\sum_{\nu=0}^{\infty}\frac{t^{\nu}}{\{\nu\}_q!}\mathscr{C}^{(n)}_{\mathrm{NWA},\lambda,\theta;\nu,q}(x), \theta\in\mathbb{N}. \tag{68}$$

Theorem 4.5. *Multiplication formula for q–Apostol-$\mathscr{C}$ polynomials*

$$\mathscr{C}^{(n)}_{\mathrm{NWA},\lambda,\theta;\nu,q}(\overline{m}_q x)=\frac{(\overline{m}_q)^{\nu}}{((\overline{m}_q)^{\theta})^n}\sum_{|\vec{j}|=n}\lambda^k\binom{n}{\vec{j}}\mathscr{C}^{(n)}_{\mathrm{NWA},\lambda^m,\theta;\nu,q}\left(x\oplus_q\frac{\overline{k}_q}{\overline{m}_q}\right), \tag{69}$$

where $k=j_1+2j_2+\cdots+(m-1)j_{m-1}$.

Proof.

$$\begin{aligned}
&\sum_{\nu=0}^{\infty}\mathscr{C}^{(n)}_{\mathrm{NWA},\lambda,\theta;\nu,q}(\overline{m}_q x)\frac{t^{\nu}}{\{\nu\}_q!}=\frac{t^{\theta n}}{(\lambda\mathrm{E}_q(t)-1)^n}\mathrm{E}_q(\overline{m}_q xt)\\
&=\frac{t^{\theta n}}{(\lambda^m\mathrm{E}_q(\overline{m}_q t)-1)^n}\left(\sum_{i=0}^{m-1}\lambda^i\mathrm{E}_q(\overline{i}_q t)\right)^n\mathrm{E}_q(\overline{m}_q xt)\\
&\overset{\text{by}(6)}{=}\left(\frac{t^{\theta}\overline{m}_q^{\theta}}{(\lambda^m\mathrm{E}_q(\overline{m}_q t)-1)}\right)^n\sum_{|\vec{j}|=n}\binom{n}{\vec{j}}\lambda^k\mathrm{E}_q\left((x\oplus_q\frac{\overline{k}_q}{\overline{m}_q})\overline{m}_q t\right)\frac{1}{((\overline{m}_q)^{\theta})^n}\\
&=\sum_{\nu=0}^{\infty}\left(\frac{(\overline{m}_q)^{\nu}}{((\overline{m}_q)^{\theta})^n}\sum_{|\vec{j}|=n}\binom{n}{\vec{j}}\lambda^k\mathscr{C}^{(n)}_{\mathrm{NWA},\lambda^m,\theta;\nu,q}\left(x\oplus_q\frac{\overline{k}_q}{\overline{m}_q}\right)\right)\frac{t^{\nu}}{\{\nu\}_q!}.
\end{aligned} \tag{70}$$

The theorem follows by equating the coefficients of $\frac{t^{\nu}}{\{\nu\}_q!}$. □

5. Discussion

This was the first multiplication formula for a q–Appell polynomial of general form; the Ward numbers replace the integers in the function argument. Certainly there are other general q–Appell polynomials with similar expansion and multiplication formulas.

Many of the proofs use the formula for a geometric sequence in q–form and the generating function for the q–Appell polynomials and the power sums. The integers i and j are crucial for the formulas; by the generating function, if λ^i,ν appears as index in a polynomial, certainly the factor $(\overline{i}_q)^{\nu}$ will also appear. If the orders of two polynomials in a formula are k and $k-1$, the last one with index λ^j, and argument $\bar{i}_q y$, surely a function $\sigma_{\mathrm{NWA},\lambda^j,m,q}(i)$ or $s_{\mathrm{NWA},\lambda^j,m,q}(i)$, together with $(\bar{j}_q)^m$ will appear. If a polynomial has λ^i, ν as index, it will have $(\bar{j}_q)$ in the function argument, and vice versa.

These considerations also hold for the case $q = 1$. Even if the reader is not interested in q–calculus, this paper is a good summary of the recent trends on Apostol type Appell polynomials; just put $q = 1$.

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