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THE RUBIN'S Q-WAVELET PACKETS

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Using the q-harmonic analysis associated with the q-Rubin operator, we study three types of q-wavelet packets and their corresponding q-wavelet transforms. We give for these wavelet transforms the related Plancherel and inversion formulas as well as their q-scale discrete scaling functions.

1. Introduction

In seismology by reflection, Morlet knew that the pulse-modulated high frequency that we send in the ground are too long to distinguish the very close strata. He, then, introduced a new tool, called nowadays wavelets, to study the analysis of seismic data. While working in theoretical physics, Grossman found in the Morlet's approach some ideas close to his work on quantum coherent states. Then the two men have reactivated a collaboration between fundamental physics and theoretical signal processing, which led to the formalization of the continuous wavelet transforms, using the classical harmonic analysis. Since then, their results were generalized to many fields and many generalized Fourier analysis. The wavelet theory is motivated by the fact that certain algorithms that decompose a signal on the whole family of scales, can be utilized as an effective

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tool for multiscale analysis. In practical applications involving fast numerical algorithms, the continuous wavelet can be computed at discrete grid points. This theory involves breaking up a complicated function into many simple pieces at different scales and positions. It allows a greatly flexibility with more desirable features such as discrimination by wavelet packets, and readiness for better implementation [4, 16, 20]. In general, wavelet packet decomposition divides the frequency space into various parts and allows better frequency localization of the signal.

The theory of q-deformation, called in some literature ”quantum calculus” provides a natural discretization in which the classical derivative is replaced by a q-difference operator, the classical integral is replaced by a discreet sum and the set of real numbers is replaced by a geometric progression.
Interest in this theory is grown at an explosive rate by both physicists and mathematicians due to the large number of its applications domain and the role played by this discretization in algorithmic field. For instance, a lot of work has been carried out while developing some q-analogues of Fourier analysis using elements of quantum calculus (see [5, 9, 10, 17–19] and references therein). Furthermore, applications of these new q-harmonic analysis in sampling theory and wavelet theory have been shown (see [1, 6]). In [8], the authors introduced and studied the q-wavelets and the q-wavelet transforms associated with the q-Rubin operator, using elements of the q-harmonic analysis, associated with this operator, developed in [18] and in [19]. In particular they provided for these transforms a Plancherel and an inversion formulas.

In this paper, we present a general construction, allowing the development of three types of q-wavelet packets starting from the so-mentioned q-continuous wavelet analysis. For each type, we study its corresponding q-wavelet packet transform and we prove for this transform a Plancherel formula and an inversion theorem. We claim out that all our results are q-analogues of the classical picture given in [20]. The methods used here are direct and constructive, and have a good resemblance with the picture developed in [20].

This paper is organized as follows: in Section 2, we present some notations and notions needed in the sequel. Section 3 is devoted to present some elements of the Rubin's q-harmonic analysis. We define and study in Sections 4, 5, 6 and 7 three types of q-wavelet packets and the corresponding q-wavelet packets transforms as well as their q-scale discrete scaling functions.

2. Notations and preliminaries

We recall some usual notions and notations used in the q-theory (see [11] and [13]). We refer to the book by G. Gasper and M. Rahman [11] for the definitions, notations and properties of the q-shifted factorials and the q-hypergeometric functions. Throughout this paper, we assume $q \in]0,1[$ and we denote

$$\mathbb{R}_q = \{\pm q^n : n \in \mathbb{Z}\}, \quad \mathbb{R}_{q,+} = \{q^n : n \in \mathbb{Z}\} \quad \text{and} \quad \widetilde{\mathbb{R}}_q = \mathbb{R}_q \cup \{0\}.$$

For complex number a, the q-shifted factorials are defined by:

$$(a;q)_0 = 1; \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), n = 1,2,\ldots; \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

$$[x]_q = \frac{1-q^x}{1-q}, \quad x \in \mathbb{C} \quad \text{and} \quad n!_q = \frac{(q;q)_n}{(1-q)^n}, \quad n \in \mathbb{N}.$$

The Rubin's q-differential operator is defined in [18, 19] by

$$\partial_q(f)(z) = \begin{cases} \dfrac{f(q^{-1}z)+f(-q^{-1}z)-f(qz)+f(-qz)-2f(-z)}{2(1-q)z} & if \quad z \neq 0 \\ \lim\limits_{x\to 0} \partial_q(f)(x) & if \quad z = 0. \end{cases} \tag{1}$$

Note that if f is differentiable at z, then $\partial_q(f)(z)$ tend to $f'(z)$ as q tends to 1.

The q-Jackson integrals from 0 to a, from 0 to $+\infty$ and from $-\infty$ to $+\infty$ are defined by (see [12])

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{\infty} f(aq^n)q^n, \tag{2}$$

$$\begin{aligned} \int_0^\infty f(x)d_qx &= (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n, \\ \int_{-\infty}^\infty f(x)d_qx &= (1-q)\sum_{n=-\infty}^{\infty} q^n f(q^n) + (1-q)\sum_{n=-\infty}^{\infty} q^n f(-q^n), \end{aligned} \tag{3}$$

provided the sums converge absolutely.
The q-Jackson integral in a generic interval $[a,b]$ is given by

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx. \tag{4}$$

In the particular case $a = bq^n$, $n \in \mathbb{N}$, the relation (4) becomes

$$\int_a^b f(x)d_qx = (1-q)b\sum_{k=0}^{n-1} f(q^kb)q^k. \tag{5}$$

For $x \in \mathbb{R}_q$, we denote by δ_x the function defined by

$$\delta_x(t) = \begin{cases} 1 & if \quad t = x \\ 0 & if \quad t \neq x. \end{cases} \tag{6}$$

We remark that for a function f on $\mathbb{R}_q$, we have

$$\int_{-\infty}^{\infty} f(t)\delta_x(t)d_qt = (1-q)|x|f(x).$$

The q-trigonometric functions (see[18, 19]) are defined on $\mathbb{C}$ by

$$\cos(x;q^2) := \sum_{n=0}^{\infty}(-1)^n b_{2n}(x;q^2) \tag{7}$$

and

$$\sin(x;q^2) := \sum_{n=0}^{\infty}(-1)^n b_{2n+1}(x;q^2), \tag{8}$$

where

$$b_n(x;q^2) = \frac{q^{[\frac{n}{2}]([\frac{n}{2}]+1)}}{n!_q}x^n \tag{9}$$

and $[x]$ is the integer part of $x \in \mathbb{R}$.
These two functions induce a ∂_q-adapted q-analogue exponential function:

$$e(z;q^2) := \cos(-iz;q^2) + i\sin(-iz;q^2) = \sum_{n=0}^{\infty} b_n(z;q^2). \tag{10}$$

$e(z;q^2)$ is absolutely convergent for all z in the plane, and we have

$$\lim_{q\to 1^-} e(z;q^2) = e^z$$

point-wise and uniformly on compacta. Note that we have

Lemma 2.1. *(see [18])*

$$\textit{For all} \quad \lambda \in \mathbb{C}, \ \partial_q e(\lambda z;q^2) = \lambda e(\lambda z;q^2). \tag{11}$$

$$\textit{For all} \quad x \in \mathbb{R}_q, \quad |e(ix;q^2)| \leq \frac{2}{(q;q)_\infty}. \tag{12}$$

In the sequel, we will need the following sets and spaces.

• $\mathcal{C}_{q,0}(\mathbb{R}_q)$ the space of bounded functions on $\mathbb{R}_q$, continued at 0 and vanishing

at ∞.
• $\mathcal{S}_q(\mathbb{R}_q)$ the space of functions on $\mathbb{R}_q$ such that

$$\forall n,m \in \mathbb{N}, \qquad \sup_{x\in\mathbb{R};0\le k\le n} \mid (1+x^2)^m \partial_q^k f(x) \mid < +\infty.$$

• For $p \in [1,+\infty]$, and we denote by $L_q^p(\mathbb{R}_q)$, the set of all functions defined on $\mathbb{R}_q$ such that

$$\|f\|_{q,p} < \infty,$$

where

$$\|f\|_{q,p} = \begin{cases} \left(\int_{-\infty}^{\infty} \mid f(x) \mid^p d_q x\right)^{\frac{1}{p}} & if \quad p \ge 1 \\ \sup_{x\in\mathbb{R}_q} |f(x)| & if \quad p = +\infty \end{cases}$$

3. Elements of Rubin's q-harmonic analysis

In [18, 19], R. L. Rubin defined the q^2-analogue Fourier transform as

$$\mathcal{F}_q(f)(x) = K \int_{-\infty}^{\infty} f(t) e(-itx;q^2) d_q t, \ \ x \in \widetilde{\mathbb{R}}_q,$$

where

$$K = \frac{(q;q^2)_\infty}{2(q^2;q^2)_\infty (1-q)^{1/2}}.$$

Letting $q \uparrow 1$ subject to the condition

$$\frac{Log(1-q)}{Log(q)} \in 2\mathbb{Z}, \tag{13}$$

gives, at least formally, the classical Fourier transform. In the remainder of this paper, we assume that the condition (13) holds.

It was shown in [2, 18, 19] that the q^2-analogue Fourier transform $\mathcal{F}_q$ verifies the following properties:

Theorem 3.1.

1. *If $f \in L_q^1(\mathbb{R}_q)$, then*

$$\mathcal{F}_q(f) \in \mathcal{C}_{q,0}(\mathbb{R}_q) \qquad and \qquad \|\mathcal{F}_q(f)\|_{q,\infty} \le \frac{2K}{(q;q)_\infty} \|f\|_{q,1}.$$

2. *If f, $\partial_q f \in L_q^1(\mathbb{R}_q)$, then* $\qquad \mathcal{F}_q(\partial_q f)(\lambda) = i\lambda \mathcal{F}_q(f)(\lambda), \quad \lambda \in \widetilde{\mathbb{R}}_q.$

Theorem 3.2.
1) $\mathcal{F}_q$ is an isomorphism of $L_q^2(\mathbb{R}_q)$ into itself, satisfying for $f \in L_q^2(\mathbb{R}_q)$

$$\|\mathcal{F}_q(f)\|_{q,2} = \|f\|_{q,2}$$

and

$$\forall t \in \mathbb{R}_q, \quad f(t) = K\int_{-\infty}^{\infty} \mathcal{F}_q(f)(x)e(itx;q^2)d_qx.$$

2) For $f, g \in L_q^2(\mathbb{R}_q)$, we have

$$\int_{-\infty}^{\infty} \overline{f}(x)g(x)d_qx = \int_{-\infty}^{\infty} \overline{\mathcal{F}_q(f)}(\lambda)\mathcal{F}_q(g)(\lambda)d_q\lambda.$$

The q-translation operator $T_{q,x}, \ x \in \widetilde{\mathbb{R}}_q$ is defined (see [2, 19]) by

$$T_{q,x}(f)(y) = K\int_{-\infty}^{\infty} \mathcal{F}_q(f)(t)e(itx;q^2)e(ity;q^2)d_qt, \quad y \in \mathbb{R}_q, \tag{14}$$

$$T_{q,0}(f)(y) = f(y). \tag{15}$$

It verifies the following properties (see [2, 19]), for $f, g \in L_q^2(\mathbb{R}_q)$,

$$T_{q,x}(f)(y) = T_{q,y}(f)(x), \ \ x, y \in \mathbb{R}_q,$$

$$\int_{-\infty}^{\infty} T_{q,x}(f)(y)d_qy = \int_{-\infty}^{\infty} f(y)d_qy, \ \ x \in \mathbb{R}_q,$$

$$\int_{-\infty}^{\infty} T_{q,x}(f)(y)g(y)d_qy = \int_{-\infty}^{\infty} f(y)T_{q,-x}(g)(y)d_qy, \ \ x \in \mathbb{R}_q,$$

$$T_{q,x}e(ity;q^2) = e(itx;q^2)e(ity;q^2), \ \ x, y, t \in \mathbb{R}_q.$$

It was shown in [2, 19] that for $f \in L_q^2(\mathbb{R}_q)$, we have for all $x \in \widetilde{\mathbb{R}}_q$, $T_{q,x}f \in L_q^2(\mathbb{R}_q)$ and

$$\|T_{q,x}f\|_{q,2} \leq \frac{2}{(q;q)_\infty}\|f\|_{q,2}. \tag{16}$$

The q-convolution product is defined (see [2, 19]), by:

$$f *_q g(x) = K\int_{-\infty}^{\infty} T_{q,-y}f(x)g(y)d_qy. \tag{17}$$

Theorem 3.3. *For $f, g \in L_q^1(\mathbb{R}_q) \cap L_q^2(\mathbb{R}_q)$, we have*

$$\mathcal{F}_q(f *_q g) = \mathcal{F}_q(f)\mathcal{F}_q(g). \tag{18}$$

For $f \in L_q^2(\mathbb{R}_q)$, we have

$$\mathcal{F}_q(T_{q,x}f)(\lambda) = e(i\lambda x;q^2)\mathcal{F}_q(f)(\lambda), \quad x \in \widetilde{\mathbb{R}}_q, \ \ \lambda \in \mathbb{R}_q. \tag{19}$$

Moreover, the following result was shown in [19].

Proposition 3.4. *Let f and g be in $L^2_q(\mathbb{R}_q)$. Then*

1. $f *_q g \in L^2_q(\mathbb{R}_q)$ *iff* $\mathcal{F}_q(f)\mathcal{F}_q(g) \in L^2_q(\mathbb{R}_q)$,

2. $$\int_{-\infty}^{\infty} | f *_q g(x) |^2 \, d_q x = \int_{-\infty}^{\infty} | \mathcal{F}_q(f)(x) |^2 | \mathcal{F}_q(g)(x) |^2 \, d_q x, \tag{20}$$
   where both sides are finite or infinite.

The dilatation operators are defined by

$$H_a(f)(x) = \frac{1}{a} f\left(\frac{x}{a}\right), \quad a \in \mathbb{R}_{q,+}. \tag{21}$$

They satisfy the following properties.

Proposition 3.5.

1. $H_1 = id; \quad H_a o H_b = H_{ab}; \quad H_a^{-1} = H_{a^{-1}}, \quad a, b \in \mathbb{R}_{q,+}.$
2. *For all $a \in \mathbb{R}_{q,+}$, the operator H_a is an automorphism of $L^1_q(\mathbb{R}_q)$ (resp. $L^2_q(\mathbb{R}_q)$) onto itself and for all $f \in L^1_q(\mathbb{R}_q)$ (resp. $f \in L^2_q(\mathbb{R}_q)$), we have*
   $$\|H_a f\|_1 = \|f\|_1 \quad (resp. \quad \|H_a f\|_2 = \frac{1}{\sqrt{a}} \|f\|_2)$$
   and
   $$\mathcal{F}_q[H_a(f)](x) = \mathcal{F}_q(f)(ax). \tag{22}$$

Proof. (1) follows from the definition of the dilatation operator.
(2) The change of variables $u = \frac{t}{a}$ gives the result. □

4. q-Wavelet Packets

We recall that a Rubin's q-wavelet is a square q-integrable function g on $\mathbb{R}_q$ satisfying the following admissibility condition:

$$0 < C_g = \int_0^{\infty} | \mathcal{F}_q(g)(a) |^2 \frac{d_q a}{a} = \int_0^{\infty} | \mathcal{F}_q(g)(-a) |^2 \frac{d_q a}{a} < \infty. \tag{23}$$

We consider a Rubin's q-wavelet g and a strictly decreasing scale sequence $(r_j)_{j\in\mathbb{Z}}$ of $\mathbb{R}_{q,+}$ satisfying $\lim_{j\to-\infty} r_j = +\infty$, $\lim_{j\to+\infty} r_j = 0$. We state the following introductory result.

Proposition 4.1. *For all $j \in \mathbb{Z}$, we have :*

1. *the function* $\lambda \longmapsto \left(\frac{1}{C_g}\int_{r_{j+1}}^{r_j}|\mathcal{F}_q(H_a(g))(\lambda)|^2\frac{d_qa}{a}\right)^{\frac{1}{2}}$ *belongs to* $L^2_q(\mathbb{R}_q)$,

2. *there exists a function* $g_j^P \in L^2_q(\mathbb{R}_q)$ *such that for all* $\lambda \in \mathbb{R}_q$,

$$\mathcal{F}_q(g_j^P)(\lambda) = \left(\frac{1}{C_g}\int_{r_{j+1}}^{r_j}|\mathcal{F}_q(H_a(g))(\lambda)|^2\frac{d_qa}{a}\right)^{\frac{1}{2}}.$$

Proof. Fix $j \in \mathbb{Z}$.
(1) On the one hand, r_j and r_{j+1} are two elements of $\mathbb{R}_{q,+}$ satisfying $r_{j+1} < r_j$, then there exists a positive integer n such that $r_{j+1} = q^n r_j$. So, using the relation (5) and Proposition 3.5, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty}\left(\frac{1}{C_g}\int_{r_{j+1}}^{r_j}|\mathcal{F}_q(H_a(g))(\lambda)|^2\frac{d_qa}{a}\right)d_q\lambda &= \frac{1-q}{C_g}\int_{-\infty}^{\infty}\sum_{k=0}^{n-1}|\mathcal{F}_q(g)(\lambda q^k r_j)|^2 d_q\lambda \\ &= \frac{1-q}{C_g}\sum_{k=0}^{n-1}\int_{-\infty}^{\infty}|\mathcal{F}_q(g)(\lambda q^k r_j)|^2 d_q\lambda.\end{aligned}$$

On the other hand, the change of variable $u = \lambda q^k r_j$, ($0 \le k \le n-1$), together with Theorem 3.2 lead to

$$\begin{aligned}\int_{-\infty}^{\infty}\left(\frac{1}{C_g}\int_{r_{j+1}}^{r_j}|\mathcal{F}_q(H_a(g))(\lambda)|^2\frac{d_qa}{a}\right)d_q\lambda &= \frac{1-q}{C_g}\sum_{k=0}^{n-1}\int_{-\infty}^{\infty}\frac{|\mathcal{F}_q(g)(u)|^2}{r_jq^k}d_qu \\ &= \frac{q}{C_g}\left(\frac{1}{r_{j+1}}-\frac{1}{r_j}\right)\|g\|^2_{q,2}.\end{aligned}$$

(2) The result follows from Theorem 3.2. □

Definition 4.2. i) The sequence $(g_j^P)_{j\in\mathbb{Z}}$ is called Rubin's q-wavelet packet.
ii) The function $g_j^P,\ \ j \in \mathbb{Z}$, is called Rubin's q-wavelet packet's member of step j.

We have the following immediate properties.

Proposition 4.3. *For all $\lambda \in \mathbb{R}_q$, we have*

$$0 \le \mathcal{F}_q(g_k^P)(\lambda) \le 1, \quad k \in \mathbb{Z} \qquad \textit{and} \qquad \sum_{j=-\infty}^{+\infty}[\mathcal{F}_q(g_j^P)(\lambda)]^2 = 1.$$

Example

Using the Euler q-analogue of the exponential function (see [11], and [13])

$$exp_{q^2}(x) = \frac{1}{((1-q^2)x;q^2)_\infty},$$

consider the function

$$G(x) = c_q exp_{q^2}\left(-\frac{qx^2}{(1+q)^2}\right),$$

where

$$c_q = \frac{\left(-\frac{q(1-q)}{1+q}, -\frac{q(1+q)}{1-q}; q^2\right)_\infty}{(1-q)^{\frac{1}{2}}\left(-\frac{q^2(1-q)}{1+q}, -\frac{1+q}{1-q}; q^2\right)_\infty}$$

It was shown in [8] that the function

$$g(x) = \partial_q^2 G(x) = \frac{c_q}{q(1+q)}\left(1-\frac{x^2}{1+q}\right) exp_{q^2}\left(-\frac{x^2}{q(1+q)^2}\right)$$

is a Rubin's q-wavelet in $\mathcal{S}_q(\mathbb{R}_q)$ satisfying

$$\mathcal{F}_q(g)(x) = -x^2\mathcal{F}_q(G)(x) = -x^2 exp_{q^2}(-x^2), \qquad x \in \mathbb{R}_q.$$

Now, for $j \in \mathbb{Z}$, put $r_j = q^j$. It is clear that $(r_j)_{j\in\mathbb{Z}}$ is a strictly decreasing sequence of $\mathbb{R}_{q,+}$, $\lim_{j\to-\infty} r_j = +\infty$ and $\lim_{j\to+\infty} r_j = 0$. The Rubin's q-wavelet packet $(g_j^P)_{j\in\mathbb{Z}}$ is given by :

$$g_j^P = -\sqrt{\frac{1-q}{C_g}} H_{q^j} g, \quad j \in \mathbb{Z}.$$

Indeed, for all $x \in \mathbb{R}_q$, we have

$$\begin{aligned}
\mathcal{F}_q(g_j^P)(x) &= \sqrt{\frac{1-q}{C_g}} \mathcal{F}_q(-g)(q^j x) \\
&= \left(\frac{1-q}{C_g}\left|\mathcal{F}_q(g)(q^j x)\right|^2\right)^{\frac{1}{2}} \\
&= \left(\frac{1}{C_g}\int_{r_j}^{r_{j+1}} \left|\mathcal{F}_q(g)(ax)\right|^2 \frac{d_q a}{a}\right)^{\frac{1}{2}}
\end{aligned}$$

▼

Let $(g_j^P)_{j\in\mathbb{Z}}$ be a Rubin's q-wavelet packet. We introduce for all $j\in\mathbb{Z}$ and $x\in\mathbb{R}_q$, the function $g_{j,x}^P$ as

$$g_{j,x}^P(y) = T_{q,y}(g_j^P)(x), \ \ y\in\widetilde{\mathbb{R}}_q. \tag{24}$$

Some properties of these functions are summarized in the following result that its proof follows easily from the properties of the q- translation operator and the definition of the Rubin's q-wavelet packets.

Proposition 4.4. *For all $j\in\mathbb{Z}$ and $x\in\mathbb{R}_q$, the function $g_{j,x}^P$ belongs to $L_q^2(\mathbb{R}_q)$ and we have for all $\lambda\in\widetilde{\mathbb{R}}_q$,*

- $\mathcal{F}_q(g_{j,x}^P)(\lambda) = e(i\lambda x;q^2)\mathcal{F}_q(g_j^P)(\lambda)$.
- $\|g_{j,x}^P\|_{q,2} \leq \dfrac{2\|g_j^P\|_{q,2}}{(q;q)_\infty}$.

Definition 4.5. Let $(g_j^P)_{j\in\mathbb{Z}}$ be a Rubin's q-wavelet packet. We define the Rubin's q-wavelet packet transform $\Psi_{q,g}^P$ by

$$\Psi_{q,g}^P(f)(j,y) = K\int_{-\infty}^{\infty} f(x)\overline{g_{j,y}^P}(x)d_qx, \ j\in\mathbb{Z}, \ y\in\widetilde{\mathbb{R}}_q \quad \text{and} \quad f\in L_q^2(\mathbb{R}_q). \tag{25}$$

Remark 4.6. The equality (25) is equivalent to

$$\Psi_{q,g}^P(f)(j,y) = \check{f}*_q\overline{g_j^P}(y) = \mathcal{F}_q(\mathcal{F}_q(\check{f}*_q\overline{g_j^P}))(-y) = \mathcal{F}_q[\mathcal{F}_q(\check{f}).\mathcal{F}_q(\overline{g_j^P})](-y), \tag{26}$$

where $\check{f}(x) = f(-x)$.

The following proposition provides some useful properties of $\Psi_{q,g}^P$.

Proposition 4.7. *Let $(g_j^P)_{j\in\mathbb{Z}}$ be a Rubin's q-wavelet packet and $f\in L_q^2(\mathbb{R}_q)$. Then,*

1. *for all $j\in\mathbb{Z}$, $b\in\widetilde{\mathbb{R}}_q$, we have*
$$|\Psi_{q,g}^P(f)(j,b)| \leq \frac{2K}{(q;q)_\infty}\|f\|_{q,2}\left\|g_j^P\right\|_{q,2};$$
2. *for all $j\in\mathbb{Z}$, the mapping $b\mapsto\Psi_{q,g}^P(f)(j,b)$ is continuous on $\widetilde{\mathbb{R}}_q$ and we have $\lim_{b\to\infty}\Psi_{q,g}^P(f)(j,b) = 0$.*

Proof. (1) From the relation (25), Proposition 4.4 and the Cauchy-Schwarz inequality, we have for $j \in \mathbb{Z}$ and $b \in \mathbb{R}_q$

$$|\Psi^P_{q,g}(f)(j,b)| = K\left|\int_{-\infty}^{\infty} f(x)\overline{g^P_{j,b}}(x)d_qx\right| \leq K||f||_{q,2}||g^P_{j,b}||_{q,2}$$
$$\leq \frac{2K}{(q;q)_\infty}||f||_{q,2}||g^P_j||_{q,2}.$$

(2) Let $j \in \mathbb{Z}$ and $f \in L^2_q(\mathbb{R}_q)$. From Theorem 3.2, we have $\mathcal{F}_q(\check{f})$ and $\mathcal{F}_q(\overline{g^P_j})$ are in $L^2_q(\mathbb{R}_q)$ and the product $\mathcal{F}_q(\check{f})\mathcal{F}_q(\overline{g^P_j})$ is in $L^1_q(\mathbb{R}_q)$. So, the relation (26) together with Theorem 3.1 achieve the proof. □

The following result shows Plancherel and Parseval formulas for the Rubin's q-wavelet packet transform $\Psi^P_{q,g}$.

Theorem 4.8. *Let $(g^P_j)_{j\in\mathbb{Z}}$ be a Rubin's q-wavelet packet.*
(1) **Plancherel formula for $\Psi^P_{q,g}$**
For $f \in L^2_q(\mathbb{R}_q)$, we have

$$\sum_{j=-\infty}^{+\infty}\int_{-\infty}^{\infty} | \Psi^P_{q,g}(f)(j,b) |^2 \, d_qb = \|f\|^2_{q,2}. \tag{27}$$

(2) **Parseval formula for $\Psi^P_{q,g}$**
For $f_1, f_2 \in L^2_q(\mathbb{R}_q)$, we have

$$\int_{-\infty}^{\infty} f_1(x)\overline{f_2(x)}d_qx = \sum_{j=-\infty}^{+\infty}\int_{-\infty}^{\infty} \Psi^P_{q,g}(f_1)(j,b)\overline{\Psi^P_{q,g}(f_2)(j,b)}d_qb. \tag{28}$$

Proof. (1) From the relations (20) and (26), we obtain

$$\int_{-\infty}^{\infty}|\Psi^P_{q,g}(f)(j,b)|^2d_qb = \int_{-\infty}^{\infty}|\mathcal{F}_q(\check{f})(a)|^2[\mathcal{F}_q(g^P_j(a)]^2d_qa.$$

So, the use of the Fubini's theorem gives

$$\sum_{j=-\infty}^{+\infty}\int_{-\infty}^{\infty}|\Psi^P_{q,g}(f)(j,b)|^2d_qb = \int_{-\infty}^{\infty}|\mathcal{F}_q(\check{f})(a)|^2\sum_{j=-\infty}^{+\infty}[\mathcal{F}_q(g^P_j(a))]^2d_qa.$$

Since

$$\sum_{j=-\infty}^{+\infty}[\mathcal{F}_q(g^P_j)(\lambda)]^2 = 1,$$

then,

$$\sum_{j=-\infty}^{+\infty}\int_{-\infty}^{\infty}|\Psi^P_{q,g}(f)(j,b)|^2d_qb = \int_{-\infty}^{\infty}|\mathcal{F}_q(\check{f})(a)|^2d_qa.$$

Thus, (27) follows from Theorem 3.2.
(2) The result is a direct consequence of assertion (1). □

Theorem 4.9. *Let $(g_j^P)_{j\in\mathbb{Z}}$ be a Rubin's q-wavelet packet. For $f \in L_q^2(\mathbb{R}_q)$, one has the following reconstruction formula :*

$$f(x) = K\sum_{j=-\infty}^{+\infty}\int_{-\infty}^{\infty}\Psi_{q,g}^P(f)(j,b)g_{j,b}^P(x)d_qb, \quad x\in\mathbb{R}_q.$$

Proof. For $x \in \mathbb{R}_q$, we have $h = \delta_x$ belongs to $L_q^2(\mathbb{R}_q)$. Then, according to the relation (28), the definition of $\Psi_{q,g}^P$ and the definition of the q-Jackson's integral, we have

$$\begin{aligned}(1-q)|x|f(x) &= \sum_{j=-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi_{q,g}^P(f)(j,b)\overline{\Psi_{q,g}^P(h)}(j,b)d_qb \\ &= K\sum_{j=-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi_{q,g}^P(f)(j,b)\left(\int_{-\infty}^{\infty}\overline{h}(t)g_{j,b}^P(t)d_qt\right)d_qb \\ &= (1-q)|x|K\sum_{j=-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi_{q,g}^P(f)(j,b)g_{j,b}^P(x)d_qb,\end{aligned}$$

which is equivalent to

$$f(x) = K\sum_{j=-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi_{q,g}^P(f)(j,b)g_{j,b}^P(x)d_qb.$$

□

5. Rubin's q-Scale discrete scaling function

In this section, we consider a Rubin's q-wavelet packet $(g_j^P)_{j\in\mathbb{Z}}$.

Proposition 5.1. *1. For all $m\in\mathbb{Z}$ and $x\in\mathbb{R}_q$, we have*

$$\sum_{j=-\infty}^{m-1}[\mathcal{F}_q(g_j^P)(x)]^2 = \frac{1}{C_g}\int_{r_m}^{\infty}|\mathcal{F}_q(H_a(g))(x)|^2\frac{d_qa}{a}. \quad (29)$$

2. For all $m\in\mathbb{Z}$, the function $x\mapsto\left(\sum_{j=-\infty}^{m-1}[\mathcal{F}_q(g_j^P)(x)]^2\right)^{\frac{1}{2}}$ belongs to $L_q^2(\mathbb{R}_q)$.

3. For all $m\in\mathbb{Z}$ there exists a function G_m^P in $L_q^2(\mathbb{R}_q)$ such that for all $x\in\mathbb{R}_q$,

$$\mathcal{F}_q(G_m^P)(x) = \left(\sum_{j=-\infty}^{m-1}[\mathcal{F}_q(g_j^P)(x)]^2\right)^{\frac{1}{2}}. \quad (30)$$

Proof. (1) follows from the definition of g_j^P.
(2) From the Fubini's theorem, the relation (29) and Proposition 3.5, we have

$$\begin{aligned}\int_{-\infty}^{\infty}\sum_{j=-\infty}^{m-1}[\mathcal{F}_q(g_j^P)(x)]^2 d_qx &= \frac{1}{C_g}\int_{-\infty}^{\infty}\int_{r_m}^{\infty}|\mathcal{F}_q(H_a(g))(x)|^2\frac{d_qa}{a}d_qx\\ &= \frac{1}{C_g}\int_{r_m}^{\infty}\left(\int_{-\infty}^{\infty}|\mathcal{F}_q(g)(ax)|^2 d_qx\right)\frac{d_qa}{a}.\end{aligned}$$

By the change of variables $u=ax$ and Theorem 3.2, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty}\left(\frac{1}{C_g}\int_{r_m}^{\infty}|\mathcal{F}_q(H_a(g))(x)|^2\frac{d_qa}{a}\right)d_qx &= \frac{1}{C_g}\int_{r_m}^{\infty}\left(\int_{-\infty}^{\infty}|\mathcal{F}_q(g)(x)|^2 d_qx\right)\frac{d_qa}{a^2}\\ &= \frac{\|g\|_{2,q}}{C_g}\int_{r_m}^{\infty}\frac{d_qa}{a^2}\\ &= \frac{q\,\|g\|_{q,2}}{C_g r_m}.\end{aligned}$$

This completes the proof of (2).
(3) We deduce the result from the previous assertion and Theorem 3.2. □

Definition 5.2. The sequence $(G_m^P)_{m\in\mathbb{Z}}$ is called Rubin's q-scale discrete scaling function.

The sequence $(G_m^P)_{m\in\mathbb{Z}}$ verifies the following trivial and easily proved properties.

Proposition 5.3.
(i) *For all $m\in\mathbb{Z}$ and $\lambda\in\mathbb{R}_q$, we have*

$$0\le\mathcal{F}_q(G_m^P)(\lambda)\le 1. \tag{31}$$

(ii) *For all $\lambda\in\mathbb{R}_q$, we have*

$$\lim_{m\to+\infty}\mathcal{F}_q(G_m^P)(\lambda)=1. \tag{32}$$

Proposition 5.4. *For $m\in\mathbb{Z}$ and $x\in\mathbb{R}_q$, the following relations*
(i)

$$\left[\mathcal{F}_q(G_m^P)(x)\right]^2+\sum_{j=m}^{\infty}[\mathcal{F}_q(g_j^P)(x)]^2=1, \tag{33}$$

(ii)

$$\left[\mathcal{F}_q(g_m^P)(x)\right]^2=\left[\mathcal{F}_q(G_{m+1}^P)(x)\right]^2-\left[\mathcal{F}_q(G_m^P)(x)\right]^2, \tag{34}$$

(iii)

$$\sum_{m=-\infty}^{\infty}\left(\left[\mathcal{F}_q(G^P_{m+1})(x)\right]^2-\left[\mathcal{F}_q(G^P_m)(x)\right]^2\right)=1 \tag{35}$$

hold.

Proof.

(i) Follows immediately from (30) and Proposition 4.3.

(ii) We deduce the result from the relation (30).

(iii) The relation is a consequence of (34) and Proposition 4.3. □

Now, let $(G^P_m)_{m\in\mathbb{Z}}$ be a Rubin's q-scale discrete scaling function and consider for all $m\in\mathbb{Z},\ x\in\mathbb{R}_q$, the function $G^P_{m,x}$ given by

$$G^P_{m,x}(y)=T_{q,y}(G^P_m)(x),\ \ \forall y\in\mathbb{R}_q. \tag{36}$$

From the properties of the q-translation, one can prove easily the following result giving some properties of the function $G^P_{m,x}$.

Proposition 5.5. *For all $m\in\mathbb{Z}$ and $x\in\mathbb{R}_q$, the function $G^P_{m,x}$ belongs to $L^2_q(\mathbb{R}_q)$ and we have*

- $\mathcal{F}_q(G^P_{m,x})(\lambda)=e(i\lambda x;q^2)\mathcal{F}_q(G^P_m)(\lambda),\ \ \lambda\in\mathbb{R}_q,$
- $\|G^P_{m,x}\|_{q,2}\leq\dfrac{2\|G^P_m\|_{q,2}}{(q;q)_\infty}.$

Definition 5.6. Let $(G^P_m)_{m\in\mathbb{Z}}$ be a Rubin's q-scale discrete scaling function. We define the Rubin's q-scale discrete scaling transform $\Theta^P_{q,G}$ on $L^2_q(\mathbb{R}_q)$, by

$$\Theta^P_{q,G}(f)(m,x)=K\int_{-\infty}^{\infty}f(b)\overline{G^P_{m,x}(b)}d_qb,\quad m\in\mathbb{Z},\text{ and }\ x\in\mathbb{R}_q. \tag{37}$$

Remark 5.7. The relation (37) is equivalent to

$$\Theta^P_{q,G}(f)(m,x)=\check{f}*_q\overline{G^P_m}(x). \tag{38}$$

In the two following results, we will provide a Plancherel and a Parseval formulas for $\Theta^P_{q,G}$.

Theorem 5.8. *Let $(G_m^P)_{m\in\mathbb{Z}}$ be a Rubin's q-scale discrete scaling function.*
(1) **Plancherel formula for $\Theta_{q,G}^P$**
For $f \in L_q^2(\mathbb{R}_q)$, we have

$$\|f\|_{q,2}^2 = \lim_{m\to+\infty} \int_{-\infty}^{\infty} |\Theta_{q,G}^P(f)(m,b)|^2 d_q b. \tag{39}$$

(2) **Parseval formula for $\Theta_{q,G}^P$**
For $f_1, f_2 \in L_q^2(\mathbb{R}_q)$, we have

$$\int_{-\infty}^{\infty} f_1(x)\overline{f}_2(x)d_q x = \lim_{m\to+\infty} \int_{-\infty}^{\infty} \Theta_{q,G}^P(f_1)(m,b)\overline{\Theta_{q,G}^P(f_2)(m,b)} d_q b. \tag{40}$$

Proof. (1) Due to the relations (38) and (20), we have for all $m \in \mathbb{Z}$,

$$\int_{-\infty}^{\infty} |\Theta_{q,G}^P(f)(m,b)|^2 d_q b = \int_{-\infty}^{\infty} |\mathcal{F}_q(\check{f})(x)|^2 [\mathcal{F}_q(G_m^P)(x)]^2 d_q x. \tag{41}$$

The relations (31) and (32), and the Lebesgue's theorem yield to

$$\lim_{m\to+\infty} \int_{-\infty}^{\infty} |\Theta_{q,G}^P(f)(m,b)|^2 d_q b = \left\|\mathcal{F}_q(\check{f})\right\|_{q,2}^2.$$

Finally, Theorem 3.2 achieves the proof of (1).
(2) The result follows from (39). □

Using the q-scale discrete scaling function $(G_m^P)_{m\in\mathbb{Z}}$ and the Rubin's q-wavelet packet transform $\Psi_{q,g}^P$, one can obtain another Plancherel formula for $\Theta_{q,G}^P$. This is the aim of the following result.

Theorem 5.9. *(1)* **Plancherel formula for $\Theta_{q,G}^P$ using $\Psi_{q,g}^P$**
For all $f \in L_q^2(\mathbb{R}_q)$, we have for all $m \in \mathbb{Z}$,

$$\|f\|_{q,2}^2 = \int_{-\infty}^{\infty} |\Theta_{q,G}^P(f)(m,b)|^2 d_q b + \sum_{j=m}^{\infty} \int_{-\infty}^{\infty} |\Psi_{q,g}^P(f)(j,b)|^2 d_q b. \tag{42}$$

(2) **Parseval formula for $\Theta_{q,G}^P$ using $\Psi_{q,g}^P$**
For $f_1, f_2 \in L_q^2(\mathbb{R}_q)$, we have for all $m \in \mathbb{Z}$,

$$\int_{-\infty}^{\infty} f_1(x)\overline{f}_2(x)d_q x = \int_{-\infty}^{\infty} \Theta_{q,G}^P(f_1)(m,b)\overline{\Theta_{q,G}^P(f_2)(m,b)} d_q b \quad + \\ \sum_{j=m}^{\infty} \int_{-\infty}^{\infty} \Psi_{q,g}^P(f_1)(j,b)\overline{\Psi_{q,g}^P(f_2)(j,b)} d_q b.$$

Proof. (1) On the one hand, from the relations (41) and (30), we have for all $m \in \mathbb{Z}$,

$$\int_{-\infty}^{\infty} |\Theta_{q,G}^{P}(f)(m,b)|^2 d_q b = \int_{-\infty}^{\infty} |\mathcal{F}_q(\check{f})(x)|^2 \left(\sum_{j=-\infty}^{m-1} \left[\mathcal{F}_q(g_j^P)(x)\right]^2 \right) d_q x.$$

On the other hand, using the relations (20) and (26), and the Fubini's theorem, we obtain

$$\sum_{j=m}^{\infty} \int_{-\infty}^{\infty} |\Psi_{q,g}^{P}(f)(j,b)|^2 d_q b = \int_{-\infty}^{\infty} |\mathcal{F}_q(\check{f})(x)|^2 \left(\sum_{j=m}^{\infty} \left[\mathcal{F}_q(g_j^P)(x)\right]^2 \right) d_q x.$$

Hence,

$$\int_{-\infty}^{\infty} |\Theta_{q,G}^{P}(f)(m,b)|^2 d_q b \quad + \quad \sum_{j=m}^{\infty} \int_{-\infty}^{\infty} |\Psi_{q,g}^{P}(f)(j,b)|^2 d_q b =$$
$$\int_{-\infty}^{\infty} |\mathcal{F}_q(\check{f})(x)|^2 \left(\sum_{j=-\infty}^{\infty} [\mathcal{F}_q(g_j^P)(x)]^2 \right) d_q x.$$

The result follows then from Proposition 4.3 and Theorem 3.2.
(2) The assertion (2) follows from (1). □

Theorem 5.10. *For $f \in L_q^2(\mathbb{R}_q)$, we have the following reconstruction formulas.*
(1) For all $x \in \mathbb{R}_q$,

$$f(x) = K \lim_{m \to +\infty} \int_{-\infty}^{\infty} \Theta_{q,G}^{P}(f)(m,b) G_{m,b}^{P}(x) d_q b. \tag{43}$$

(2) For all $x \in \mathbb{R}_q$ and all $m \in \mathbb{Z}$,

$$f(x) = K \int_{-\infty}^{\infty} \Theta_{q,G}^{P}(f)(m,b) G_{m,b}^{P}(x) d_q b + K \sum_{j=m}^{\infty} \int_{-\infty}^{\infty} \Psi_{q,g}^{P}(f)(j,b) g_{j,b}^{P}(x) d_q b. \tag{44}$$

Proof. (1) Let $f \in L_q^2(\mathbb{R}_q)$, fix $x \in \mathbb{R}_q$ and put $h = \delta_x$. By using the relation (40), we get

$$\begin{aligned}(1-q)|x| f(x) &= \lim_{m \to +\infty} \int_{-\infty}^{\infty} \Theta_{q,G}^{P}(f)(m,b) \overline{\Theta_{q,G}^{P}(h)(m,b)} d_q b \\ &= \lim_{m \to +\infty} K \int_{-\infty}^{\infty} \Theta_{q,G}^{P}(f)(m,b) \left(\int_{-\infty}^{\infty} \overline{h}(t) G_{m,b}^{P}(t) d_q t \right) d_q b \\ &= \lim_{m \to +\infty} K(1-q)|x| \int_{-\infty}^{\infty} \Theta_{q,G}^{P}(f)(m,b) G_{m,b}^{P}(x) d_q b.\end{aligned}$$

Thus,

$$f(x) = K \lim_{m\to+\infty} \int_{-\infty}^{\infty} \Theta_{q,G}^{P}(f)(m,b) G_{m,b}^{P}(x) d_q b.$$

(2) The technique of the proof is similar to (1). □

Corollary 1. For $f \in L_q^2(\mathbb{R}_q)$, one has for all $m \in \mathbb{Z}$,

$$\int_{-\infty}^{\infty} \Psi_{q,g}^{P}(f)(m,b) g_{m,b}^{P}(x) d_q b = \int_{-\infty}^{\infty} \Theta_{q,G}^{P}(f)(m+1,b) G_{m+1,b}^{P}(x) d_q b \quad - \int_{-\infty}^{\infty} \Theta_{q,G}^{P}(f)(m,b) G_{m,b}^{P}(x) d_q b.$$

Proof. For $x \in \mathbb{R}_q$, and $m \in \mathbb{Z}$, we have owing to the relation (44),

$$\int_{-\infty}^{\infty} \Theta_{q,G}^{P}(f)(m+1,b) G_{m+1,b}^{P}(x) d_q b = \frac{1}{K} f(x) - \sum_{j=m+1}^{\infty} \int_{-\infty}^{\infty} \Psi_{q,g}^{P}(f)(j,b) g_{j,b}^{P}(x) d_q b$$

and

$$\int_{-\infty}^{\infty} \Theta_{q,G}^{P}(f)(m,b) G_{m,b}^{P}(x) d_q b = \frac{1}{K} f(x) - \sum_{j=m}^{\infty} \int_{-\infty}^{\infty} \Psi_{q,g}^{P}(f)(j,b) g_{j,b}^{P}(x) d_q b.$$

Then, the difference of the two previous equations finishes the proof. □

6. Modified Rubin's q-wavelet packets

Let $(G_m^P)_{m\in\mathbb{Z}}$ be a q-scale discret scaling function. For $j \in \mathbb{Z}$, we define the functions g_j^M and $\tilde{g}_j^M$ by

$$g_j^M = G_{j+1}^P - G_j^P,$$

and

$$\tilde{g}_j^M = G_{j+1}^P + G_j^P.$$

Let us give some properties of these functions, which follow immediately from Proposition 5.3, Proposition 5.4 and Theorem 3.2.

Proposition 6.1.

(i) The functions g_j^M and $\tilde{g}_j^M$ belong to $L_q^2(\mathbb{R}_q)$.

(ii) The functions $\mathcal{F}_q(g_j^M)$ and $\mathcal{F}_q(\tilde{g}_j^M)$ are in $L_q^2(\mathbb{R}_q) \cap L_q^\infty(\mathbb{R}_q)$ and for all $\lambda \in \mathbb{R}_q$, we have

$$|\mathcal{F}_q(g_j^M)(\lambda)| \le 2, \text{ and } |\mathcal{F}_q(\tilde{g}_j^M)(\lambda)| \le 2 \tag{45}$$

and

$$\sum_{j=-\infty}^{\infty} \mathcal{F}_q(g_j^M) \mathcal{F}_q(\tilde{g}_j^M) = 1. \tag{46}$$

Definition 6.2. The sequences $(g_j^M)_{j\in\mathbb{Z}}$ and $(\tilde{g}_j^M)_{j\in\mathbb{Z}}$ are called respectively modified Rubin's q-wavelet packet and the corresponding dual modified Rubin's q-wavelet packet.

The following proposition gives a relationship between the q-scale discrete scaling function $(G_m^P)_{m\in\mathbb{Z}}$, and the modified Rubin's q-wavelet packet and its dual.

Proposition 6.3. *For all $x \in \mathbb{R}_q$ and all $m \in \mathbb{Z}$, we have*

$$\mathcal{F}_q(G_m^P)(x) = \left(\sum_{j=-\infty}^{m-1} \mathcal{F}_q(g_j^M)(x)\mathcal{F}_q(\tilde{g}_j^M)(x) \right)^{\frac{1}{2}}.$$

Proof. For $j \in \mathbb{Z}$, we have

$$\begin{aligned}\mathcal{F}_q(g_j^M)\mathcal{F}_q(\tilde{g}_j^M) &= \left(\mathcal{F}_q(G_{j+1}^P) - \mathcal{F}_q(G_j^P)\right)\left(\mathcal{F}_q(G_{j+1}^P) + \mathcal{F}_q(G_j^P)\right) \\ &= \left[\mathcal{F}_q(G_{j+1}^P)\right]^2 - \left[\mathcal{F}_q(G_j^P)\right]^2 = \left[\mathcal{F}_q(g_j^P)\right]^2.\end{aligned} \tag{47}$$

Then, the relation (30) achieves the proof. □

Let $(g_j^M)_{m\in\mathbb{Z}}$ be a modified Rubin's q-wavelet packet and $(\tilde{g}_j^M)_{m\in\mathbb{Z}}$ be its dual. For all $j \in \mathbb{Z}$, $x \in \mathbb{R}_q$ and $y \in \widetilde{\mathbb{R}}_q$, we define the functions $g_{j,x}^M$ and $\tilde{g}_{j,x}^M$ by

$$g_{j,x}^M(y) = T_{q,y}(g_j^M)(x) \quad \text{and} \quad \tilde{g}_{j,x}^M(y) = T_{q,y}(\tilde{g}_j^M)(x).$$

From the properties of the q-translation, one can prove easily the following proposition, which gives some properties of the functions $g_{j,x}^M$ and $\tilde{g}_{j,x}^M$.

Proposition 6.4. *For all $j \in \mathbb{Z}$ and $x \in \mathbb{R}_q$, the functions $g_{j,x}^M$ and $\tilde{g}_{j,x}^M$ belong to $L_q^2(\mathbb{R}_q)$ and we have*

- $\mathcal{F}_q(g_{j,x}^M)(y) = e(iyx;q^2)\mathcal{F}_q(g_j^M)(y), \ y \in \widetilde{\mathbb{R}}_q,$
- $\mathcal{F}_q(\tilde{g}_{j,x}^M)(y) = e(iyx;q^2)\mathcal{F}_q(\tilde{g}_j^M)(y), \ y \in \widetilde{\mathbb{R}}_q,$
- $\|g_{j,x}^M\|_{q,2} \le \dfrac{2\left\|g_j^M\right\|_{q,2}}{(q;q)_\infty},$
- $\|\tilde{g}_{j,x}^M\|_{q,2} \le \dfrac{2\left\|\tilde{g}_j^M\right\|_{q,2}}{(q;q)_\infty}.$

Definition 6.5. We define the modified Rubin's q-wavelet packet transform $\Psi^M_{q,g}$ (resp. the dual modified Rubin's q-wavelet packet transform $\widetilde{\Psi}^M_{q,g}$) on $L^2_q(\mathbb{R}_q)$ by

$$\Psi^M_{q,g}(f)(j,x) = K\int_{-\infty}^{\infty} f(b)\overline{g^M_{j,x}}(b)d_qb, \quad j \in \mathbb{Z} \text{ and } x \in \mathbb{R}_q$$

$$\left(\text{resp. } \widetilde{\Psi}^M_{q,g}(f)(j,x) = K\int_{-\infty}^{\infty} f(b)\overline{\tilde{g}^M_{j,x}}(b)d_qb, \quad j \in \mathbb{Z} \text{ and } x \in \mathbb{R}_q\right).$$

Remark 6.6. The transforms $\Psi^M_{q,g}$ and $\widetilde{\Psi}^M_{q,g}$ can also be written in the form

$$\Psi^M_{q,g}(f)(j,x) = \check{f} *_q \overline{g^M_j}(x), \quad j \in \mathbb{Z} \text{ and } x \in \mathbb{R}_q \tag{48}$$

and

$$\widetilde{\Psi}^M_{q,g}(f)(j,x) = \check{f} *_q \overline{\tilde{g}^M_j}(x), \quad j \in \mathbb{Z} \text{ and } x \in \mathbb{R}_q. \tag{49}$$

Theorem 6.7. (Plancherel formula) *For all $f \in L^2_q(\mathbb{R}_q)$, we have*

$$\|f\|^2_{q,2} = \sum_{j=-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi^M_{q,g}(f)(j,x)\overline{\widetilde{\Psi}^M_{q,g}(f)(j,x)}d_qx. \tag{50}$$

Proof. Due to the relations (48) and (49), and Theorem 3.2, we have for all $j \in \mathbb{Z}$,

$$\int_{-\infty}^{\infty} \Psi^M_{q,g}(f)(j,x)\overline{\widetilde{\Psi}^M_{q,g}(f)(j,x)}d_qx = \int_{-\infty}^{\infty} |\mathcal{F}_q(f)(b)|^2\mathcal{F}_q(g^M_j)(b)\mathcal{F}_q(\tilde{g}^M_j)(b)d_qb.$$

On the other hand, using the definition of g^M_j and $\tilde{g}^M_j$ and the relation (34), we deduce that for all $b \in \mathbb{R}_q$, $\mathcal{F}_q(g^M_j)(b)\mathcal{F}_q(\tilde{g}^M_j)(b) \geq 0$.

So, by the help of the Fubini-Tonelli's theorem, we obtain the result from (46) and Theorem 3.2. □

In the following theorem, we give a pointwise reconstruction formulas for the transforms $\Psi^M_{q,g}$ and $\widetilde{\Psi}^M_{q,g}$.

Theorem 6.8. *For all $f \in L^2_q(\mathbb{R}_q)$, we have for all $x \in \mathbb{R}_q$,*

$$f(x) = K\sum_{j=-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi^M_{q,g}(f)(j,b)\tilde{g}^M_{j,b}(x)d_qb \tag{51}$$

and

$$f(x) = K\sum_{j=-\infty}^{\infty}\int_{-\infty}^{\infty} \widetilde{\Psi}^M_{q,g}(f)(j,b)g^M_{j,b}(x)d_qb. \tag{52}$$

Proof. The result is a consequence of the following lemma and Theorem 4.9. □

Lemma 6.9. *Let f be in $L_q^2(\mathbb{R}_q)$. Then for all $j \in \mathbb{Z}$ and $x \in \mathbb{R}_q$, we have the following equalities*

$$\begin{aligned}\int_{-\infty}^{\infty} \Psi_{q,g}^{M}(f)(j,b)\tilde{g}_{j,b}^{M}(x)d_qb &= \int_{-\infty}^{\infty} \widetilde{\Psi}_{q,g}^{M}(f)(j,b)g_{j,b}^{M}(x)d_qb \\ &= \int_{-\infty}^{\infty} \Psi_{q,g}^{P}(f)(j,b)g_{j,b}^{P}(x)d_qb.\end{aligned}$$

Proof. Let $j \in \mathbb{Z}$ and $x \in \mathbb{R}_q$. From the relation (48), we have

$$\int_{-\infty}^{\infty} \Psi_{q,g}^{M}(f)(j,b)\tilde{g}_{j,b}^{M}(x)d_qb = \int_{-\infty}^{\infty} \check{f} *_q \overline{g_j^M}(b)\overline{\overline{\tilde{g}_{j,b}^{M}(x)}}d_qb.$$

On the one hand, by Proposition 6.1, we have $\mathcal{F}_q\left(\overline{g_j^M}\right) \in L_q^2(\mathbb{R}_q) \cap L_q^{\infty}(\mathbb{R}_q)$. Then, the fact that $\mathcal{F}_q\left(\check{f}\right) \in L_q^2(\mathbb{R}_q)$ leads to $\mathcal{F}_q\left(\check{f}\right)\mathcal{F}_q\left(\overline{g_j^M}\right) \in L_q^2(\mathbb{R}_q)$, and Proposition 3.4 leads to $\check{f} *_q \overline{g_j^M} \in L_q^2(\mathbb{R}_q)$ and $\mathcal{F}_q\left(\check{f} *_q \overline{g_j^M}\right) = \mathcal{F}_q\left(\check{f}\right)\mathcal{F}_q\left(\overline{g_j^M}\right)$. So, thanks to Theorem 3.2, we get

$$\begin{aligned}\int_{-\infty}^{\infty} \Psi_{q,g}^{M}(f)(j,b)\tilde{g}_{j,b}^{M}(x)d_qb &= \int_{-\infty}^{\infty} \mathcal{F}_q\left(\check{f} *_q \overline{g_j^M}\right)(b)\overline{\mathcal{F}_q\left(\overline{\tilde{g}_{j,b}^{M}(x)}\right)}d_qb \\ &= \int_{-\infty}^{\infty} \mathcal{F}_q\left(\check{f}\right)(b)\mathcal{F}_q\left(\overline{g_j^M}\right)(b)\overline{\mathcal{F}_q\left(\overline{\tilde{g}_{j,b}^{M}(x)}\right)}d_qb.\end{aligned}$$

But $\mathcal{F}_q(g_j^M)$ and $\mathcal{F}_q(\tilde{g}_j^M)$ are real functions, then $\mathcal{F}_q\left(\overline{g_j^M}\right)(b) = \mathcal{F}_q\left(g_j^M\right)(-b)$,

$$\overline{\tilde{g}_{j,b}^{M}(x)} = \overline{T_{q,x}(\tilde{g}_j^M)(b)} = T_{q,-x}(\tilde{g}_j^M)(-b)$$

and

$$\begin{aligned}\overline{\mathcal{F}_q\left(\overline{\tilde{g}_{j,b}^{M}(x)}\right)} &= \overline{\mathcal{F}_q\left(T_{q,-x}(\tilde{g}_j^M)\right)(-b)} = \overline{\mathcal{F}_q(\tilde{g}_j^M)(-b)e(ibx;q^2)} \\ &= \mathcal{F}_q(\tilde{g}_j^M)(-b)e(-ibx;q^2).\end{aligned}$$

Furthermore, we have $\mathcal{F}_q\left(\check{f}\right)(b) = \mathcal{F}_q(f)(-b)$. Then,

$$\int_{-\infty}^{\infty} \Psi_{q,g}^{M}(f)(j,b)\tilde{g}_{j,b}^{M}(x)d_qb =$$

$$\begin{aligned}&= \int_{-\infty}^{\infty} \mathcal{F}_q(f)(-b)\mathcal{F}_q(g_j^M)(-b)\mathcal{F}_q(\tilde{g}_j^M)(-b)e(-ibx;q^2)d_qb \\ &= \int_{-\infty}^{\infty} \mathcal{F}_q(f)(b)\mathcal{F}_q(g_j^M)(b)\mathcal{F}_q(\tilde{g}_j^M)(b)e(ibx;q^2)d_qb.\end{aligned}$$

Hence, from the relation (47), we have

$$\int_{-\infty}^{\infty} \Psi_{q,g}^{M}(f)(j,b)\tilde{g}_{j,b}^{M}(x)d_qb =$$

$$\begin{aligned} &= \int_{-\infty}^{\infty} \mathcal{F}_q(f)(b)\mathcal{F}_q(g_j^M)(b)\mathcal{F}_q(\tilde{g}_j^M)(b)e(ibx;q^2)d_qb \\ &= \int_{-\infty}^{\infty} \mathcal{F}_q(f)(b)\mathcal{F}_q(g_j^P)(b)\mathcal{F}_q(g_j^P)(b)e(ibx;q^2)d_qb. \end{aligned}$$

On the other hand, the fact that $\mathcal{F}_q(g_j^P)$ is a real function gives $\mathcal{F}_q\left(\overline{g_j^P}\right)(b) = \mathcal{F}_q(g_j^P)(-b)$, and

$$\overline{g_{j,b}^P(x)} = \overline{T_{q,x}(g_j^P)(b)} = T_{q,-x}(g_j^P)(-b)$$

and

$$\overline{\mathcal{F}_q\left(\overline{g_{j,b}^P(x)}\right)} = \overline{\mathcal{F}_q\left(T_{q,-x}(g_j^P)\right)(-b)} = \mathcal{F}_q(g_j^P)(-b)e(-ibx;q^2).$$

Thus, a new application of Theorem 3.2 gives

$$\int_{-\infty}^{\infty} \Psi_{q,g}^{M}(f)(j,b)\tilde{g}_{j,b}^{M}(x)d_qb =$$

$$\begin{aligned} &= \int_{-\infty}^{\infty} \mathcal{F}_q(f)(b)\mathcal{F}_q(g_j^P)(b)\mathcal{F}_q(g_j^P)(b)e(ibx;q^2)d_qb \\ &= \int_{-\infty}^{\infty} \mathcal{F}_q(f)(-b)\mathcal{F}_q(g_j^P)(-b)\mathcal{F}_q(g_j^P)(-b)e(-ibx;q^2)d_qb \\ &= \int_{-\infty}^{\infty} \mathcal{F}_q(\check{f})(b)\mathcal{F}_q\left(\overline{g_j^P}\right)(b)\overline{\mathcal{F}_q\left(\overline{g_{j,b}^P(x)}\right)}d_qb \\ &= \int_{-\infty}^{\infty} \mathcal{F}_q\left(\check{f} *_q \overline{g_j^P}\right)(b)\overline{\mathcal{F}_q\left(\overline{g_{j,b}^P(x)}\right)}d_qb \\ &= \int_{-\infty}^{\infty} \Psi_{q,g}^{P}(f)(j,b)g_{j,b}^{P}(x)d_qb. \end{aligned}$$

By the same way, we prove that

$$\int_{-\infty}^{\infty} \widetilde{\Psi}_{q,g}^{M}(f)(j,b)g_{j,b}^{M}(x)d_qb = \int_{-\infty}^{\infty} \Psi_{q,g}^{P}(f)(j,b)g_{j,b}^{P}(x)d_qb.$$

□

Theorem 6.10. *Let f be in $L^2_q(\mathbb{R}_q)$. Then, we have the following reconstruction formulas. For all $x \in \mathbb{R}_q$ and all $m \in \mathbb{Z}$,*

$$f(x) = K\left[\int_{-\infty}^{\infty} \Theta^P_{q,G}(f)(m,b)G^P_{m,b}(x)d_qb + \sum_{j=m}^{\infty}\int_{-\infty}^{\infty} \Psi^M_{q,g}(f)(j,b)\tilde{g}^M_{j,b}(x)d_qb\right]$$

and

$$f(x) = K\left[\int_{-\infty}^{\infty} \Theta^P_{q,G}(f)(m,b)G^P_{m,b}(x)d_qb + \sum_{j=m}^{\infty}\int_{-\infty}^{\infty} \widetilde{\Psi}^M_{q,g}(f)(j,b)g^M_{j,b}(x)d_qb\right].$$

Proof. Theorem 5.10 and Lemma 6.9 yield to the result. □

7. Rubin's S-q-wavelet packet

Definition 7.1. A sequence $(g^S_j)_{j\in\mathbb{Z}}$ in $L^2_q(\mathbb{R}_q)$ is called Rubin's S-q-wavelet packet if the following assumptions are verified:
i) For all $j \in \mathbb{Z}$, $\mathcal{F}_q(g^S_j)$ is a real valued function.
ii) q-stability conditions: there exist some positive real numbers a and b, such that for all $x \in \mathbb{R}_q$,

$$a \leq \sum_{j=-\infty}^{\infty} [\mathcal{F}_q(g^S_j)(x)]^2 \leq b. \tag{53}$$

We say that a and b are the q-stability constants.

Definition 7.2. Let $(g^S_j)_{j\in\mathbb{Z}}$ be a Rubin's S-q-wavelet packet. We define the corresponding dual Rubin's S-q-wavelet packet $(\widetilde{g}^S_j)_{j\in\mathbb{Z}}$ by

$$\mathcal{F}_q(\widetilde{g}^S_j)(x) = \frac{\mathcal{F}_q(g^S_j)(x)}{\sum_{k=-\infty}^{\infty} [\mathcal{F}_q(g^S_k)(x)]^2}, \quad x \in \mathbb{R}_q. \tag{54}$$

In the following propositions we give some immediate properties of Rubin's S-q-wavelet packet $(g^S_j)_{j\in\mathbb{Z}}$ and its dual $(\widetilde{g}^S_j)_{j\in\mathbb{Z}}$.

Proposition 7.3. *For all $x \in \mathbb{R}_q$ and all $j \in \mathbb{Z}$, we have*

$$\left|\mathcal{F}_q(g^S_j)(x)\right| \leq b^{\frac{1}{2}}, \tag{55}$$

$$\left|\mathcal{F}_q(\widetilde{g}^S_j)(x)\right| \leq \frac{b^{\frac{1}{2}}}{a}, \tag{56}$$

$$\sum_{j=-\infty}^{\infty} \mathcal{F}_q(g^S_j)(x)\mathcal{F}_q(\widetilde{g}^S_j)(x) = 1 \tag{57}$$

and

$$\sum_{j=-\infty}^{\infty}\left[\mathcal{F}_q(g_j^S)(x)\right]^2=\left(\sum_{j=-\infty}^{\infty}\left[\mathcal{F}_q(\widetilde{g}_j^S)(x)\right]^2\right)^{-1}. \tag{58}$$

Proposition 7.4. *Let $m\in\mathbb{Z}$ and $x\in\mathbb{R}_q$. Then*

1.

$$\frac{\sum_{j=-\infty}^{m-1}\left[\mathcal{F}_q(\widetilde{g}_j^S)(x)\right]^2}{\sum_{j=-\infty}^{\infty}\left[\mathcal{F}_q(\widetilde{g}_j^S)(x)\right]^2}=\frac{\sum_{j=-\infty}^{m-1}\left[\mathcal{F}_q(g_j^S)(x)\right]^2}{\sum_{j=-\infty}^{\infty}\left[\mathcal{F}_q(g_j^S)(x)\right]^2}. \tag{59}$$

2.

$$\sum_{j=-\infty}^{m-1}\mathcal{F}_q(g_j^S)(x)\mathcal{F}_q(\widetilde{g}_j^S)(x)=\frac{\sum_{j=-\infty}^{m-1}\left[\mathcal{F}_q(g_j^S)(x)\right]^2}{\sum_{j=-\infty}^{\infty}\left[\mathcal{F}_q(g_j^S)(x)\right]^2}. \tag{60}$$

Proof. The equalities are consequences of the relations (54) and (58). □

Proposition 7.5. *1. The dual Rubin's S-wavelet packet $(\widetilde{g}_j^S)_{j\in\mathbb{Z}}$ verifies the following inequalities:*

$$\frac{1}{b}\leq\sum_{j=-\infty}^{\infty}[\mathcal{F}_q(\widetilde{g}_j^S)(x)]^2\leq\frac{1}{a}, \tag{61}$$

where a and b are the q-stability constants.

2. Suppose that for all $m\in\mathbb{Z}$, the function $x\mapsto\left(\sum_{j=-\infty}^{m-1}(\mathcal{F}_q(g_j^S)(x))^2\right)^{\frac{1}{2}}$ is in $L_q^2(\mathbb{R}_q)$ then,

$x\mapsto\left(\sum_{j=-\infty}^{m-1}(\mathcal{F}_q(\widetilde{g}_j^S)(x))^2\right)^{\frac{1}{2}}$ and $x\mapsto\left(\sum_{j=-\infty}^{m-1}\mathcal{F}_q(g_j^S)(x)\mathcal{F}_q(\widetilde{g}_j^S)(x)\right)^{\frac{1}{2}}$ belong to $L_q^2(\mathbb{R}_q)$.

The previous result allows us to state the following definition.

Definition 7.6. Let $(g_j^S)_{j\in\mathbb{Z}}$ be a Rubin's S-q-wavelet packet and $(\widetilde{g}_j^S)_{j\in\mathbb{Z}}$ be its dual. We suppose that for all $m\in\mathbb{Z}$, the function $x\mapsto\left(\sum_{j=-\infty}^{m-1}(\mathcal{F}_q(g_j^S)(x))^2\right)^{\frac{1}{2}}$ is in $L_q^2(\mathbb{R}_q)$. We define the q-scale discrete scaling function $(G_m^S)_{m\in\mathbb{Z}}$ by

$$\mathcal{F}_q(G_m^S)(x)=\left(\sum_{j=-\infty}^{m-1}\mathcal{F}_q(g_j^S)(x)\mathcal{F}_q(\widetilde{g}_j^S)(x)\right)^{\frac{1}{2}}. \tag{62}$$

Some properties of the sequence $(G_m^S)_{m\in\mathbb{Z}}$ are given in the following proposition.

Proposition 7.7. *For $m\in\mathbb{Z}$, the function G_m^S belongs to $L_q^2(\mathbb{R}_q)$ and for all $x\in\mathbb{R}_q$, we have*

$$0\leq\mathcal{F}_q(G_m^S)(x)\leq 1 \tag{63}$$

and

$$\lim_{m\to+\infty}\mathcal{F}_q(G_m^S)(x)=1. \tag{64}$$

Proof. The relation (63) follows from the relations (54) and (57) and (64) can be easily deduced from (60). □

Let $(g_j^S)_{j\in\mathbb{Z}}$ be a Rubin's S-q-wavelet packet and $(\widetilde{g}_j^S)_{j\in\mathbb{Z}}$ be its dual. For all $j\in\mathbb{Z}$ and $y\in\mathbb{R}_q$, we define the functions $g_{j,y}^S$ and $\widetilde{g}_{j,y}^S$ on $x\in\mathbb{R}_q$ by

$$g_{j,y}^S(x)=T_{q,x}(g_j^S)(y),$$

$$\widetilde{g}_{j,y}^S(x)=T_{q,x}(\widetilde{g}_j^S)(y).$$

The following proposition provides some properties of these functions.

Proposition 7.8. *For all $j\in\mathbb{Z}$ and $y\in\mathbb{R}_q$, the functions $g_{j,y}^S$ and $\widetilde{g}_{j,y}^S$ belong to $L_q^2(\mathbb{R}_q)$ and we have for all $x\in\mathbb{R}_q$,*

- $\mathcal{F}_q(g_{j,y}^S)(x)=e(ixy;q^2)\mathcal{F}_q(g_j^S)(x)$.
- $\|g_{j,y}^S\|_{q,2}\leq\dfrac{2\|g_j^S\|_{q,2}}{(q;q)_\infty}$.
- $\mathcal{F}_q(\widetilde{g}_{j,y}^S)(x)=e(ixy;q^2)\mathcal{F}_q(\widetilde{g}_j^S)(x)$.
- $\|\widetilde{g}_{j,y}^S\|_{q,2}\leq\dfrac{2\|\widetilde{g}_j^S\|_{q,2}}{(q;q)_\infty}$.

Proof. The assertions follow from the relations (16) and (19), and the properties of the q-translation operator $T_{q,x}$. □

Definition 7.9. We define the Rubin's S-q-wavelet transform $\Psi^S_{q,g}$ (resp. the dual Rubin's S-q-wavelet transform $\widetilde{\Psi}^S_{q,g}$) for all $f \in L^2_q(\mathbb{R}_q)$, by

$$\Psi^S_{q,g}(f)(j,y) = K\int_{-\infty}^{\infty} f(x)\overline{g^S_{j,y}}(x)d_qx, \;\; j \in \mathbb{Z}, \;\; y \in \mathbb{R}_q \tag{65}$$

$$(resp. \;\; \widetilde{\Psi}^S_{q,g}(f)(j,y) = K\int_{-\infty}^{\infty} f(x)\overline{\widetilde{g}^S_{j,y}}(x)d_qx, \;\; j \in \mathbb{Z}, \;\; y \in \mathbb{R}_q) \tag{66}$$

Remark 7.10. The transform $\Psi^S_{q,g}$ (resp. $\widetilde{\Psi}^S_{q,g}$) can be written in the form

$$\Psi^S_{q,g}(f)(j,y) = \check{f} *_q \overline{g^S_j}(y) \tag{67}$$

$$(resp. \;\; \widetilde{\Psi}^S_{q,g}(f)(j,y) = \check{f} *_q \overline{\widetilde{g}^S_j}(y)). \tag{68}$$

Proposition 7.11. *For f in $L^2_q(\mathbb{R}_q)$, one has:*

1. *for all $j \in \mathbb{Z}$, $y \in \mathbb{R}_q$,*

$$|\Psi^S_{q,g}(f)(j,y)| \leq \frac{2K}{(q;q)_\infty} \|f\|_{q,2} \left\|g^S_j\right\|_{q,2}$$

and

$$|\widetilde{\Psi}^S_{q,g}(f)(j,y)| \leq \frac{2K}{(q;q)_\infty} \|f\|_{q,2} \left\|\widetilde{g}^S_j\right\|_{q,2};$$

2. *for all $j \in \mathbb{Z}$, the functions $y \mapsto \Psi^S_{q,g}(f)(j,y)$ and $y \mapsto \widetilde{\Psi}^S_{q,g}(f)(j,y)$ are continuous on $\widetilde{\mathbb{R}}_q$ and we have $\lim_{y\to\infty} \Psi^S_{q,g}(f)(j,y) = 0$, as well as $\lim_{y\to\infty} \widetilde{\Psi}^S_{q,g}(f)(j,y) = 0$.*

The following proposition is a direct deduction from the relations (55), (56) and Proposition 15.

Proposition 7.12. *Let f be in $L^2_q(\mathbb{R}_q)$. Then, for all $j \in \mathbb{Z}$, the functions $y \mapsto \Psi^S_{q,g}(f)(j,y)$ and $y \mapsto \widetilde{\Psi}^S_{q,g}(f)(j,y)$ belong to $L^2_q(\mathbb{R}_q)$ and for all $x \in \mathbb{R}_q$, we have*

$$\mathcal{F}_q(\Psi^S_{q,g}(f)(j,.))(x) = \mathcal{F}_q(f)(-x)\mathcal{F}_q(g^S_j)(-x) \tag{69}$$

and

$$\mathcal{F}_q(\widetilde{\Psi}^S_{q,g}(f)(j,.))(x) = \mathcal{F}_q(f)(-x)\mathcal{F}_q(\widetilde{g}^S_j)(-x).$$

The following theorems provide Plancherel and reconstruction formulas for the transforms $\Psi^S_{q,g}$ and $\widetilde{\Psi}^S_{q,g}$. They can be proved by the same ways as in the precedent section.

Theorem 7.13. *(Plancherel formula)*
For all $f \in L^2_q(\mathbb{R}_q)$, we have

$$\|f\|^2_{q,2} = \sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi^S_{q,g}(f)(j,y)\overline{\widetilde{\Psi}^S_{q,g}(f)(j,y)}d_qy. \tag{70}$$

Theorem 7.14. *Let f be in $L^2_q(\mathbb{R}_q)$. Then, we have the following reconstruction formulas:*

$$f(x) = K\sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi^S_{q,g}(f)(j,y)\widetilde{g}^S_{j,y}(x)d_qy;$$

$$f(x) = K\sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} \widetilde{\Psi}^S_{q,g}(f)(j,y)g^S_{j,y}(x)d_qy.$$

For this end, let us introduce, for all $m \in \mathbb{Z}$, the function

$$G^S_{m,y}(x) = T_{q,x}(G^S_m)(y), \ \ \forall y \in \mathbb{R}_q. \tag{71}$$

We have the following result.

Theorem 7.15. *Let $(G^S_m)_{m\in\mathbb{Z}}$ be the q-scale discrete scaling function. For all f in $L^2_q(\mathbb{R}_q)$, we have the following reconstruction formula*

$$f(x) = K\lim_{m\to+\infty} \int_{-\infty}^{\infty} (\check{f} *_q \overline{G^S_m})(y)G^S_{m,y}(x)d_qy.$$

Remark 7.16. In the case $a = b$, we have $\widetilde{g}^s_j = \frac{1}{a}g^s_j$, which proves that decomposition and reconstitution are given by the same formula.

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