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SOME RESULTS OF p-VALENT ANALYTIC FUNCTIONS DEFINED BY INTEGRAL OPERATOR

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In this paper we study different applications of the inclusion relationships, radius problems and some other interesting properties of p-valent functions which are defined by integral operator.

1. Introduction

Let $\mathcal{H}(\mathbb{U})$ be the class of functions which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{H}[a,p]$ be the subclass of $\mathcal{H}(\mathbb{U})$ consisting of functions of the form:

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \dots (a \in \mathbb{C};\ p \in \mathbb{N} = \{1,2,\dots\}).$$

Also, let $\mathcal{A}(p)$ be the subclass of the functions $f \in \mathcal{H}(\mathbb{U})$ of the form:

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (p \in \mathbb{N}). \tag{1}$$

Let $f \in \mathcal{A}(p)$ be given by (1) and $g \in \mathcal{A}(p)$ is given by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p}.$$

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The Hadamard product (or convolution) of f and g is defined by

$$(f*g)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} b_{k+p} z^{k+p} = (g*f)(z) \qquad (z \in \mathbb{U}).$$

Also, let $\mathcal{P}_k(\rho)$ be the class of functions $h(z) = 1 + \sum_{k=1}^{\infty} c_{k+p} z^{k+p}$ which are analytic in $\mathbb{U}$ and satisfying the properties $h(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\Re\{h(z)\} - \rho}{1-\rho} \right| d\theta \leq k\pi, \tag{2}$$

where $z = re^{i\theta}$, $k \geq 2$ and $0 \leq \rho < 1$. This class was introduced by Padmanabhan and Parvatham [8].
We note that:
(i) $\mathcal{P}_k(0) = \mathcal{P}_k$ $(k \geq 2)$ (see Pinchuk [9]);
(ii) $\mathcal{P}_2(\rho) = \mathcal{P}(\rho)$, the class of analytic functions with positive real part greater than ρ;
(iii) $\mathcal{P}_2(0) = \mathcal{P}$, the class of functions with positive real part. Let $h(z) \in \mathcal{P}_k(\rho)$, then we can write $h(z)$ of the form

$$h(z) = \left(\frac{k}{4} + \frac{1}{2}\right) h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) h_2(z) \quad (z \in \mathbb{U};\ h_1, h_2 \in \mathcal{P}(\rho)).$$

Let us consider the integral operator (see Aouf et al. [1]):

$$\begin{aligned} &\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z) \\ &= z^p + \frac{\Gamma(p+\alpha+\beta-\varepsilon+1)}{\Gamma(p+\beta)} \sum_{k=1}^{\infty} \left[\frac{\Gamma(p+\beta+k)}{\Gamma(p+\alpha+\beta+k-\varepsilon+1)} \right] a_{k+p} z^{k+p} \\ &\qquad (\beta > -p; \alpha \geq \varepsilon - 1; \varepsilon > 0; p \in \mathbb{N}; z \in \mathbb{U}). \end{aligned} \tag{3}$$

From (3), it is easy to verify that

$$cz(\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z))' = (p+\alpha+\beta-\varepsilon) \mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon} f(z) - (\alpha+\beta-\varepsilon) \mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z). \tag{4}$$

Remark 1.1.

(i) Putting $\varepsilon = 1$, $\mathcal{Q}_{\beta,p}^{\alpha,1} f(z) = \mathcal{Q}_{\beta,p}^{\alpha} f(z)$

$$\begin{aligned} &= z^p + \frac{\Gamma(p+\alpha+\beta)}{\Gamma(p+\beta)} \sum_{k=1}^{\infty} \left[\frac{\Gamma(p+\beta+k)}{\Gamma(p+\alpha+\beta+k)} \right] a_{k+p} z^{k+p} \\ &(\beta > -p; \alpha \geq 0; p \in \mathbb{N}; z \in \mathbb{U}), \end{aligned} \tag{5}$$

where the operator $\mathcal{Q}^{\alpha}_{\beta,p}$ is defined by Liu and Owa. [4];

(ii) Putting $\alpha=\varepsilon$ and $\beta=c$,

$$\begin{aligned}\mathcal{Q}^{\alpha,\alpha}_{c,p}f(z)=J_{c,p}&=\frac{c+p}{z^{c}}\int_0^z t^{c-1}f(t)dt\\ &=z^p+\sum_{k=1}^{\infty}\left(\frac{c+p}{c+k+p}\right)a_{k+p}z^{k+p}\\ &(c>-p;p\in\mathbb{N};z\in\mathbb{U}),\end{aligned}\tag{6}$$

where $J_{c,p}$ is the familiar integral operator which was defined by Saitoh et al. [11];

(iii) Putting $p=\varepsilon=1$, $\mathcal{Q}^{\alpha,1}_{\beta,1}f(z)=\mathcal{Q}^{\alpha}_{\beta}f(z)$

$$\begin{aligned}&=z+\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{k=2}^{\infty}\left[\frac{\Gamma(\beta+k)}{\Gamma(\alpha+\beta+k)}\right]a_kz^k\\ &(\beta>-1;\alpha>0;z\in\mathbb{U}),\end{aligned}$$

where the operator $\mathcal{Q}^{\alpha}_{\beta}$ is defined by Jung et al. [3].
Using the operator $\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p}$, we now define a subclass of $\mathcal{A}(p)$ as follows [12–14]:

Definition 1.2. Let $\beta>-1, \alpha\geq\varepsilon-1$, $\varepsilon>0$, $\mu>0$, $\lambda\in\mathbb{C}^*=\mathbb{C}\backslash\{0\}, p\in\mathbb{N}$, we say that a function $f(z)\in\mathcal{A}(p)$ is in the class $\mathcal{F}^{\alpha,\varepsilon}_{\beta,p,k}(\lambda,\mu,\rho)$ if it satisfies

$$\begin{aligned}&\left\{(1-\lambda)\left(\frac{\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p}f(z)}{\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p}g(z)}\right)^{\mu}+\lambda\left(\frac{\mathcal{Q}^{\alpha-1,\varepsilon}_{\beta,p}f(z)}{\mathcal{Q}^{\alpha-1,\varepsilon}_{\beta,p}g(z)}\right)\left(\frac{\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p}f(z)}{\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p}g(z)}\right)^{\mu-1}\right\}\\ &\in\mathcal{P}_k(\rho)\quad(z\in\mathbb{U}),\end{aligned}\tag{7}$$

where $k\geq 2$, $0\leq\rho<1$ and $g\in\mathcal{A}(p)$ satisfies the condition

$$\left(\frac{\mathcal{Q}^{\alpha-1,\varepsilon}_{\beta,p}g(z)}{\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p}g(z)}\right)\in\mathcal{P}(\eta)\quad(0\leq\eta<1;z\in\mathbb{U}).\tag{8}$$

We note that $\mathcal{F}^{\alpha,1}_{\beta,p,k}(\lambda,\mu,\rho)=\mathcal{F}^{\alpha}_{\beta,p,k}(\lambda,\mu,\rho)$ (see [6]).
In the present paper, we investigate some inclusion relationships, radius problem and some interesting properties of p-valent functions which are defined by integral operator $\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p}$.

2. Preliminaries

In this section, we recall some known results.

Lemma 2.1 ([5]). *Let $\theta(u,v)$ be a complex-valued function such that*

$$\theta : D \to \mathbb{C}, D \subset \mathbb{C}\times\mathbb{C} \qquad (\mathbb{C} \textit{ is the complex plane})$$

and let $u = u_1 + iu_2$ and $v = v_1 + iv_2$. Suppose that $\theta(u,v)$ satisfies the following conditions :
(i) $\theta(u,v)$ is continuous in D;
(ii) $(1,0) \in D$ and $\Re\{\theta(1,0)\} > 0$;
(iii) for all $(iu_2, v_1) \in D$ such that

$$v_1 \le -\frac{1}{2}(1+u_2^2) \quad \Re\{\theta(iu_2,v_1)\} \le 0.$$

Let

$$q(z) = 1 + q_1 z + q_2 z^2 + \dots$$

be analytic in $\mathbb{U}$ such that $(q(z), zq'(z)) \in D\,(z \in \mathbb{U})$. If

$$\Re\left\{\theta(q(z), zq'(z))\right\} > 0 \ \ (z \in \mathbb{U}),$$

then

$$\Re\{q(z)\} > 0 \ \ (z \in \mathbb{U}).$$

Lemma 2.2 ([10]). *If $\varphi(z)$ is analytic in $\mathbb{U}$ with $\varphi(0) = 1$ and λ_1 is a complex number satisfying $\Re(\lambda_1) \ge 0$ $(\lambda_1 \neq 0)$, then*

$$\Re\{\varphi(z) + \lambda_1 z\varphi'(z)\} > \sigma \quad (0 \le \sigma < 1).$$

Implies

$$\Re\varphi(z) > \sigma + (1-\sigma)(2\gamma_1 - 1),$$

where γ_1 is given by

$$\gamma_1 = \gamma_1(\Re(\lambda_1)) = \int_0^1 \left(1 + t^{\Re(\lambda_1)}\right)^{-1} dt,$$

which is increasing function of $\Re(\lambda_1)$ and $\frac{1}{2} \le \gamma_1 < 1$. The estimate is sharp in the sense that the bound cannot be improved.

Lemma 2.3 ([2]). *let $q(z)$ be analytic in $\mathbb{U}$ with $q(0) = 1$ and $\Re(q(z)) > 0$ $(z \in \mathbb{U})$. Then, for $|z| = r, z \in \mathbb{U}$,*

(i) $\frac{1-r}{1+r} \le \Re(q(z)) \le |q(z)| \le \frac{1+r}{1-r}$;
(ii) $|q'(z)| \le \frac{2\Re(q(z))}{1-r^2}$.

3. Main Results

Theorem 3.1. *Let* $f \in \mathcal{F}^{\alpha,\varepsilon}_{\beta,p,k}(\lambda,\mu,\rho)$ *and* $\Re(\lambda) > 0$. *Then* $\left(\frac{\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p}f(z)}{\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p}g(z)}\right)^{\mu} \in \mathcal{P}_k(\gamma)$, *where*

$$\gamma = \frac{2\mu(p+\alpha+\beta-\varepsilon)\rho + \Re(\lambda)\,\delta}{2\mu(p+\alpha+\beta-\varepsilon) + \Re(\lambda)\,\delta}, \tag{9}$$

and $g \in \mathcal{A}(p)$ *satisfies the condition (8) and*

$$\delta = \frac{\Re h_0(z)}{|h_0(z)|^2}, h_0(z) = \left(\frac{\mathcal{Q}^{\alpha-1,\varepsilon}_{\beta,p}g(z)}{\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p}g(z)}\right).$$

Proof. Set

$$\left(\frac{\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p}f(z)}{\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p}g(z)}\right)^{\mu} = (1-\gamma)h(z)+\gamma, \tag{10}$$

$h(0) = 1$, and $h(z)$ is analytic in $\mathbb{U}$ and we can write

$$h(z) = \left(\frac{k}{4}+\frac{1}{2}\right)h_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)h_2(z) \quad (h_1, h_2 \in \mathcal{P}(\gamma)). \tag{11}$$

Differentiating (10) with respect to z and using the identity (4), we have

$$\begin{aligned}
&\left\{(1-\lambda)\left(\frac{\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p}f(z)}{\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p}g(z)}\right)^{\mu} + \lambda\left(\frac{\mathcal{Q}^{\alpha-1,\varepsilon}_{\beta,p}f(z)}{\mathcal{Q}^{\alpha-1,\varepsilon}_{\beta,p}g(z)}\right)\left(\frac{\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p}f(z)}{\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p}g(z)}\right)^{\mu-1}\right\} \\
&= \left(\frac{k}{4}+\frac{1}{2}\right)\left\{[(1-\gamma)\,h_1(z)+\gamma] - \rho + \frac{\lambda\,(1-\gamma)\,zh_1'(z)}{\mu(p+\alpha+\beta-\varepsilon)h_0(z)}\right\} \\
&- \left(\frac{k}{4}-\frac{1}{2}\right)\left\{[(1-\gamma)\,h_2(z)+\gamma] - \rho + \frac{\lambda\,(1-\gamma)\,zh_2'(z)}{\mu(p+\alpha+\beta-\varepsilon)h_0(z)}\right\}.
\end{aligned}$$

Now, we form the functional $\psi(u,v)$ by choosing $u = h_i(z) = u_1 + iu_2$ and $v = zh_i'(z) = v_1 + iv_2$. Thus

$$\psi(u,v) = [(1-\gamma)\,u+\gamma] - \rho + \frac{\lambda\,(1-\gamma)\,v}{\mu(p+\alpha+\beta-\varepsilon)h_0(z)}.$$

The first and second conditions of Lemma 2.1 are satisfied by using $\psi(u,v)$. To prove the third condition:

$$\begin{aligned}
\Re\{\psi(iu_2,v_1)\} &= \gamma - \rho + \frac{\Re(\lambda)\,(1-\gamma)\,v_1\Re h_0(z)}{\mu(p+\alpha+\beta-\varepsilon)\,|h_0(z)|^2} \\
&= \gamma - \rho + \frac{\Re(\lambda)\,(1-\gamma)\,v_1}{\mu(p+\alpha+\beta-\varepsilon)}\delta, \quad \delta = \frac{\Re h_0(z)}{|h_0(z)|^2}
\end{aligned}$$

By using $v_1 \leq -\frac{1}{2}(1+u_2^2)$, we have

$$\begin{aligned}\Re\{\psi(iu_2,v_1)\} &\leq \gamma-\rho-\frac{\Re(\lambda)(1-\gamma)(1+u_2^2)}{2\mu(p+\alpha+\beta-\varepsilon)}\delta \\ &= \frac{2\mu(p+\alpha+\beta-\varepsilon)(\gamma-\rho)-\Re(\lambda)\delta(1-\gamma)-\Re(\lambda)\delta(1-\gamma)u_2^2}{2\mu(p+\alpha+\beta-\varepsilon)} \\ &= \frac{C+D}{2E}, E>0, \\ C &= 2\mu(p+\alpha+\beta-\varepsilon)(\gamma-\rho)-\Re(\lambda)\delta(1-\gamma) \\ D &= -\Re(\lambda)\delta(1-\gamma) \leq 0.\end{aligned}$$

Now, $\Re\{\psi(iu_2,v_1)\} \leq 0$, if $C \leq 0$ and this gives us γ as defined by (9). By applying Lemma 2.1 to conclude that $h_i \in \mathcal{P}$ for $z \in \mathbb{U}$ and thus $h \in \mathcal{P}_k$ which gives us the result. □

Remark 3.2. We note that $\gamma = \rho$ when $\lambda = 0$.

Theorem 3.3. *For $\lambda \geq 1$, let $f \in \mathcal{F}_{\beta,p,k}^{\alpha,\varepsilon}(\lambda,\mu,\rho)$. Then*

$$\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon}f(z)}{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon}g(z)}\right) \in \mathcal{P}_k(\rho) \quad (z \in \mathbb{U}).$$

Proof. We can write

$$\begin{aligned}\lambda\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon}f(z)}{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon}g(z)}\right) &= \left\{(1-\lambda)\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon}f(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon}g(z)}\right)+\lambda\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon}f(z)}{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon}g(z)}\right)\right\}+(\lambda-1)\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon}f(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon}g(z)}\right).\end{aligned}$$

This implies that

$$\begin{aligned}\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon}f(z)}{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon}g(z)}\right) &= \frac{1}{\lambda}\left\{(1-\lambda)\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon}f(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon}g(z)}\right)+\lambda\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon}f(z)}{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon}g(z)}\right)\right\}+(1-\frac{1}{\lambda})\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon}f(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon}g(z)}\right) \\ &= \frac{1}{\lambda}H_1(z)+\left(1-\frac{1}{\lambda}\right)H_2(z) \quad (\lambda \geq 1).\end{aligned}$$

Since $H_1(z), H_2(z) \in \mathcal{P}_k(\rho)$, by Theorem 3.1, Definition 1.2 and since $\mathcal{P}_k(\rho)$ is a convex set (see [7]), the proof of Theorem 3.3 is completed. □

Putting $g(z) = z^p$ in Theorem 3.3, we obtain the following theorem:

Theorem 3.4. *Let $\lambda \in \mathbb{C}^*$ with $\Re(\lambda) > 0$. If $f \in \mathcal{A}(p)$ satisfies the condition*

$$\left\{(1-\lambda)\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{z^p}\right)^{\mu} + \lambda\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon} f(z)}{z^p}\right)\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{z^p}\right)^{\mu-1}\right\} \in \mathcal{P}_k(\rho)$$

$$(\mu > 0; z \in \mathbb{U}),$$

then

$$\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{z^p}\right)^{\mu} \in \mathcal{P}_k(\sigma),$$

where

$$\sigma = \rho + (1-\rho)\left[2F_1(1,1,\frac{\mu(p+\alpha+\beta-\varepsilon)}{\lambda_1}+1;\frac{1}{2})-1\right].$$

The value of σ is the best possible and cannot be improved.

Proof. Setting

$$\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{z^p}\right)^{\mu} = h(z) = \left(\frac{k}{4}+\frac{1}{2}\right)h_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)h_2(z),$$

where $h(0) = 1$ and h is analytic in U. By using (4) and some computations, we have

$$\left\{(1-\lambda)\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{z^p}\right)^{\mu} + \lambda\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon} f(z)}{z^p}\right)\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{z^p}\right)^{\mu-1}\right\}$$
$$= \left\{h(z) + \frac{\lambda z h'(z)}{\mu(p+\alpha+\beta-\varepsilon)}\right\} \in \mathcal{P}_k(\rho) \quad (z \in \mathbb{U}).$$

By using Lemma 2.2, we have $h_i(z) \in \mathcal{P}_k(\sigma)$, where

$$\sigma = \rho + (1-\rho)(2\sigma_1 - 1), \tag{12}$$

with

$$\begin{aligned}\sigma_1 &= \int_0^1 (1 + t^{\frac{\Re(\lambda)}{\mu(p+\alpha+\beta-\varepsilon)}})^{-1} dt \\ &= \frac{\mu(p+\alpha+\beta-\varepsilon)}{\lambda_1}\int_0^1 u^{\frac{\mu(p+\alpha+\beta-\varepsilon)}{\lambda_1}-1}(1+u)^{-1} du, \quad (\lambda_1 = \Re(\lambda) > 0) \\ &= 2F_1(1,\frac{\mu(p+\alpha+\beta-\varepsilon)}{\lambda_1},\frac{\mu(p+\alpha+\beta-\varepsilon)}{\lambda_1}+1;-1) \\ &= 2F_1(1,1,\frac{\mu(p+\alpha+\beta-\varepsilon)}{\lambda_1}+1;\frac{1}{2}).\end{aligned}$$

Now, the proof of Theorem 3.4 is completed. □

Next, we consider the operator defined by

$$F_c = \left(\frac{p\mu + c}{z^c} \int_0^z t^{c-1}(f(t))^\mu dt \right)^{\frac{1}{\mu}} \quad (c > -p\mu; z \in \mathbb{U}). \tag{13}$$

It is clear that the function $F_c \in \mathcal{A}(p)$ and

$$z^c(\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p} F_c(z))^\mu = (p\mu + c) \int_0^z t^{c-1}(\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p} f(t))^\mu dt \quad (z \in \mathbb{U}). \tag{14}$$

Theorem 3.5. *Let $\lambda > 0, \mu > 0$ and $c > -p\mu$. If $f \in \mathcal{A}(p)$ satisfies the condition*

$$\left\{ (1-\lambda) \left(\frac{\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p} f(z)}{z^p} \right)^\mu + \lambda \frac{(\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p} f(z))'}{pz^{p-1}} \left(\frac{\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p} f(z)}{z^p} \right)^{\mu-1} \right\} \in \mathcal{P}_k(\rho) \tag{15}$$

$$(\mu > 0; z \in \mathbb{U}),$$

then

$$\left\{ (1-\lambda) \left(\frac{\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p} F_c(z)}{z^p} \right)^\mu + \lambda \frac{(\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p} F_c(z))'}{pz^{p-1}} \left(\frac{\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p} F_c(z)}{z^p} \right)^{\mu-1} \right\} \in \mathcal{P}_k(\alpha_1), \tag{16}$$

where $F_c(z)$ is given by (13) and

$$\alpha_1 = \rho + (1-\rho)\left[2F_1(1,1,p\mu+c+1;\frac{1}{2}) - 1 \right].$$

The value of α_1 is the best possible.

Proof. It is clear that $F_c \in \mathcal{A}(p)$ and differentiating both sides of (14), we have

$$(p\mu + c) z^{c-1} \left(\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p} f(z) \right)^\mu = cz^{c-1} \left(\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p} f(z) \right)^\mu + \mu z^c \left(\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p} f(z) \right)^{\mu-1} \left(\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p} f(z) \right)',$$

divide both sides by z^{c-1} and multiply by $z^{-p\mu}$, we get

$$(p\mu + c) \left(\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p} f(z) \right)^\mu = c \left(\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p} f(z) \right)^\mu + \mu z \left(\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p} f(z) \right)^{\mu-1} (\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p} F_c(z))',$$

$$(p\mu + c) \left(\frac{\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p} f(z)}{z^p} \right)^\mu = c \left(\frac{\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p} f(z)}{z^p} \right)^\mu + p\mu z^p \frac{(\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p} F_c(z))'}{pz^{p-1}} \frac{\left(\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p} f(z) \right)^{\mu-1}}{z^{p\mu}},$$

hence

$$(p\mu + c) \left(\frac{\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p} f(z)}{z^p} \right)^\mu = c \left(\frac{\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p} F_c(z)}{z^p} \right)^\mu + p\mu \frac{(\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p} F_c(z))'}{pz^{p-1}} \left(\frac{\mathcal{Q}^{\alpha,\varepsilon}_{\beta,p} F_c(z)}{z^p} \right)^{\mu-1}. \tag{17}$$

Putting

$$G(z)=\left\{(1-\lambda)\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon}F_c(z)}{z^p}\right)^{\mu}+\lambda\frac{(\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon}F_c(z))'}{pz^{p-1}}\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon}F_c(z)}{z^p}\right)^{\mu-1}\right\}, \tag{18}$$

where

$$G(z)=\left(\frac{k}{4}+\frac{1}{2}\right)g_1(z)-\left(\frac{k}{4}-\frac{1}{2}\right)g_2(z).$$

Then $G(z)$ is analytic in $\mathbb{U}$ with $G(0)=1$. By differentiating (18) and using (17) in the resulting equation, we have

$$\left\{(1-\lambda)\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon}f(z)}{z^p}\right)^{\mu}+\lambda\frac{(\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon}f(z))'}{pz^{p-1}}\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon}f(z)}{z^p}\right)^{\mu-1}\right\}$$
$$=\left\{G(z)+\frac{zG'(z)}{p\mu+c}\right\}\in\mathcal{P}_k(\rho)\ \ (z\in\mathbb{U}).$$

Using Lemma 2.2, we note that $g_i(z)\in\mathcal{P}_k(\alpha_1)$, where

$$\alpha_1=\rho+(1-\rho)(2\sigma_2-1), \tag{19}$$

with

$$\sigma_2=2F_1(1,1,p\mu+c+1;\frac{1}{2}).$$

The proof of Theorem 3.5 is completed. □

Theorem 3.6. *For* $0\le\lambda_2<\lambda_1$,

$$\mathcal{F}_{\beta,p,k}^{\alpha,\varepsilon}(\lambda_1,\mu,\rho)\subset\mathcal{F}_{\beta,p,k}^{\alpha,\varepsilon}(\lambda_2,\mu,\rho).$$

Proof. If $\lambda_2=0$, then the proof is immediate from Theorem 3.1. Let $\lambda_2>0$ and $f\in\mathcal{F}_{\beta,p,k}^{\alpha,\varepsilon}(\lambda_1,\mu,\rho)$. Then there exist two functions $H_1,H_2\in\mathcal{P}_k(\rho)$ such that

$$(1-\lambda_1)\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon}f(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon}g(z)}\right)^{\mu}+\lambda_1\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon}f(z)}{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon}g(z)}\right)\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon}f(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon}g(z)}\right)^{\mu-1}=H_1(z),$$

and

$$\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon}f(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon}g(z)}\right)^{\mu}=H_2(z).$$

Then

$$(1-\lambda_2)\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon}f(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon}g(z)}\right)^{\mu}+\lambda_2\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon}f(z)}{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon}g(z)}\right)\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon}f(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon}g(z)}\right)^{\mu-1}$$

$$=\frac{\lambda_2}{\lambda_1}H_1(z)+(1-\frac{\lambda_2}{\lambda_1})H_2(z), \tag{20}$$

and since $\mathcal{P}_k(\rho)$ is a convex set (see [8]), it follows that the right hand side of (20) belongs to $\mathcal{P}_k(\rho)$ and this completes the proof of Theorem 3.6. □

Next, we can give the converse of Theorem 3.1 as follows:

Theorem 3.7. *Let* $\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon}f(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon}g(z)}\right)^{\mu}\in\mathcal{P}_k(\rho)$, *with* $\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon}g(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon}g(z)}\right)\in\mathcal{P}(\eta)$, *for* $z\in\mathbb{U}$, $0\le\eta<1$.
Then $f\in\mathcal{F}_{\beta,p,k}^{\alpha,\varepsilon}(\lambda,\mu,\rho)$ *for* $|z|<r$, *where r is given by*

$$r=\mu(p+\alpha+\beta-\varepsilon)/\Big(\{(1-\eta)\mu(p+\alpha+\beta-\varepsilon)+|\lambda|\}$$

$$+\sqrt{\mu^2\eta^2(p+\alpha+\beta-\varepsilon)^2+|\lambda|^2+2|\lambda|(1-\eta)\mu(p+\alpha+\beta-\varepsilon)}\Big). \tag{21}$$

Proof. Let

$$\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon}f(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon}g(z)}\right)^{\mu}=H,\quad\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon}g(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon}g(z)}\right)=H_0,$$

then $H\in\mathcal{P}_k(\rho), H_0\in\mathcal{P}_k(\eta)$.
Proceeding as in Theorem 3.1, for $\beta>-1,\alpha\ge0,\mu>0,\lambda\in\mathbb{C},k\ge2,\rho\ge0,\eta<1$ and

$$H=(1-\rho)h+\rho,\quad H_0=(1-\eta)h_0+\eta,\text{ with } h\in\mathcal{P}_k, h_0\in\mathcal{P}$$

we have

$$\frac{1}{1-\rho}\left\{(1-\lambda)\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon}f(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon}g(z)}\right)^{\mu}+\lambda\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon}f(z)}{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon}g(z)}\right)\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon}f(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon}g(z)}\right)^{\mu-1}-\rho\right\}$$

$$=\left\{h(z)+\frac{\lambda}{\mu(p+\alpha+\beta-\varepsilon)}\frac{zh'(z)}{\{(1-\eta)h_0(z)+\eta\}}\right\}$$

$$=\left(\frac{k}{4}+\frac{1}{2}\right)\left[h_1(z)+\frac{\lambda}{\mu(p+\alpha+\beta-\varepsilon)}\frac{zh_1'(z)}{\{(1-\eta)h_0(z)+\eta\}}\right]$$

$$-\left(\frac{k}{4}-\frac{1}{2}\right)\left[h_2(z)+\frac{\lambda}{\mu(p+\alpha+\beta-\varepsilon)}\frac{zh_2'(z)}{\{(1-\eta)h_0(z)+\eta\}}\right].$$

Using well known estimates, (see [3]), for $h_i \in \mathcal{P}$,

$$|zh_i(z)| \leq \frac{2r\Re h_i(z)}{1-r^2}, \quad \frac{1-r}{1+r} \leq |h_i(z)| \leq \frac{1+r}{1-r},$$

we have

$$\begin{aligned}
&\Re\left\{h_i(z) + \frac{\lambda}{\mu\,(p+\alpha+\beta-\varepsilon)} \frac{zh_i'(z)}{\{(1-\eta)h_0(z)+\eta\}}\right\} \\
&\geq \Re\,(h_i(z))\left\{1 - \frac{2r|\lambda|}{\mu\,(p+\alpha+\beta-\varepsilon)} \frac{1}{1-r^2}\left(\frac{1+r}{(1-r(1-2\eta))}\right)\right\} \\
&\geq \Re\,(h_i(z))\left\{1 - \frac{2r|\lambda|}{\mu\,(p+\alpha+\beta-\varepsilon)} \frac{1}{1-r}\left(\frac{1+r}{(1-r(1-2\eta))}\right)\right\} \\
&\geq \Re\,(h_i(z))\left[\frac{\mu\,(p+\alpha+\beta-\varepsilon)\,[(1-r-(1-2\eta)r+r^2(1-2\eta)]-2r|\lambda|}{\mu\,(p+\alpha+\beta-\varepsilon)\,(1-r)\,\{1-(1-2\eta)r\}}\right] \\
&\geq \Re\,(h_i(z))\left[\frac{\mu\,(p+\alpha+\beta-\varepsilon)\,r^2(1-2\eta)-2[(1-\eta)\mu\,(p+\alpha+\beta-\varepsilon)+|\lambda|]r}{\mu\,(p+\alpha+\beta-\varepsilon)\,(1-r)\,\{1-(1-2\eta)r\}}\right. \\
&\left. + \frac{\mu\,(p+\alpha+\beta-\varepsilon)}{\mu\,(p+\alpha+\beta-\varepsilon)\,(1-r)\,\{1-(1-2\eta)r\}}\right]
\end{aligned}$$

The right hand side of the last inequality is positive for $|z| < r$, where r is given by (21).

This completes the proof of Theorem 3.7. □

Remark 3.8. Putting $\varepsilon = 1$ in the above results, we obtain the results obtained by Muhammad [6].

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