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ON STABILITY FOR NONLINEAR IMPLICIT FRACTIONAL DIFFERENTIAL EQUATIONS

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The purpose of this paper is to establish some types of Ulam stability: Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability for a class of implicit fractional-order differential equation.

1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary order (non-integer). See, for example, the books such as [2–4, 7, 27, 31, 37] and the articles [9, 10], and references therein.

In recent years, fractional differential equations arise naturally in various fields such as rheology, fractals, chaotic dynamics, modeling and control theory, signal processing, bioengineering and biomedical applications, etc; Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. We refer the reader, for example, to the books such as [8, 23, 25, 39], and references therein.

"Under what conditions does there exist an additive mapping near an approximately additive mapping?" This is the stability problem of functional equation (of group homomorphisms) which was raised by Ulam in 1940 in a talk

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given at Wisconsin University [40, 41]. In 1941, Hyers [17] gave the first answer to the question of Ulam (for the additive mapping) in the case Banach spaces. Between 1982 and 1998 Rassias established the Hyers-Ulam stability of linear and nonlinear mappings. Jung [19, 20] investigated in 1998, the Hyers-Ulam stability of more general mappings on restricted domains. Obloza [26], in 1997, is the first author who has investigated the Hyers-Ulam stability of linear differential equations. After, many articles and books on this subject have been published in order to generalize the results of Hyers in many directions. For more detailed definitions of the Hyers-Ulam stability and the generalized Hyers-Ulam stability, we refer the reader to the papers [1, 5, 6, 16, 18, 21, 22, 24, 32, 36, 43–45] and the books [12, 33, 34].

Integer order implicit differential equations of arbitrary orders have been considered extensively in the literature; see for instance [11, 13–15, 28–30, 35, 38, 46].

The purpose of this paper, is to establish four types of Ulam stability : Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability for the equation, for the following initial value problem for implicit fractional-order differential equation

$$^cD^\alpha y(t) = f(t, y(t), ^cD^\alpha y(t)), \ \forall\, t \in J, \ 0 < \alpha \leq 1, \tag{1}$$

$$y(0) = y_0, \tag{2}$$

where $^cD^\alpha$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function space, $y_0 \in \mathbb{R}$ and $J = [0,T]$, $T > 0$.

This paper intiates the study of Ulam stability for such class of problems.

2. Preliminaries

Definition 2.1 ([31])**.** The fractional (arbitrary) order integral of the function $h \in L^1([0,T], \mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds,$$

where Γ is the Euler gamma function defined by $\Gamma(\alpha) = \displaystyle\int_0^\infty t^{\alpha-1} e^{-t} dt, \ \alpha > 0.$

Definition 2.2 ([23])**.** For a function h given on the interval $[0,T]$, the Caputo fractional-order derivative of order α of h, is defined by

$$(^cD^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} h^{(n)}(s) ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of the real number α.

Lemma 2.3 ([23]). *Let $\alpha \geq 0$ and $n = [\alpha] + 1$. Then*

$$I^{\alpha}({}^{c}D^{\alpha} f(t)) = f(t) - \sum_{k=0}^{n-1} \frac{f^{k}(0)}{k!} t^{k}.$$

We state the following generalization of Gronwall's lemma for singular kernels.

Lemma 2.4 ([42]). *Let $v : [0,T] \to [0,+\infty)$ be a real function and $w(.)$ is a nonnegative, locally integrable function on $[0,T]$ and there are constants $a > 0$ and $0 < \alpha < 1$ such that*

$$v(t) \leq w(t) + a \int_{0}^{t} (t-s)^{-\alpha} v(s) ds,$$

Then, there exists a constant $K = K(\alpha)$ such that

$$v(t) \leq w(t) + Ka \int_{0}^{t} (t-s)^{-\alpha} w(s) ds, \textit{ for every } t \in [0,T].$$

We adopt the definitions in Rus [36]: Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability for the equation, for the implicit fractional-order differential equation (1).

Definition 2.5. The equation (1) is *Ulam-Hyers stable* if there exists a real number $c_f > 0$ such that for each $\varepsilon > 0$ and for each solution $z \in C^1(J,\mathbb{R})$ of the inequality

$$|{}^{c}D^{\alpha} z(t) - f(t, z(t), {}^{c}D^{\alpha} z(t))| \leq \varepsilon, \ t \in J, \tag{3}$$

there exists a solution $y \in C^1(J,\mathbb{R})$ of equation (1) with

$$|z(t) - y(t)| \leq c_f \varepsilon, \ t \in J.$$

Definition 2.6. The equation (1) is *generalised Ulam-Hyers stable* if there exists $\psi_f \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\psi_f(0) = 0$, such that for each solution $z \in C^1(J,\mathbb{R})$ of the inequality (3) there exists a solution $y \in C^1(J,\mathbb{R})$ of the equation (1) with

$$|z(t) - y(t)| \leq \psi_f(\varepsilon), \ t \in J.$$

Definition 2.7. The equation (1) is *Ulam-Hyers-Rassias stable* with respect to $\varphi \in C(J,\mathbb{R}_+)$ if there exists a real number $c_f > 0$ such that for each $\varepsilon > 0$ and for each solution $z \in C^1(J,\mathbb{R})$ of the inequality

$$|{}^{c}D^{\alpha} z(t) - f(t, z(t), {}^{c}D^{\alpha} z(t))| \leq \varepsilon \varphi(t), \ t \in J, \tag{4}$$

there exists a solution $y \in C^1(J,\mathbb{R})$ of equation (1) with

$$|z(t) - y(t)| \leq c_f \varepsilon \varphi(t), \ t \in J.$$

Definition 2.8. The equation (1) is *generalised Ulam-Hyers-Rassias stable* with respect to $\varphi \in C(J,\mathbb{R}_+)$ if there exists a real number $c_{f,\varphi} > 0$ such that for each solution $z \in C^1(J,\mathbb{R})$ of the inequality

$$|^cD^\alpha z(t) - f(t,z(t),^cD^\alpha z(t))| \le \varphi(t),\ t \in J, \tag{5}$$

there exists a solution $y \in C^1(J,\mathbb{R})$ of equation (1) with

$$|z(t) - y(t)| \le c_{f,\varphi}\varphi(t),\ t \in J.$$

Remark 2.9. A function $z \in C^1(J,\mathbb{R})$ is a solution of of the inequality (3) if and only if there exists a function $g \in C(J,\mathbb{R})$ (which depend on y) such that

(i) $|g(t)| \le \varepsilon$, $\forall t \in J$.

(ii) $^cD^\alpha z(t) = f(t,z(t),^cD^\alpha z(t)) + g(t), t \in J$.

Remark 2.10. Clearly,

(i) Definition 2.5 $\Longrightarrow$ Definition 2.6.

(ii) Definition 2.7 $\Longrightarrow$ Definition 2.8.

Remark 2.11. A solution of the implicit differential inequation (3) with fractional order is called an fractional ε-solution of the implicit fractional differential equation (1).

So, the Ulam stabilities of the implicit differential equations with fractional order are some special types of data dependence of the solutions of fractional implicit differential equations.

3. Existence and Ulam-Hyers Stability

Definition 3.1. A function $u \in C^1(J)$ is said to be a solution of the problem $(1)-(2)$ is u satisfied equation (1) and condition (2) on J.

Lemma 3.2. *Let a function $f(t,u,v) : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous. Then the problem $(1)-(2)$ is equivalent to the problem*

$$y(t) = y_0 + I^\alpha g(t), \tag{6}$$

where $g \in C(J,\mathbb{R})$ satisfies the functional equation

$$g(t) = f(t, y_0 + I^\alpha g(t), g(t)).$$

Proof. If ${}^cD^\alpha y(t) = g(t)$ then $I^\alpha\ {}^cD^\alpha y(t) = I^\alpha g(t)$. So we obtain $y(t) = y_0 + I^\alpha g(t)$. □

Lemma 3.3 ([9])**.** *Assume*

(H1) The function $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.

(H2) There exist constants $K > 0$ and $0 < L < 1$ such that

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \le K|u-\bar{u}| + L|v-\bar{v}| \text{ for any } u,v,\bar{u},\bar{v} \in \mathbb{R} \text{ and } t \in J.$$

If

$$\frac{KT^\alpha}{(1-L)\Gamma(\alpha+1)} < 1, \tag{7}$$

then there exists a unique solution for the IVP $(1)-(2)$ on J.

Theorem 3.4. *Assume that the assumptions (H1), (H2) and (7) hold. Then the equation (1) is Ulam-Hyers stable.*

Proof. Let $z \in C(J,\mathbb{R})$ be a solution of the inequation (3), i.e.

$$|{}^cD^\alpha z(t) - f(t,z(t),{}^cD^\alpha z(t))| \le \varepsilon,\ t \in J. \tag{8}$$

Let us denote by $y \in C(J,\mathbb{R})$ the unique solution of the Cauchy problem

$${}^cD^\alpha y(t) = f(t,y(t),{}^cD^\alpha y(t)),\ \forall\, t \in J,\ 0 < \alpha \le 1,$$

$$y(0) = z(0).$$

By using Lemma 3.2, we have

$$y(t) = z(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g_y(s)ds,$$

where $g_y \in C(J,\mathbb{R})$ satisfies the functional equation

$$g_y(t) = f(t, y(0) + I^\alpha g_y(t), g_y(t)).$$

But, by integration of the formula (8) we obtain

$$\begin{aligned}\left|z(t) - z(0) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g_z(s)ds\right| &\le \frac{\varepsilon t^\alpha}{\Gamma(\alpha+1)}\\ &\le \frac{\varepsilon T^\alpha}{\Gamma(\alpha+1)},\end{aligned} \tag{9}$$

where $g_z \in C(J,\mathbb{R})$ satisfies the functional equation

$$g_z(t) = f(t, z(0) + I^\alpha g_z(t), g_z(t)).$$

On the other hand, we have, for each $t \in J$

$$\begin{aligned}
|z(t) - y(t)| &= \left| z(t) - z(0) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_y(s)ds \right| \\
&= \left| z(t) - z(0) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_z(s)ds \right. \\
&\left. + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(g_z(s) - g_y(s)\right) ds \right| \\
&\leq \left| z(t) - z(0) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_z(s)ds \right| \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |g_z(s) - g_y(s)| ds, \qquad (10)
\end{aligned}$$

where

$$g_y(t) = f(t, y(t), g_y(t)),$$

and

$$g_z(t) = f(t, z(t), g_z(t)).$$

By $(H2)$, we have, for each $t \in J$

$$\begin{aligned}
|g_z(t) - g_y(t)| &= |f(t, z(t), g_z(t)) - f(t, y(t), g_y(t))| \\
&\leq K|z(t) - y(t)| + L|g_z(t) - g_y(t)|.
\end{aligned}$$

Then

$$|g_z(t) - g_y(t)| \leq \frac{K}{1-L} |z(t) - y(t)|. \qquad (11)$$

Thus, by (9), (10), and (11) we get

$$|z(t) - y(t)| \leq \frac{\varepsilon T^\alpha}{\Gamma(\alpha+1)} + \frac{K}{(1-L)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |z(s) - y(s)| ds.$$

Then Lemma 2.4 implies that for each $t \in J$

$$|z(t) - y(t)| \leq \frac{\varepsilon T^\alpha}{\Gamma(\alpha+1)} \left[1 + \frac{\gamma K T^\alpha}{(1-L)\Gamma(\alpha+1)} \right] := c\varepsilon, \qquad (12)$$

where $\gamma = \gamma(\alpha)$ is a constant. So, the equation (1) is Ulam-Hyers stable. This completes the proof. □

By putting $\psi(\varepsilon) = c\varepsilon$, $\psi(0) = 0$ yields that the equation (1) is generalized Ulam-Hyers stable.

4. Ulam-Hyers-Rassias Stability.

Theorem 4.1. *Assume (H1), (H2) and*

(H3) The function $\varphi \in C(J,\mathbb{R}_+)$ is increasing and there exists $\lambda_\varphi > 0$ such that, for each $t \in J$, we have

$$I^\alpha \varphi(t) \le \lambda_\varphi \varphi(t).$$

Then the equation (1) *is Ulam-Hyers-Rassias stable with respect to φ.*

Proof. Let $z \in C(J,\mathbb{R})$ be a solution of the inequation (4), i.e.

$$|{}^cD^\alpha z(t) - f(t,z(t),{}^cD^\alpha z(t))| \le \varepsilon\varphi(t),\ t \in J,\ \varepsilon > 0. \tag{13}$$

Let us denote by $y \in C(J,\mathbb{R})$ the unique solution of the Cauchy problem

$$\begin{gathered}{}^cD^\alpha y(t) = f(t,y(t),{}^cD^\alpha y(t)),\ \forall\, t \in J,\ 0 < \alpha \le 1,\\ y(0) = z(0).\end{gathered}$$

By using Lemma 3.2, we have

$$y(t) = z(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g_y(s)ds,$$

where $g_y \in C(J,\mathbb{R})$ satisfies the functional equation

$$g_y(t) = f(t,y(0)+I^\alpha g_y(t),g_y(t)).$$

But, by integration of the formula (13) and by (H3), we obtain

$$\begin{aligned}\left|z(t) - z(0) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g_z(s)ds\right| &\le \frac{\varepsilon}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi(s)ds\\ &\le \varepsilon\lambda_\varphi\varphi(t),\end{aligned} \tag{14}$$

where $g_z \in C(J,\mathbb{R})$ satisfies the functional equation

$$g_z(t) = f(t,z(0)+I^\alpha g_z(t),g_z(t)).$$

On the other hand, we have, for each $t \in J$

$$\begin{aligned}|z(t)-y(t)| &= \left|z(t) - z(0) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g_y(s)ds\right|\\ &= \left|z(t) - z(0) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g_z(s)ds\right.\\ &\quad \left.+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(g_z(s)-g_y(s)\right)ds\right|\\ &\le \left|z(t) - z(0) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g_z(s)ds\right|\\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|g_z(s)-g_y(s)|ds,\end{aligned} \tag{15}$$

where

$$g_y(t) = f(t, y(t), g_y(t)),$$

and

$$g_z(t) = f(t, z(t), g_z(t)).$$

By $(H2)$, we have

$$\begin{aligned}|g_z(t) - g_y(t)| &= |f(t, z(t), g_z(t)) - f(t, y(t), g_y(t))| \\ &\leq K|z(t) - y(t)| + L|g_z(t) - g_y(t)|.\end{aligned}$$

Then

$$|g_z(t) - g_y(t)| \leq \frac{K}{1-L}|z(t) - y(t)|. \tag{16}$$

Thus, by (14), (15), and (16)

$$|z(t) - y(t)| \leq \varepsilon\lambda_\varphi \varphi(t) + \frac{K}{(1-L)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|z(s) - y(s)|ds.$$

Then Lemma 2.4 implies that, for each $t \in J$

$$|z(t) - y(t)| \leq \varepsilon\lambda_\varphi \varphi(t) + \frac{\gamma_1 \varepsilon \lambda_\varphi K}{(1-L)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\varphi(s)ds, \tag{17}$$

where constant $\gamma_1 = \gamma_1(\alpha)$ is a constant.
Thus, by (H3) and (17), we obtain

$$\begin{aligned}|z(t) - y(t)| &\leq \varepsilon\lambda_\varphi \varphi(t) + \frac{\gamma_1 K \varepsilon \lambda_\varphi^2 \varphi(t)}{(1-L)} \\ &= \left(1 + \frac{\gamma_1 K \lambda_\varphi}{1-L}\right)\lambda_\varphi \varepsilon \varphi(t).\end{aligned}$$

Then, for each $t \in J$

$$|z(t) - y(t)| \leq \left[\left(1 + \frac{\gamma_1 K \lambda_\varphi}{1-L}\right)\lambda_\varphi\right]\varepsilon\varphi(t) := c\varepsilon\varphi(t). \tag{18}$$

So, the equation (1) is Ulam-Hyers-Rassias stable. This completes the proof. □

5. Examples

Example 5.1. Consider the following Cauchy problem

$$ {}^{c}D^{\frac{1}{2}}y(t) = \frac{1}{100}(t\cos y(t) - y(t)\sin(t)) + \frac{|{}^{c}D_t^{\frac{1}{2}}y(t)|}{50+|{}^{c}D_t^{\frac{1}{2}}y(t)|}, \ \forall\, t \in [0,1], \tag{19} $$

$$ y(0) = 1. \tag{20} $$

Set

$$ f(t,u,v) = \frac{1}{100}(t\cos u - u\sin(t)) + \frac{v}{50+v}, \ \ t \in [0,1], \ u,v \in \mathbb{R}. $$

Clearly, the function f is jointly continuous.
For any $u,v,\bar{u},\bar{v} \in \mathbb{R}$ and $t \in [0,1]$:

$$ \begin{aligned} |f(t,u,v) - f(t,\bar{u},\bar{v})| &\le \frac{1}{100}|t||\cos u - \cos\bar{u}| \\ &+ \frac{1}{100}|\sin t||u-\bar{u}| + \frac{50|v-\bar{v}|}{(50+v)(50+\bar{v})} \\ &\le \frac{1}{100}|u-\bar{u}| + \frac{1}{100}|u-\bar{u}| + \frac{1}{50}|v-\bar{v}| \\ &\le \frac{1}{50}|u-\bar{u}| + \frac{1}{50}|v-\bar{v}|. \end{aligned} $$

Hence condition (H2) is satisfied with $K = L = \frac{1}{50}$.
Thus condition

$$ \frac{KT^{\alpha}}{(1-L)\Gamma(\alpha+1)} = \frac{\frac{1}{50}}{(1-\frac{1}{50})\Gamma(\frac{3}{2})} = \frac{2}{49\sqrt{\pi}} < 1, $$

is satisfied. It follows from Theorem 3.3 that the problem (19)-(20) has a unique solution on J, and from Theorem 3.4, equation (19) is Ulam-Hyers stable.

Example 5.2. Consider the following Cauchy problem

$$ {}^{c}D^{\frac{1}{2}}y(t) = \frac{2+|y(t)|+|{}^{c}D^{\frac{1}{2}}y(t)|}{120e^{t+10}(1+|y(t)|+|{}^{c}D^{\frac{1}{2}}y(t)|)}, \ \forall\, t \in [0,1], \tag{21} $$

$$ y(0) = 1. \tag{22} $$

Set

$$ f(t,u,v) = \frac{2+|u|+|v|}{120e^{t+10}(1+|u|+|v|)}, \ \ t \in [0,1], \ u,v \in \mathbb{R}. $$

Clearly, the function f is jointly continuous.
For any $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in [0,1]$

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \leq \frac{1}{120e^{10}}(|u-\bar{u}| + |v-\bar{v}|).$$

Hence condition (H2) is satisfied with $K = L = \frac{1}{120e^{10}}$.
Let $\varphi(t) = t^2$. We have

$$I^{\alpha}\varphi(t) \leq \frac{2}{\Gamma(\frac{7}{2})}t^2 := \lambda_{\varphi}\varphi(t).$$

Thus condition (H3) is satisfied with $\varphi(t) = t^2$ and $\lambda_{\varphi} = \frac{2}{\Gamma(\frac{7}{2})} = \frac{16}{15\sqrt{\pi}}$ It follows from Theorem 3.3 that the problem (21)-(22) as a unique solution on J, and from Theorem 4.1 equation (21) is Ulam-Hyers-Rassias stable.

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