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SOME NEW ESTIMATES OF HERMITE-HADAMARD INEQUALITIES VIA HARMONICALLY r-CONVEX FUNCTIONS

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In this paper, we introduce the class of harmonically r-convex functions. We derive some Hermite-Hadamard type inequalities for this class of harmonic convex functions.

1. Introduction

In recent decades, the concepts of classical convex sets and convex functions have been extended and generalized in various directions using novel and innovative ideas, for example see [1, 2, 5, 6, 8–14, 16, 17]. Recently Iscan et al. introduced the concept of harmonically convex functions, see [5]. For some recent investigations on harmonically convex functions interested readers are referred to [6, 12]. Pearce et al. [16] improved the Hermite-Hadamard type inequality for r-convex function. For some useful details on r-convex functions, see [4, 15, 18].

Theory of convex functions is closely related with theory of inequalities. Many classical inequalities are proved for convex functions, see [8, 17]. One of the extensively studied inequality in the literature is Hermite-Hadamard's inequality. It reads as: A function $f : I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ is convex if and only if the

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following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2},$$

holds.
For some generalizations and extensions of Hermite-Hadamard's type inequality interested readers are referred to [2]-[18].
In this paper, we introduce the class of harmonically r-convex functions, we derive some interesting Hermite-Hadamard type inequalities for this new class of harmonically convex functions. This is the main motivation of this paper.

2. Preliminaries

In this section, we recall some previous known concepts.

Definition 2.1 ([5]). Let $I \subset \mathbb{R} \setminus \{0\}$ be interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex function, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq (1-t)f(x)+tf(y), \quad \forall x,y \in I, t \in [0,1]. \tag{1}$$

We note that for $t = \frac{1}{2}$, we have the definition of Jensen type of harmonic convex functions, that is

$$f\left(\frac{2xy}{x+y}\right) \leq \frac{f(x)+f(y)}{2}, \quad \forall x,y \in I. \tag{2}$$

Definition 2.2 ([12]). Let $I \subset \mathbb{R} \setminus \{0\}$ be interval. A function $f : I \to \mathbb{R}$ is said to be harmonically log-convex function, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq (f(x))^{1-t}(f(y))^{t}, \quad \forall x,y \in I, t \in [0,1]. \tag{3}$$

Definition 2.3 ([16, 17]). A function f is said to be r-convex positive function, if $\forall x,y \in [a,b]$ and $t \in [0,1]$, we have

$$f(tx+(1-t)y) \leq \begin{cases} \left(t[f(x)]^r+(1-t)[f(y)]^r\right)^{\frac{1}{r}}, & r \neq 0, \\ f^t(x)f^{1-t}(y), & r = 0. \end{cases}$$

One can see that 0-convex is classical log-convex function and 1-convex is classical convex function.
Now we define the harmonically r-convex functions.

Definition 2.4. A function f is said to be harmonically r-convex positive function, if $\forall x, y \in [a,b]$ and $t \in [0,1]$, we have

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq \begin{cases} \left((1-t)[f(x)]^r + t[f(y)]^r\right)^{\frac{1}{r}}, & r \neq 0, \\ f^{1-t}(x)f^t(y), & r = 0. \end{cases}$$

Note that for $r=1$, we have classical harmonic convex functions and for $r=0$, we have harmonically log-convex functions.
Logarithmic mean $L(x,y)$ of two positive numbers x, y is given by

$$L(x,y) = \begin{cases} \dfrac{x-y}{\ln x - \ln y}, & x \neq y, \\ x, & x = y. \end{cases}$$

Generalized logarithmic means of order r of positive numbers x, y defined by

$$L_r(x,y) = \begin{cases} \dfrac{r}{r+1}\dfrac{x^{r+1}-y^{r+1}}{x^r-y^r}, & r \neq \{-1,0\}, x \neq y, \\ \dfrac{x-y}{\ln x - \ln y}, & r = 0, x \neq y, \\ xy\dfrac{\ln x - \ln y}{x-y}, & r = -1, x \neq y, \\ x, & x = y. \end{cases}$$

Minkowski's Inequality is stated as follows;
Let

$$p \geq 1, 0 < \int_a^b f(x)^p dx < \infty, 0 < \int_a^b w(x)^p dx < \infty.$$

Then

$$\left(\int_a^b (f(x)+w(x))^p dx\right)^{\frac{1}{p}} \leq \left(\int_a^b f(x)^p dx\right)^{\frac{1}{p}} + \left(\int_a^b w(x)^p dx\right)^{\frac{1}{p}}.$$

3. Main Results

In this section, we derive our main results.

Proposition 3.1. *Let f and w be two harmonically r-convex function. Then, for $r \neq 0$, the product of f and w is harmonically r-convex, if both f and w are similarly ordered and for $r=0$ the product of f and w is harmonically r-convex.*

Proof. Let f and w be harmonically r-convex function. We consider both cases:
(i) If $r \neq 0$, then

$$\begin{aligned}
& f^r\Big(\frac{xy}{tx+(1-t)y}\Big)w^r\Big(\frac{xy}{tx+(1-t)y}\Big) \\
& \leq (1-t)^2 f^r(x)w^r(x) + t^2 f^r(y)w^r(y) + t(1-t)[f^r(x)w^r(y) + f^r(y)w^r(x)] \\
& \leq (1-t)^2 f^r(x)w^r(x) + t^2 f^r(y)w^r(y) + t(1-t)[f^r(x)w^r(x) + f^r(x)w^r(x)] \\
& = (1-t)f^r(x)w^r(x) + tf^r(y)w^r(y).
\end{aligned}$$

(ii) The case $r = 0$ is obvious.
This completes the proof. □

Theorem 3.2. *Let $f : I \to \mathbb{R}$ be harmonically r-convex function, where $a, b \in I$ and $a < b$. Then for $r \neq 0$, we have*

$$f^r\left(\frac{2xy}{x+y}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f^r(x)}{x^2}dx \leq \frac{f^r(a)+f^r(b)}{2}.$$

Proof. Let f be harmonically r-convex function. For $t = \frac{1}{2}$, we have

$$f^r\left(\frac{2xy}{x+y}\right) \leq \frac{1}{2}[f^r(x) + f^r(y)].$$

This implies that

$$f^r\left(\frac{2ab}{a+b}\right) \leq \frac{1}{2}\left[\left\{f^r\left(\frac{ab}{(1-t)a+tb}\right)\right\} + \left\{f^r\left(\frac{ab}{ta+(1-t)b}\right)\right\}\right].$$

Integrating above inequality with respect to t on $[0,1]$, we have

$$f^r\left(\frac{2ab}{a+b}\right) \leq \frac{1}{2}\left[\int_0^1 f^r\left(\frac{ab}{(1-t)a+tb}\right)dt + \int_0^1 f^r\left(\frac{ab}{ta+(1-t)b}\right)dt\right].$$

This implies that

$$f^r\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f^r(x)}{x^2}dx. \tag{4}$$

Also

$$\frac{ab}{b-a}\int_a^b \log\left(\frac{f^r(x)}{x^2}\right)dx = \int_0^1 f^r\left(\frac{ab}{ta+(1-t)b}\right)dt \leq \frac{f^r(a)+f^r(b)}{2}. \tag{5}$$

Combining (4) and (5) completes the proof. □

Theorem 3.3. *Let $f,w : I \to \mathbb{R}$ be harmonically r-convex functions on $I = [a,b]$ with $a < b$. Then for $r \neq 0$, we have*

$$2f^r\left(\frac{2ab}{a+b}\right)w^r\left(\frac{2ab}{a+b}\right) - \left[\frac{1}{6}M(a,b) + \frac{1}{3}N(a,b)\right]$$
$$\leq \frac{ab}{b-a}\int_a^b \frac{f^r(x)w^r(x)}{x^2}dx \leq \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b),$$

where

$$M(a,b) = f^r(a)w^r(a) + f^r(b)w^r(b), \tag{6}$$
$$N(a,b) = f^r(a)w^r(b) + f^r(b)w^r(a). \tag{7}$$

Proof. Let f and w be harmonically r-convex functions. Then

$$\begin{aligned}
& f^r\left(\frac{2ab}{a+b}\right)w^r\left(\frac{2ab}{a+b}\right) \\
& \leq \frac{1}{4}\left[f^r\Big(\frac{ab}{(1-t)a+tb}\Big)w^r\Big(\frac{ab}{(1-t)a+tb}\Big)\right. \\
& \qquad\qquad \left. + f^r\Big(\frac{ab}{ta+(1-t)b}\Big)w^r\Big(\frac{ab}{ta+(1-t)b}\Big)\right] \\
& \quad + \frac{1}{4}\left[f^r\Big(\frac{ab}{(1-t)a+tb}\Big)w^r\Big(\frac{ab}{ta+(1-t)b}\Big)\right. \\
& \qquad\qquad \left. + f^r\Big(\frac{ab}{ta+(1-t)b}\Big)w^r\Big(\frac{ab}{(1-t)a+tb}\Big)\right] \\
& \leq \frac{1}{4}\left[f^r\Big(\frac{ab}{(1-t)a+tb}\Big)w^r\Big(\frac{ab}{(1-t)a+tb}\Big)\right. \\
& \qquad\qquad \left. + f^r\Big(\frac{ab}{ta+(1-t)b}\Big)w^r\Big(\frac{ab}{ta+(1-t)b}\Big)\right] \\
& \quad + \frac{1}{4}\left[2t(1-t)(f^r(a)w^r(a) + f^r(b)w^r(b))\right. \\
& \qquad\qquad \left. + (t^2 + (1-t)^2)(f^r(b)w^r(a) + f^r(a)w^r(b))\right].
\end{aligned}$$

Integrating with respect to t on $[0,1]$, we have

$$\begin{aligned}
& f^r\left(\frac{2ab}{a+b}\right)w^r\left(\frac{2ab}{a+b}\right) \\
& \quad \leq \frac{1}{4}\left[\frac{ab}{b-a}\int_a^b \frac{f^r(x)w^r(x)}{x^2}dx\right] + \frac{1}{2}\left[\frac{1}{6}M(a,b) + \frac{1}{3}N(a,b)\right].
\end{aligned}$$

This implies that

$$
\begin{aligned}
&2f^r\left(\frac{2ab}{a+b}\right)w^r\left(\frac{2ab}{a+b}\right)-\left[\frac{1}{6}M(a,b)+\frac{1}{3}N(a,b)\right]\\
&\leq \frac{ab}{b-a}\int_a^b\Big(\frac{f^r(x)w^r(x)}{x^2}\Big)dx\\
&=\int_0^1 f^r\Big(\frac{ab}{(1-t)a+tb}\Big)w^r\Big(\frac{ab}{(1-t)a+tb}\Big)dt\\
&\leq \int_0^1 [tf^r(a)+(1-t)f^r(b)][tw^r(a)+(1-t)w^r(b)]dt=\frac{1}{3}M(a,b)+\frac{1}{6}N(a,b).
\end{aligned}
$$

This completes the proof. □

Theorem 3.4. *Let $f: I \to \mathbb{R}$ be harmonically r-convex function, where $a,b \in I$ and $a < b$. Then*

$$
\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \begin{cases} L_r(f(a),f(b)), & r\neq\{0,-1\}, f(a)\neq f(b),\\ L_r(f(a),f(a)), & r\neq\{0,-1\}, f(a)= f(b),\\ L_{-1}(f(a),f(b)), & r=-1, f(a)\neq f(b),\\ L_{-1}(f(a),f(a)), & r=-1, f(a)= f(b). \end{cases}
$$

Proof. Since f is harmonically r-convex function, so we have

$$
\begin{aligned}
\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx &= \int_0^1 f\Big(\frac{ab}{ta+(1-t)b}\Big)dt\\
&\leq \int_0^1 \{(1-t)f^r(a)+tf^r(b)\}^{\frac{1}{r}}dt\\
&=\big(\frac{r}{r+1}\big)\frac{f^{r+1}(b)-f^{r+1}(a)}{f^r(b)-f^r(a)}=L_r(f(a),f(b)).
\end{aligned}
$$

The case $r\neq\{0,-1\}, f(a)=f(b)$ can be proved similarly.

Now, when $r=-1, f(a)\neq f(b)$, we have

$$\begin{aligned}\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx &= \int_0^1 f\Big(\frac{ab}{ta+(1-t)b}\Big)dt\\ &\leq \int_0^1 \{(1-t)f^{-1}(a)+tf^{-1}(b)\}^{-1}dt\\ &= \frac{f(a)f(b)}{f(b)-f(a)}\int_{f(a)}^{f(b)} \frac{1}{u}du = L_{-1}(f(a),f(b)).\end{aligned}$$

The case $r=-1, f(a)=f(b)$ can be proved on the similar lines. □

Theorem 3.5. *Let $f: I\to\mathbb{R}$ be harmonically r-convex function, where $a,b\in I$ and $a<b$. Then for $0<r\leq 1$, we have*

$$\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \Big(\frac{r}{r+1}\Big)\Big[(f^r(a)+f^r(b))\Big]^{\frac{1}{r}}.$$

Proof. Using Minkowski's inequality and the fact that f is harmonically r-convex function, we have

$$\begin{aligned}\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx &= \int_0^1 f\Big(\frac{ab}{(1-t)a+tb}\Big)dt\\ &\leq \int_0^1 \{tf^r(a)+(1-t)f^r(b)\}^{\frac{1}{r}}dt\\ &\leq \Bigg[\Big(\int_0^1 t^{\frac{1}{r}}f(a)dt\Big)^r+\Big(\int_0^1 (1-t)^{\frac{1}{r}}f(b)dt\Big)^r\Bigg]^{\frac{1}{r}}\\ &= \Bigg[\Big(\frac{r}{r+1}\Big)^r(f^r(a)+f^r(b))\Bigg]^{\frac{1}{r}}\\ &= \Big(\frac{r}{r+1}\Big)\Big[(f^r(a)+f^r(b))\Big]^{\frac{1}{r}}.\end{aligned}$$

This completes the proof. □

Theorem 3.6. *Let f and w be harmonically r_1-convex function and harmonically r_2-convex function respectively, then for $0 < r_1, r_2 \leq 2$, we have*

$$\begin{aligned}\frac{ab}{b-a}\int_a^b \frac{f(x)w(x)}{x^2}dx &\leq \frac{1}{2}\Big(\frac{r_1}{r_1+2}\Big)(f^{r_1}(a)+f^{r_1}(b))^{\frac{1}{r_1}}\\ &+\frac{1}{2}\Big(\frac{r_2}{r_2+2}\Big)(w^{r_2}(a)+w^{r_2}(b))^{\frac{1}{r_2}}.\end{aligned}$$

Proof. Using the fact that f and w are harmonically r_1-convex function and harmonically r_2-convex function respectively and also using Cauchy's inequality and Minkowski's inequality, we have

$$\begin{aligned}&\frac{ab}{b-a}\int_a^b \frac{f(x)w(x)}{x^2}dx\\ &=\int_0^1 f\Big(\frac{ab}{(1-t)a+tb}\Big)w\Big(\frac{ab}{(1-t)a+tb}\Big)dt\\ &\leq \int_0^1 (tf^{r_1}(a)+(1-t)f^{r_1}(b))^{\frac{1}{r_1}}(tw^{r_2}(a)+(1-t)w^{r_2}(b))^{\frac{1}{r_2}}dt\\ &\leq \frac{1}{2}\int_0^1 (tf^{r_1}(a)+(1-t)f^{r_1}(b))^{\frac{2}{r_1}}dt+\frac{1}{2}\int_0^1 (tw^{r_2}(a)+(1-t)w^{r_2}(b))^{\frac{2}{r_2}}dt\\ &\leq \frac{1}{2}\Bigg[\Big(\int_0^1 t^{\frac{2}{r_1}}f^2(a)dt\Big)^{\frac{r_1}{2}}+\Big(\int_0^1 (1-t)^{\frac{2}{r_1}}f^2(b)dt\Big)^{\frac{r_1}{2}}\Bigg]^{\frac{2}{r_1}}\\ &\quad+\frac{1}{2}\Bigg[\Big(\int_0^1 t^{\frac{2}{r_2}}w^2(a)dt\Big)^{\frac{r_2}{2}}+\Big(\int_0^1 (1-t)^{\frac{2}{r_2}}w^2(b)dt\Big)^{\frac{r_2}{2}}\Bigg]^{\frac{2}{r_2}}\\ &=\frac{1}{2}\Big(\frac{r_1}{r_1+2}\Big)(f^{r_1}(a)+f^{r_1}(b))^{\frac{1}{r_1}}+\frac{1}{2}\Big(\frac{r_2}{r_2+2}\Big)(w^{r_2}(a)+w^{r_2}(b))^{\frac{1}{r_2}}.\end{aligned}$$

This completes the proof. □

Corollary 3.7. *Under the conditions of Theorem 3.6, if $r_1 = 2 = r_2$, we have*

$$\frac{ab}{b-a}\int_a^b \frac{f(x)w(x)}{x^2}dx \leq \frac{1}{4}(f^2(a)+f^2(b))^{\frac{1}{2}}+\frac{1}{4}(w^2(a)+w^2(b))^{\frac{1}{2}}.$$

Corollary 3.8. *Under the conditions of Theorem 3.6, if $r_1 = 2 = r_2$ and $f(x) = w(x)$, we have*

$$\frac{ab}{b-a}\int_a^b \frac{f^2(x)}{x^2}dx \le \frac{1}{2}(f^2(a)+f^2(b))^{\frac{1}{2}}.$$

Theorem 3.9. *Let f and w be harmonically r_1-convex function and harmonically r_2-convex function respectively. Then for $r_1 > 1$ and $\frac{1}{r_1}+\frac{1}{r_2} = 1$, we have*

$$\frac{ab}{b-a}\int_a^b \frac{f(x)w(x)}{x^2}dx \le \left\{\frac{f^{r_1}(a)+f^{r_1}(b)}{2}\right\}^{\frac{1}{r_1}}\left\{\frac{w^{r_2}(a)+w^{r_2}(b)}{2}\right\}^{\frac{1}{r_2}}.$$

Proof. Using the fact that f and w are harmonically r_1-convex function and harmonically r_2-convex function respectively and also using Holder's inequality, we have

$$\begin{aligned}
&\frac{ab}{b-a}\int_a^b \frac{f(x)w(x)}{x^2}dx \\
&= \int_0^1 f\Big(\frac{ab}{(1-t)a+tb}\Big)w\Big(\frac{ab}{(1-t)a+tb}\Big)dt \\
&\le \int_0^1 (tf^{r_1}(a)+(1-t)f^{r_1}(b))^{\frac{1}{r_1}}(tw^{r_2}(a)+(1-t)w^{r_2}(b))^{\frac{1}{r_2}}dt \\
&\le \left\{\int_0^1 (tf^{r_1}(a)+(1-t)f^{r_1}(b))dt\right\}^{\frac{1}{r_1}}\left\{\int_0^1 (tw^{r_2}(a)+(1-t)w^{r_2}(b))dt\right\}^{\frac{1}{r_2}} \\
&= \left\{\frac{f^{r_1}(a)+f^{r_1}(b)}{2}\right\}^{\frac{1}{r_1}}\left\{\frac{w^{r_2}(a)+w^{r_2}(b)}{2}\right\}^{\frac{1}{r_2}}.
\end{aligned}$$

This completes the proof. □

Corollary 3.10. *Under the conditions of Theorem 3.9, if $r_1 = 2 = r_2$, we have*

$$\frac{ab}{b-a}\int_a^b \frac{f(x)w(x)}{x^2}dx \le \left\{\frac{f^2(a)+f^2(b)}{2}\right\}^{\frac{1}{2}}\left\{\frac{w^2(a)+w^2(b)}{2}\right\}^{\frac{1}{2}}.$$

Corollary 3.11. *Under the conditions of Theorem 3.9, if $r_1 = 2 = r_2$ and $f(x) = w(x)$, we have*

$$\frac{ab}{b-a}\int_a^b \frac{f^2(x)}{x^2}dx \leq \left\{\frac{f^2(a)+f^2(b)}{2}\right\}^{\frac{1}{2}}.$$

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