LE MATEMATICHE
Vol. LXI (2006) - Fasc. II, pp. 385-394

A NEW ENERGY FUNCTIONAL FOR NONLINEAR STABILITY OF THE CLASSICAL BÉNARD PROBLEM

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Nonlinear stability of motionless state of the classical Bénard problem in case of stress-free boundaries is studied for 2-dimensional disturbances, by the Liapunov's second method. For Rayleigh number smaller than $27\pi^4/4$ the motionless state is proved to be unconditionally and exponentially stable with respect to a new Liapunov function which is essentially stronger than the kinetic energy.

1. Introduction.

Bénard problem is well suited to illustrate many physical and mathematical facets of the general theory of hydrodynamics stability and convection problems. For this reason many mathematicians have selected this problem for the mathematical treatment and much work has been done for the problem with rotation or magnetic field (see: [1-5]). In this paper we consider again the classical Bénard problem.

The classical Bénard problem refers to an infinite horizontal layer

Entrato in redazione il 6 dicembre 2006.
Key Words: Nonlinear stability, Rayleigh number, Bénard problem.
MR 2000: 46N20, 35Q35, 76E06.

filled with incompressible fluid. The layer is heated from below so that an adverse temperature gradient is maintained. If the temperature gradient exceeds a certain value the instability will set in. Lord Rayleigh showed that what decided the stability is the numerical value of the dimensionless parameter, Rayleigh number. It is given by $R = \frac{g\alpha\beta}{\kappa\nu}d^3$, where $g, \beta, d, \alpha, \kappa, \nu$ are gravity acceleration, the adverse temperature gradient maintained between the layers, the thickness of the layer, the coefficients of volume expansion, thermometric conductivity, and kinematic viscosity, respectively.

The linear stability of the classical Bénard problem is thoroughly studied in [6]. It is shown that the motionless state is linearly stable for $R < 27\pi^4/4$, when the boundaries are stress free. Many papers proved that $R < 27\pi^4/4$ also guarantees nonlinear stability and meanwhile perturbation will decay exponentially to zero, with respect to kinetic energy:

$$E(t) = \int_\Omega |u|^2 dV + p_r \int_\Omega |\theta|^2 dV$$

(see [7-9]).

In this paper, by taking advantage of decomposition of a solenoidal field into poloidal field, toroidal field and the mean flow, we will show that 2-D perturbation will also decay exponentially to zero with respect to an energy functional being essentially stronger than kinetic energy. So far we haven't seen any other energy functional which is essentially stronger than kinetic energy, and with it one can still prove unconditional stability for this problem. In three-dimensional case we can only obtain a conditional stability result.

2. Mathematical formulation of the problem.

Let us consider an infinite horizontal layer $\mathbb{R}^2 \times (0, d)$ in a Cartesian reference frame Oxyz with unit vector $\mathbf{k} = (0, 0, 1)^T$ for the z-axis being against gravity. The layer is filled with an incompressible fluid and heated from below. Then the non-dimensionalized equations for a perturbation ($\mathbf{u}$, θ, P) of the motionless state ($\mathbf{u} = 0, T = -\beta z + T_0$, where T_0 is the temperature to be held at the bottom of the layer) are given as follows, see e.g. [9]

$$
\text{(1)} \qquad \begin{cases} \partial_t \mathbf{u} - \Delta \mathbf{u} - \sqrt{R}\theta \mathrm{k} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = 0 \\ \nabla \cdot \mathbf{u} = 0 \cdot \\ p_r \partial_t \theta - \Delta \theta - \sqrt{R}\mathbf{u} \cdot \mathbf{k} + p_r \mathbf{u} \cdot \nabla \theta = 0 \end{cases}
$$

where $\mathbf{u} = (u, v, w)^T$, $p_r > 0$ is the Prandtl number, $R > 0$ is the Rayleigh number. $\mathbf{u}$, θ and P represent the velocity, temperature and pressure field of the perturbation, $\triangle$ is the three dimensional Laplacian, $.^T$ refers to transposition. $\mathbf{u}$, θ and P are assumed to be $x, y-$periodic with respect to a rectangle $\mathscr{P} = \left(-\frac{\pi}{\alpha}, \frac{\pi}{\alpha}\right) \times \left(-\frac{\pi}{\beta}, \frac{\pi}{\beta}\right)$ with wave numbers α, β in $x-$ and $y-$direction, respectively, and $z \in \left(-\frac{1}{2}, \frac{1}{2}\right)$.

The boundary conditions on $\mathbf{u}$ at $z = \pm\frac{1}{2}$ are

$$
\text{(2)} \qquad \partial_z u = \partial_z v = w = \theta = 0
$$

this is so called the stress free boundaries. In what follows we use the decomposition of a periodic solenoidal vector field $\boldsymbol{u}$ into a poloidal field, a toroidal one and a mean flow

$$
\begin{aligned} \mathbf{u} &= \mathrm{curlcurl}\varphi\mathbf{k} + \mathrm{curl}\psi\mathbf{k} + \mathbf{f} \\ &= \boldsymbol{\delta}\varphi + \boldsymbol{\varepsilon}\psi + \mathbf{f}. \end{aligned}
$$

where

$$
\boldsymbol{\delta}\cdot = \mathrm{curlcurl}.\mathbf{k} = (\partial_{xz}\cdot, \partial_{yz}\cdot, -\Delta_2\cdot)^T,
$$

$$
\boldsymbol{\varepsilon}\cdot = \mathrm{curl}.\mathbf{k} = (\partial_y\cdot, -\partial_x\cdot, 0)^T, \quad \mathbf{f} = (f_1, f_2, f_3)^T
$$

is a vector field depending only on z and it has a constant third component, together with the boundary conditions (2) we have $f_3 = 0$. Then, the system (1) can be transformed into an equivalent one for $\boldsymbol{\Phi} = (\varphi, \psi, \theta, f_1, f_2)^T$. It has the form (see [9,10])

$$
\text{(3)} \qquad \begin{cases} \partial_t(-\Delta)(-\Delta_2)\varphi = -(-\Delta)^2(-\Delta_2)\varphi + \sqrt{R}(-\Delta_2)\theta - \boldsymbol{\delta} \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) \\ \partial_t(-\Delta_2)\psi = -(-\Delta)(-\Delta_2)\psi + \boldsymbol{\varepsilon} \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) \\ p_r \partial_t \theta = -(-\Delta)\theta + \sqrt{R}(-\Delta_2)\varphi - p_r \mathbf{u} \cdot \nabla \theta \\ \partial_t f_1 = -(-\partial_z^2) f_1 - \dfrac{1}{|\mathscr{P}|} \displaystyle\int_{\mathscr{P}} \tilde{\mathbf{u}} \cdot \nabla \tilde{\mathbf{u}}_1 dx dy \\ \partial_t f_2 = -(-\partial_z^2) f_2 - \dfrac{1}{|\mathscr{P}|} \displaystyle\int_{\mathscr{P}} \tilde{\mathbf{u}} \cdot \nabla \tilde{\mathbf{u}}_2 dx dy \end{cases}
$$

where $\tilde{\mathbf{u}} = \boldsymbol{\delta}\varphi + \boldsymbol{\varepsilon}\psi$, $\tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2$ refer to first and second component of $\tilde{\mathbf{u}}$, respectively. φ, ψ are uniquely determined if we require them to be periodic in x, y and to have vanishing mean value over $\mathcal{P}$, i.e. $\int_{\mathcal{P}} \varphi dxdy = \int_{\mathcal{P}} \psi dxdy = 0$.

The corresponding boundary conditions for the new dependent variables are

$$\varphi = \partial_z^2 \varphi = \partial_z \psi = \partial_z f_1 = \partial_z f_2 = \theta = 0 \quad \text{at} \quad z = \pm\frac{1}{2}. \tag{4}$$

We study the new system (3) for $(\varphi, \psi, \theta, f_1, f_2)$ in the Hilbert spaces:

$$H = L^2_M(\Omega) \times L^2_M(\Omega) \times L^2(\Omega) \times \left(L^2_M\left(-\frac{1}{2}, \frac{1}{2}\right)\right)^2$$

where

$$L^2_M(\Omega) = \left\{\hat{\varphi} \middle| \hat{\varphi} \in L^2(\Omega); \quad \int_{\mathcal{P}} \hat{\varphi} dxdy = 0\right\},$$

$$L^2_M\left(-\frac{1}{2}, \frac{1}{2}\right) = \left\{\hat{f} \middle| \hat{f} \in L^2\left(-\frac{1}{2}, \frac{1}{2}\right); \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} f(z)\, dz = 0\right\},$$

$$\boldsymbol{\Omega} = \mathcal{P} \times \left(-\frac{1}{2}, \frac{1}{2}\right).$$

For both $L^2(\Omega)$ and $L^2\left(-\frac{1}{2}, \frac{1}{2}\right)$ we define the inner product by

$$\langle f, g\rangle = \int_{\Omega} f \cdot \bar{g}\, dx\, dy\, dz,$$

then, for $f, g \in L^2(-\frac{1}{2}, \frac{1}{2})$ the inner product is given by

$$\langle f, g\rangle = |\mathcal{P}| \int_{-\frac{1}{2}}^{\frac{1}{2}} f \cdot \bar{g}\, dz,$$

with $\|\cdot\|$ we denote the corresponding norm to the inner product. If no confusion can arise we also apply symbol $\langle\cdot,\cdot\rangle$ as inner product and $\|\cdot\|$ as norm for any Hilbert space.

It is clear that

$$\frac{1}{\sqrt{|\mathcal{P}|}} e^{i(\alpha\kappa_1 x + \beta\kappa_2 y)}, \quad \boldsymbol{\kappa} = (\kappa_1, \kappa_2)^T \in \mathbb{Z}^2$$

form a complete orthonormal system in $L^2(\mathscr{P})$, $\mathbb{Z}$ is the set of integer. Thus, $\varphi, \psi \in L^2_M(\Omega)$ and $\theta \in L^2(\Omega)$ can be expanded in series

$$\varphi(x, y, z, t) = \frac{1}{\sqrt{|\mathscr{P}|}} \sum_{\boldsymbol{K} \in \mathbb{Z}^2 - \{0\}} a_{\boldsymbol{K}}(z, t) e^{i(\alpha\kappa_1 x + \beta\kappa_2 y)}$$

$$\psi(x, y, z, t) = \frac{1}{\sqrt{|\mathscr{P}|}} \sum_{\boldsymbol{K} \in \mathbb{Z}^2 - \{0\}} b_{\boldsymbol{K}}(z, t) e^{i(\alpha\kappa_1 x + \beta\kappa_2 y)}$$

$$\theta(x, y, z, t) = \frac{1}{\sqrt{|\mathscr{P}|}} \sum_{\boldsymbol{K} \in \mathbb{Z}^2} c_{\boldsymbol{K}}(z, t) e^{i(\alpha\kappa_1 x + \beta\kappa_2 y)}$$

One may ask for the reason to use complicated equations (3) instead of (1). The system (3) seems to be used almost exclusively by physicists, in particular when there is no mean flow. One of the main reason for the usage of (3) is that the pressure is eliminated and that at the same time the nonlinearity is almost local, except for the subsystem for mean flows. We don't use the classical tool of projecting $L^2(\Omega)$ on its divergence-free part to eliminate ∇P, because the projection just mentioned is nonlocal and therefore yields a nonlocal nonlinearity, with which it is difficult for us to obtain unconditional stability.

Now we try to give a definition of energy functional. It is a generalization of the kinetic energy.

Definition 2.1. Let H be the above Hilbert space. Let S be the set of the weak solutions of system (3) in H. A real valued function $F : S \longrightarrow [0, \infty)$ is said to be an energy functional for system (3) iff, for any fixed time t, F satisfies:

(i) $F(\Phi) > 0$, for $\Phi \neq 0$ in $S \subset H$,

(ii) $F(\Phi) = 0$, iff $\Phi = 0$ in H.

It is clear that an energy functional for system (3) is a function of time t. Taking account of the Fourier expansion and the boundary conditions, one can easily show that kinetic energy

$$E(t) = \|\mathbf{u}\|^2 + p_r\|\theta\|^2 = \|\boldsymbol{\delta}\varphi\|^2 + \|\boldsymbol{\varepsilon}\psi\|^2 + \|f_1\|^2 + \|f_2\|^2 + p_r\|\theta\|^2$$

is an energy functional for system (3). Of course, one can define an energy functional as one pleases, but it is very difficult to find an energy functional with which one can obtain a meaningful result for nonlinear stability.

Definition 2.2. Let $\mathcal{E}_1(t)$ and $\mathcal{E}_2(t)$ be two energy functionals for system (3), $\mathcal{E}_2(t)$ is said to be essentially stronger than $\mathcal{E}_1(t)$, iff there exists a positive constant C such that $\mathcal{E}_1(t) < C\mathcal{E}_2(t)$, but there exists no positive constant C such that $\mathcal{E}_2(t) < C\mathcal{E}_1(t)$.

3. Nonlinear stability.

In this section we will define an energy functional to study the nonlinear stability of the Bénard problem for 2-dimensional disturbances. A sufficient condition for unconditional stability of the motionless state will be derived, that is the following theorem.

Theorem 3.1. *Let the perturbation be 2-D ($\partial_y \cdot = 0$). We define an energy functional by*

$$\mathcal{E}(t) = \frac{1}{2}\{\|\Delta\partial_x\varphi\|^2 + \|\partial_x\psi\|^2 + \|\partial_z f_1\|^2 + \|f_2\|^2 + \lambda p_r\|\theta\|^2\},$$

then

$$\mathcal{E}(t) \le \mathcal{E}(0)e^{\frac{-2\min\{\alpha^2,\pi^2\}}{\max\{1,p_r\}}\left(1-\sqrt{\frac{R}{R_E}}\right)t}$$

if $R < R_E = \frac{27}{4}\pi^4$, where $\lambda = \frac{3\pi^2}{2}$. That means $\mathcal{E}(t)$ decays exponentially for any initial value whenever $R < R_E$.

Proof. From equations (3) and boundary conditions (4) we have

$$\frac{1}{2}\frac{d}{dt}\|\Delta\partial_x\varphi\|^2 = -\|\partial_x\nabla\Delta\varphi\|^2 + \sqrt{R}\langle\partial_x^2\theta, \Delta\varphi\rangle - \langle\boldsymbol{\delta}\cdot(\mathbf{u}\cdot\nabla\mathbf{u}), -\Delta\varphi\rangle$$

$$\frac{1}{2}\frac{d}{dt}\|\partial_x\psi\|^2 = -\|\partial_x\nabla\psi\|^2 + \langle\boldsymbol{\varepsilon}\cdot(\mathbf{u}\cdot\nabla\mathbf{u}), \psi\rangle$$

$$\frac{1}{2}\lambda p_r\frac{d}{dt}\|\theta\|^2 = -\lambda\|\nabla\theta\|^2 + \lambda\sqrt{R}\langle-\partial_x^2\varphi, \theta\rangle - \lambda p_r\langle\mathbf{u}\cdot\nabla\theta, \theta\rangle$$

$$\frac{1}{2}\frac{d}{dt}\|\partial_z f_1\|^2 = -\|\partial_z^2 f_1\|^2 - \langle\tilde{\mathbf{u}}\cdot\nabla\tilde{\mathbf{u}}_1, -\partial_z^2 f_1\rangle$$

$$\frac{1}{2}\frac{d}{dt}\|f_2\|^2 = -\|\partial_z f_2\|^2 - \langle\tilde{\mathbf{u}}\cdot\nabla\tilde{\mathbf{u}}_2, f_2\rangle$$

λ is a positive coupling parameter and it will be selected to get the best stability boundary R_E. With the identity $\boldsymbol{\delta}\cdot(\boldsymbol{\delta}\varphi\cdot\nabla\boldsymbol{\delta}\varphi) = \partial_x(\boldsymbol{\delta}\varphi\cdot\nabla\partial_x\Delta\varphi)$ and boundary conditions (4) we find

$$\langle\boldsymbol{\delta}\cdot(\mathbf{u}\cdot\nabla\mathbf{u}), -\Delta\varphi\rangle + \langle\tilde{\mathbf{u}}\cdot\nabla\tilde{\mathbf{u}}_1, -\partial_z^2 f_1\rangle$$
$$= \langle\boldsymbol{\delta}\cdot(\tilde{\mathbf{u}}\cdot\nabla\mathbf{f}+\mathbf{f}\cdot\nabla\tilde{\mathbf{u}}), -\Delta\varphi\rangle + \langle\tilde{\mathbf{u}}\cdot\nabla\partial_{xz}\varphi, -\partial_z^2 f_1\rangle$$
$$= \partial_z^2 f_1\cdot(-\partial_x^3\varphi)+f_1\cdot\partial_x^3\Delta\varphi, -\Delta\varphi\rangle+\langle\partial_{xz}\varphi\cdot\partial_{x^2z}\varphi+(-\partial_x^2\varphi)\cdot\partial_{xz^2}\varphi, (-\partial_z^2)f_1\rangle$$
$$= \langle\partial_z^2 f_1\cdot(-\partial_x^3\varphi), -\Delta\varphi\rangle + \langle(-\partial_x^2\varphi)\cdot\partial_{xz^2}\varphi, -\partial_z^2 f_1\rangle$$
$$= \langle\partial_z^2 f_1\cdot(-\partial_x^3\varphi), -\partial_z^2\varphi\rangle + \langle(-\partial_x^2\varphi)(-\partial_z^2 f_1), \partial_{xz^2}\varphi\rangle = 0$$

$$\langle\boldsymbol{\varepsilon}\cdot(\mathbf{u}\cdot\nabla\mathbf{u}), \psi\rangle - \langle\tilde{\mathbf{u}}\cdot\nabla\tilde{\mathbf{u}}_2, f_2\rangle$$
$$\text{with}\quad \langle\boldsymbol{\varepsilon}\cdot(\tilde{\mathbf{u}}\cdot\nabla\tilde{\mathbf{u}}), \psi\rangle = 0$$
$$= \langle\mathbf{f}\cdot\nabla\tilde{\mathbf{u}}_2+\tilde{\mathbf{u}}\cdot\nabla f_2, \partial_x\psi\rangle - \langle\tilde{\mathbf{u}}\cdot\nabla\tilde{\mathbf{u}}_2, f_2\rangle$$
$$= \langle f_1(-\partial_x^2\psi)+\partial_z f_2\cdot(-\partial_x^2\varphi), \partial_x\psi\rangle$$
$$-\langle\partial_{xz}\varphi\cdot(-\partial_x^2\psi)+(-\partial_x^2\varphi)(-\partial_{xz}\psi), f_2\rangle = 0$$

and $\langle\mathbf{u}\cdot\nabla\theta, \theta\rangle = 0$, we obtain by differentiating $\mathcal{E}(t)$ the relation

$$\frac{d}{dt}\mathcal{E}(t) = -D + I \leq -D(1-\mu) \tag{5}$$

where

$$\mu = \max_{\hat{H}}\frac{I}{D}, \tag{6}$$

$$\hat{H} = \{\varphi, \psi, \theta, \mathbf{f}|\varphi, \psi, \theta \text{ are periodic, in } x, y \text{ and } \varphi, \theta, \mathbf{f} \text{ satisfy boundary conditions (4)}\},$$
$$D = \|\partial_x\nabla\Delta\varphi\|^2 + \|\partial_x\nabla\psi\|^2 + \lambda\|\nabla\theta\|^2 + \|\partial_z^2 f_1\|^2 + \|\partial_z f_2\|^2,$$
$$I = \sqrt{R}\langle\partial_x^2\theta, \Delta\varphi\rangle - \lambda\sqrt{R}\langle\partial_x^2\theta, \varphi\rangle.$$

The Euler-Lagrange equations for (6) are found to be

$$\begin{cases} 2\mu(-\Delta)^3(-\partial_x^2)\varphi = \sqrt{R}(-\Delta)(-\partial_x^2)\theta + \lambda\sqrt{R}(-\partial_x^2)\theta \\ 2\mu\lambda(-\Delta)\theta = \sqrt{R}(-\Delta)(-\partial_x^2)\varphi + \lambda\sqrt{R}(-\partial_x^2)\varphi \\ \psi = \mathbf{f} = 0. \end{cases} \tag{7}$$

By eliminating θ between $(7)_1$ and $(7)_2$ we obtain

$$4\mu^2\lambda(-\Delta)^4\varphi = R\big[\Delta^2(-\partial_x^2)\varphi + 2\lambda(-\Delta)(-\partial_x^2)\varphi + \lambda^2(-\partial_x^2)\varphi\big]. \tag{8}$$

From equation $(3)_1$ and boundary conditions (4) it follows that $\partial_z^4\varphi|_{z=\pm\frac{1}{2}} = 0$. From equation $(7)_2$ we have $\partial_z^2\theta|_{z=\pm\frac{1}{2}} = 0$, together with $(7)_1$ we obtain $\partial_z^6\varphi|_{z=\pm\frac{1}{2}} = 0$. With equation (8) we have $\partial_z^8\varphi|_{z=\pm\frac{1}{2}} = 0$. Differentiate equation (8) twice with respect to z, we conclude $\partial_z^{10}\varphi|_{z=\pm\frac{1}{2}} = 0$. By further differentiations of equation (8) we can obtain, successively, that all the even derivatives of φ vanish on the boundaries. Thus

$$\partial_z^{2m}\varphi|_{z=\pm\frac{1}{2}} = 0, \quad m = 1, 2, \cdots$$

So we consider φ of the form

$$\varphi = A_0 \sin n\pi \left(z + \frac{1}{2}\right) e^{i\alpha\kappa_1 x} \tag{9}$$

where A_0 is constant, $n = 1, 2, \cdots$ and $\kappa_1 \in Z - \{0\}$.

By putting (9) into (8), setting $r^2 = \alpha^2\kappa_1^2$, we have

$$\mu^2 = \frac{Rr^2[\lambda + n^2\pi^2 + r^2]^2}{4\lambda(n^2\pi^2 + r^2)^4}.$$

The basic flow is nonlinearly asymptotically stable whenever $\mu^2\langle 1$. This is equivalent to requiring that

$$R\langle \frac{4\lambda(n^2\pi^2 + r^2)^4}{r^2(\lambda + n^2\pi^2 + r^2)^2}.$$

Denoting

$$R(n^2\pi^2, r^2; \lambda) = \frac{4\lambda(n^2\pi^2 + r^2)^4}{r^2(\lambda + n^2\pi^2 + r^2)^2},$$

then the critical Rayleigh number for nonlinear stability is given by

$$R_E = \max_{\lambda\in\mathbb{R}^+} \min_{\substack{n\in\mathbb{N} \\ r^2\in(0,+\infty)}} R(n^2\pi^2, r^2; \lambda).$$

Since $\dfrac{dR(n^2\pi^2, r^2; \lambda)}{dn^2}\rangle 0$, we have $R_E = \max_{\lambda\in\mathbb{R}^+} \min_{r^2\in\mathbb{R}^+} R(\pi^2, r^2; \lambda)$.

Solving the following equations

$$\begin{cases} \dfrac{dR(\pi^2, r^2; \lambda)}{dr^2} = 0 \\ \dfrac{dR(\pi^2, r^2; \lambda)}{d\lambda} = 0 \end{cases}$$

we get only one root $r_c^2 = \dfrac{\pi^2}{2}, \lambda_c = \dfrac{3\pi^2}{2}$. It is easy to justify that $R_E = \dfrac{27}{4}\pi^4$. Moreover, for $R\langle R_E$ there is

$$1 - \mu = 1 - \frac{\sqrt{R}|r|(\lambda + n^2\pi^2 + r^2)}{2\sqrt{\lambda}(n^2\pi^2 + r^2)^2} \rangle 1 - \frac{\sqrt{R}}{\sqrt{R_E}}. \tag{10}$$

By using Wirtinger and Poincaré inequalities, see e.g. [1, 5], we have

$$D \geq \frac{2\min\{\alpha^2, \pi^2\}}{\max\{1, p_r\}} \mathcal{E}(t).$$

Together with (5) and (10), the assertion is proved. □

Theorem 3.2. *The energy functional $\mathcal{E}(t)$, defined in theorem 3.1, is essentially stronger than kinetic energy $E(t)$.*

Proof. By applying again Wirtinger and Poincaré inequalities we have

$$\|\boldsymbol{\delta}\varphi\|^2 = \|\nabla\partial_x\varphi\|^2 \leq \max\{\alpha^{-2}, \pi^{-2}\}\|\Delta\partial_x\varphi\|^2, \quad \|f_1\|^2 \leq \pi^{-2}\|\partial_z f_1\|^2$$

so that

$$E(t) \leq \max\{\alpha^{-2}, \pi^{-2}\}\|\Delta\partial_x\varphi\|^2 + \|\partial_x\psi\|^2 + \pi^{-2}\|\partial_z f_1\|^2 + \|f_2\|^2$$
$$+\lambda p_r\|\theta\|^2 \leq C\mathcal{E}(t)$$

where $C = 2\max\left\{\alpha^{-2}, \dfrac{3}{2}\pi^2\right\}$.

Let $f(z) = \cos n\pi(z + \frac{1}{2})$, $n = 1, 2, \cdots$. It is easy to see that $f \in L_M^2\left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$, $\|f\|^2 = \dfrac{1}{2}$ and $\|\partial_z f\|^2 = \dfrac{1}{2}n^2\pi^2$. From this example it follows that there exists no constant C such that $\mathcal{E}(t)\langle CE(t)$. According to definition 3.2 the theorem is proved. □

Remark. From theorem 3.1 one can see that the decay rate of a perturbation depends on wave numbers α. Moreover, if the 2-dimensional disturbances are x-independent i.e. $\partial_x \cdot = 0$, it is easy to prove the similar result.

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