SOME STRUCTURE THEOREMS ON LOCALLY CONVEX CONES OF LINEAR OPERATORS
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In this paper we investigate the structure of $\mathcal{C}(\mathcal{P}, \mathcal{Q})$ (the cone of all continuous linear operators from locally convex cone $(\mathcal{P}, \mathcal{U})$ into locally convex cone $(\mathcal{Q}, \mathcal{W})$), when $(\mathcal{P}, \mathcal{U})$ or $(\mathcal{Q}, \mathcal{W})$ are inductive or projective limit locally convex cones. We consider some special convex quasiuniform structures on $\mathcal{C}(\mathcal{P}, \mathcal{Q})$, and prove some structure theorems.

1. Introduction

The theory of locally convex cones as developed in [5] and [13] uses an order theoretical concept or a convex quasiuniform structure to introduce a topological structure on a cone. In this paper we use the later. For recent researches see [1–3, 9, 12].

A cone is a set $\mathcal{P}$ endowed with an addition and a scalar multiplication for nonnegative real numbers. The addition is assumed to be associative and commutative, and there is a neutral element $0 \in \mathcal{P}$. For the scalar multiplication the usual associative and distributive properties hold, that is $\alpha(\beta a) = (\alpha\beta)a$, $(\alpha + \beta)a = \alpha a + \beta a$, $\alpha(a + b) = \alpha a + \alpha b$, $1a = a$ and $0a = 0$ for all $a, b \in \mathcal{P}$ and $\alpha, \beta \geq 0$.
Let $\mathcal{P}$ be a cone. A collection $\mathcal{U}$ of convex subsets $U \subseteq \mathcal{P}^2 = \mathcal{P} \times \mathcal{P}$ is called a convex quasiuniform structure on $\mathcal{P}$, if the following properties hold:

$(U_1)$ $\Delta \subseteq U$ for every $U \in \mathcal{U}$ ($\Delta = \{(a, a) : a \in \mathcal{P}\}$);

$(U_2)$ for all $U, V \in \mathcal{U}$ there is a $W \in \mathcal{U}$ such that $W \subseteq U \cap V$;

$(U_3)$ $\lambda U \circ \mu U \subseteq (\lambda + \mu) U$ for all $U \in \mathcal{U}$ and $\lambda, \mu > 0$;

$(U_4)$ $\alpha U \in \mathcal{U}$ for all $U \in \mathcal{U}$ and $\alpha > 0$.

Here, for $U, V \subseteq \mathcal{P}^2$, by $U \circ V$ we mean the set of all $(a, b) \in \mathcal{P}^2$ such that there is some $c \in \mathcal{P}$ with $(a, c) \in U$ and $(c, b) \in V$.

Let $\mathcal{P}$ be a cone and $\mathcal{U}$ be a convex quasiuniform structure on $\mathcal{P}$. We shall say $(\mathcal{P}, \mathcal{U})$ is a locally convex cone if

$(U_5)$ for each $a \in \mathcal{P}$ and $U \in \mathcal{U}$ there is some $\rho > 0$ such that $(0, a) \in \rho U$.

We say that the convex subset $E$ of $\mathcal{P}^2$ is uniformly convex whenever $E$ has properties (U1) and (U3). The uniformly convex subsets play an important role in the construction of a convex quasiuniform structure. With every collection of uniformly convex subsets we can obtain a convex quasiuniform structure (see [1], Proposition 2.2). With every convex quasiuniform structure $\mathcal{U}$ on $\mathcal{P}$ we associate two topologies: The neighborhood bases for an element $a$ in the upper and lower topologies are given by the sets

$$U(a) = \{b \in \mathcal{P} : (b, a) \in U\}, \quad \text{resp.} \quad (a)U = \{b \in P : (a, b) \in U\}, \quad U \in \mathcal{U}.$$ 

The common refinement of the upper and lower topologies is called symmetric topology. A neighborhood base for $a \in \mathcal{P}$ in this topology is given by the sets

$$U(a)U = U(a) \cap (a)U, \quad U \in \mathcal{U}.$$ 

Let $\mathcal{U}$ and $\mathcal{W}$ be convex quasiuniform structures on $\mathcal{P}$. We say that $\mathcal{U}$ is finer than $\mathcal{W}$ if for every $W \in \mathcal{W}$ there is $U \in \mathcal{U}$ such that $U \subseteq W$.

In locally convex cone $(\mathcal{P}, \mathcal{U})$ the closure of $a \in \mathcal{P}$ is defined to be the set

$$\overline{a} = \bigcap_{U \in \mathcal{U}} U(a)$$

(see [5], chapter I). The locally convex cone $(\mathcal{P}, \mathcal{U})$ is called separated if $\overline{a} = \overline{b}$ implies $a = b$ for $a, b \in \mathcal{P}$. It is proved in [5] that the locally convex cone $(\mathcal{P}, \mathcal{U})$ is separated if and only if its symmetric topology is Hausdorff.

The extended real number system $\mathbb{R} = \mathbb{R} \cup \{+\infty\}$ is a cone endowed with the usual algebraic operations, in particular $a + \infty = +\infty$ for all $a \in \mathbb{R}$, $\alpha \cdot (+\infty) = +\infty$ for all $\alpha > 0$ and $0.(+\infty) = 0$. We set $\tilde{\mathcal{V}} = \{\tilde{\varepsilon} : \varepsilon > 0\}$, where

$$\tilde{\varepsilon} = \{(a, b) \in \mathbb{R}^2 : a \leq b + \varepsilon\}.$$
Then \( \tilde{V} \) is a convex quasiuniform structure on \( \mathbb{R} \) and \((\mathbb{R}, \tilde{V})\) is a locally convex cone. For \( a \in \mathbb{R} \) the intervals \((-\infty, a + \varepsilon]\) are the upper and the intervals \([a - \varepsilon, +\infty)\) are the lower neighborhoods, while for \( a = +\infty \) the entire cone \( \mathbb{R} \) is the only upper neighborhood, and \( \{ +\infty \} \) is open in the lower topology. The symmetric topology is the usual topology on \( \mathbb{R} \) with \(+\infty\) as an isolated point.

For cones \( P \) and \( Q \), a mapping \( T : P \to Q \) is called a linear operator if \( T(a + b) = T(a) + T(b) \) and \( T(\alpha a) = \alpha T(a) \) hold for all \( a, b \in P \) and \( \alpha \geq 0 \). If both \((P, \mathcal{U})\) and \((Q, \mathcal{W})\) are locally convex cones, the operator \( T \) is called (uniformly) continuous if for every \( W \in \mathcal{W} \) one can find \( U \in \mathcal{U} \) such that \((T \times T)(U) \subseteq W\), where \((T \times T)(U) = \{(T(a), T(b)) : (a, b) \in U\}\).

A linear functional on \( P \) is a linear operator \( \mu : P \to \mathbb{R} \). The dual cone \( P^* \) of a locally convex cone \((P, \mathcal{U})\) consists of all continuous linear functionals on \( P \).

Let \((P, \mathcal{U})\) be a locally convex cone. We shall say that the subset \( F \) of \( P^2 \) is \( u \)-bounded if it is absorbed by each \( U \in \mathcal{U} \). The subset \( B \) of \( P \) is called bounded below (or above) whenever \( \{0\} \times B \) (or \( B \times \{0\} \)) is \( u \)-bounded. The subset \( B \) is called bounded if it is bounded below and above. An element \( a \in P \) is called bounded below (or above) whenever \( \{a\} \) is so (recall that every \( a \in P \) is required to be bounded below by \((U_5)\)).

The locally convex cone \((P, \mathcal{U})\) is called a uc-cone whenever \( \mathcal{U} = \{\alpha U : \alpha > 0\} \) for some \( U \in \mathcal{U} \). It is proved in [1] that the locally convex cone \((P, \mathcal{U})\) is a uc-cone if and only if \( \mathcal{U} \) has a \( u \)-bounded element.

Let \((P, \mathcal{U})\) and \((Q, \mathcal{W})\) be locally convex cones. The linear operator \( T : P \to Q \) is called \( u \)-bounded whenever for every \( u \)-bounded subset \( B \) of \( P^2 \), \((T \times T)(B) \) is \( u \)-bounded. The locally convex cone \((P, \mathcal{U})\) is called bornological if every \( u \)-bounded linear operator from \((P, \mathcal{U})\) into any locally convex cone is continuous.

The projective and inductive limits of locally convex cones have been investigated in [10]. Also, the strict inductive limit of locally convex cones has been defined in [9]. The products and direct sums as a special case of projective and inductive limits have been investigated in [8]. The dual of projective and inductive limits of locally convex cones have been investigated in [7]. In this paper we want to study the structure of \( C(P, Q) \) (the cone of continuous linear operators), when \((P, \mathcal{U})\) or \((Q, \mathcal{W})\) are the inductive or projective limit locally convex cones. The structure of \( C(P, Q) \), when \( P \) or \( Q \) are products or direct sums of some locally convex cones is an interesting special case that investigated in this paper. We review some results from [10]. For every \( \gamma \in \Gamma \) let \((P_\gamma, \mathcal{U}_\gamma)\) be a locally convex cone. If \( P \) is a cone and for every \( \gamma \in \Gamma \), \( u_\gamma \) is a linear mapping of \( P \) into \( P_\gamma \), then there is a coarsest convex quasiuniform structure \( \mathcal{U} \) on \( P \) that makes all \( u_\gamma \) continuous. \((P, \mathcal{U})\) is a locally convex cone
and it is called the projective limit of the locally convex cones \((P_\gamma, U_\gamma), \gamma \in \Gamma\). If \(P = \prod_{\gamma \in \Gamma} P_\gamma\), then \(P\) can be made into a locally convex cone by regarding it as the projective limit of the locally convex cones \((P_\gamma, U_\gamma)\) by the projections mapping \(\pi_\gamma : P \to P_\gamma, \pi_\gamma((x_\gamma)_{\gamma \in \Gamma}) = x_\gamma\).

For each \(\gamma \in \Gamma\), let \((P_\gamma, U_\gamma)\) be a locally convex cone. Suppose \(P\) is a cone and for every \(\gamma \in \Gamma\), \(\nu_\gamma : P_\gamma \to P\) is a linear mapping such that \(P = span(\bigcup_{\gamma \in \Gamma} \nu_\gamma(P_\gamma))\). Then there is the finest convex quasiuniform structure \(U\) on \(P\) that makes all \(\nu_\gamma\) continuous. \((P, U)\) is a locally convex cone and it is called the inductive limit of locally convex cones \((P_\gamma, U_\gamma), \gamma \in \Gamma\). The subcone of \(P = \prod_{\gamma \in \Gamma} P_\gamma\) spanned by \(\bigcup_{\gamma \in \Gamma} j_\gamma(P_\gamma)\), where \(j_\gamma : P_\gamma \to \prod_{\gamma \in \Gamma} P_\gamma\) is the injection mapping, is called the direct sum of cones \(P_\gamma, \gamma \in \Gamma\) and denoted by \(\bigoplus_{\gamma \in \Gamma} P_\gamma\).

If we consider the product convex quasiuniform structure on \(\bigoplus_{\gamma \in \Gamma} P_\gamma\), then it induces the original convex quasiuniform structure on each \(P_\gamma\). The finest such convex quasiuniform structure on \(\bigoplus_{\gamma \in \Gamma} P_\gamma\) is obtained by regarding \(\bigoplus_{\gamma \in \Gamma} P_\gamma\) as the inductive limit of locally convex cones \((P_\gamma, U_\gamma), \gamma \in \Gamma\) (see [8]).

Let \((P, U)\) be a locally convex cone and \(P^*\) be its dual. In the following we denote by \(U_\sigma(P, P^*)\) the coarsest convex quasiuniform structure on \(P\) that makes all \(\mu \in P^*\) continuous. Similarly, \(U_\sigma(P^*, P)\) is the coarsest convex quasiuniform structure that makes all \(a \in P\) continuous, as linear functionals on \(P^*\). In fact, \((P, U_\sigma(P, P^*))\) is the projective limit of \((\overline{\mathbb{R}}, V)\) by the functionals \(\mu \in P^*\).

2. Some structure theorems

Let \((P, U)\) and \((Q, W)\) be locally convex cones. We denote the cone of all continuous linear operators from \(P\) into \(Q\) by \(C(P, Q)\). If \((Q, W) = (\overline{\mathbb{R}}, V)\), then \(C(P, Q) = P^*\). We define a convex quasiuniform structure on \(C(P, Q)\).

Let \(B\) be a collection of bounded below subsets of \((P, U)\) such that

\[
\text{for every } A, B \in B \text{ there is } C \in B \text{ such that } A \cup B \subseteq C. 
\]

(UW)

For \(B \in B\) and \(W \in W\) we set

\[
V_{B,W} = \{(S, T) \in C(P, Q) \times C(P, Q) : (S(b), T(b)) \in W\}.
\]

Then \(V_{B,W} = \{V_{B,W} : B \in B, W \in W\}\) is a convex quasiuniform structure on \(C(P, Q)\). We prove that the elements of \(C(P, Q)\) are bounded below with respect to the convex quasiuniform structure \(V_{B,W}\). Let \(V_{B,W} \in V_{B,W}\) and \(T \in C(P, Q)\).

Since \(B\) is bounded below and \(T\) is continuous, we realize that \(T(B)\) is bounded below in \((Q, W)\). Then there is \(\lambda > 0\) such that \((0, T(b)) \in \lambda W\) for all \(b \in B\). This shows that \((0, T) \in \lambda V_{B,W}\). Therefore \((C(P, Q), V_{B,W})\) is a locally convex cone.
Let \((\mathcal{P}, \mathcal{U})\) and \((\mathcal{Q}, \mathcal{W})\) be locally convex cones. If \(\mathcal{B}\) is the collection of all bounded below or bounded subsets of \((\mathcal{P}, \mathcal{U})\), then we denote the corresponding convex quasiuniform structure on \(C(\mathcal{P}, \mathcal{Q})\) by \(\mathcal{V}_{b\mathcal{B}}\) or \(\mathcal{V}_{\mathcal{B}}\). Obviously, \(\mathcal{V}_{b\mathcal{B}}\) is finer than \(\mathcal{V}_{\mathcal{B}}\), since every bounded subset of \(\mathcal{P}\) is bounded below.

**Proposition 2.1.** Let \((\mathcal{P}, \mathcal{U})\) and \((\mathcal{Q}, \mathcal{W})\) be uc-cones. Then \(C(\mathcal{P}, \mathcal{Q})\) is a uc-cone if endowed with the convex quasiuniform structures \(\mathcal{V}_{b\mathcal{B}}\) and \(\mathcal{V}_{\mathcal{B}}\).

**Proof.** Let \(\mathcal{U} = \{\alpha U : \alpha > 0\}\) and \(\mathcal{W} = \{\alpha W : \alpha > 0\}\). We set \(B = (0)U\). We shall prove that \(\mathcal{V}_{b\mathcal{B}}\) is equivalent to the convex quasiuniform structure \(\{\epsilon V_{\mathcal{B},\mathcal{W}} : \epsilon > 0\}\) and \(\mathcal{V}_{\mathcal{B}}\) is finer than \(\mathcal{V}_{b\mathcal{B}}\). Let \(V_{A,\alpha W} \in \mathcal{V}_{b\mathcal{B}}\). Then we have \(A \subseteq \lambda B\) for some \(\lambda > 0\). We claim that \(\frac{\alpha}{\lambda} V_{\mathcal{B},\mathcal{W}} \subseteq V_{A,\alpha W}\). Let \((S, T) \in \frac{\alpha}{\lambda} V_{\mathcal{B},\mathcal{W}}\). Then \((S(\lambda a), T(\lambda a)) \in \alpha W\) for all \(a \in B\). If we set \(b = \lambda a\), then we have \((S(b), T(b)) \in \alpha W\) for all \(b \in \lambda B\). Since \(A \subseteq \lambda B\), this shows that \((S(b), T(b)) \in \alpha W\) for all \(b \in A\). Therefore \((S, T) \in V_{A,\alpha W}\).

In a similar way one can prove that \(\mathcal{V}_{\mathcal{B}}\) is equivalent to the convex quasiuniform structure \(\{\alpha V_{B,W} : \alpha > 0\}\), where \(B' = U(0)U\).

**Example 2.2.** Suppose that \((\mathcal{P}, \mathcal{U}) = (\mathcal{Q}, \mathcal{W}) = (\mathbb{R}, \tilde{V})\). Then \(C(\mathbb{R}, \mathbb{R}) = \mathbb{R}^* = [0, +\infty) \cup \{0\}\), where \(\tilde{V}\) is a functional on \((\mathbb{R}, \tilde{V})\) acting as follows:

\[
\tilde{V}(a) = \begin{cases} 
0 & a \in \mathbb{R} \\
+\infty & \text{else}.
\end{cases}
\]

In this example we have \(\mathcal{V}_{b\mathcal{B}} = \{\alpha V_{[-1, +\infty], 1} : \alpha > 0\}\) and \(\mathcal{V}_{\mathcal{B}} = \{\alpha V_{[-1, +\infty], 1} : \alpha > 0\}\) by Proposition 2.1. The upper, lower and symmetric neighborhoods of \(0\) in \((C(\mathbb{R}, \mathbb{R}), \mathcal{V}_{b\mathcal{B}})\) are as follows:

\[
V_{[-1, +\infty], 1}(0) = \{0, \tilde{V}\}, \quad \tilde{V}V_{[-1, +\infty], 1} = \{0\} \quad \text{and} \quad V_{[-1, +\infty], 1}(\tilde{V})V_{[-1, +\infty], 1} = \{\tilde{V}\}.
\]

Then the functional \(\tilde{V}\) is an isolated point in the lower and symmetric topologies of \((C(\mathbb{R}, \mathbb{R}), \mathcal{V}_{b\mathcal{B}})\). Similarly in \((C(\mathbb{R}, \mathbb{R}), \mathcal{V}_{\mathcal{B}})\) we have

\[
V_{[-1, +\infty], 1}(0) = \{0, \tilde{V}\}, \quad \tilde{V}V_{[-1, +\infty], 1} = \{0, \tilde{V}\} \quad \text{and} \quad V_{[-1, +\infty], 1}(\tilde{V})V_{[-1, +\infty], 1} = \{0, \tilde{V}\}.
\]

We shall say that a subset \(H\) of \(C(\mathcal{P}, \mathcal{Q})\) is equicontinuous whenever for each \(W \in \mathcal{W}\) there is \(U \in \mathcal{U}\) such that \((S \times S)(U) \subseteq W\) for all \(S \in H\). Every equicontinuous subset \(H\) of \(C(\mathcal{P}, \mathcal{Q})\) is bounded below with respect to the convex quasiuniform structure \(\mathcal{V}_{\mathcal{B},\mathcal{W}}\). Indeed, let \(V_{\mathcal{B},\mathcal{W}} \in \mathcal{V}_{\mathcal{B},\mathcal{W}}\). Then there is \(U \in \mathcal{U}\) such that \((S \times S)(U) \subseteq W\) for all \(S \in H\). Also, there is \(\lambda > 0\) such that \((\{0\} \times B) \subseteq \lambda U\), since \(B\) is bounded below in \((\mathcal{P}, \mathcal{U})\). We claim that \(\{0\} \times H \subseteq \lambda V_{\mathcal{B},\mathcal{W}}\). Let \(S \in H\). Then \((S \times S)(U) \subseteq W\). This shows that \((S \times S)(\frac{1}{\lambda}(\{0\} \times B)) \subseteq W\), since \(\frac{1}{\lambda}(\{0\} \times B) \subseteq U\). Therefore \((0, \frac{1}{\lambda} S(b)) \in W\) for all \(b \in B\), yields \((0, S) \in \lambda V_{\mathcal{B},\mathcal{W}}\).
Proposition 2.3. Let \((\mathcal{P}, \mathcal{U})\) and \((\mathcal{Q}, \mathcal{W})\) be locally convex cones and let \(\mathcal{B}\) be a collection of bounded below subsets of \(\mathcal{P}\) which has property \((\mathcal{UW})\). If \(\mathcal{P} = \bigcup_{B \in \mathcal{B}} B\), then for every \(a \in \mathcal{P}\) the linear operator \(\delta_a : \mathcal{C}(\mathcal{P}, \mathcal{Q}) \to \mathcal{Q}, \delta_a(T) = T(a)\) is continuous.

Proof. Let \(W \in \mathcal{W}\) and \(a \in \mathcal{P}\). There is \(B \in \mathcal{B}\) such that \(a \in B\). We prove that \((\delta_a \times \delta_a)(V_{B,W}) \subseteq W\). Let \((S,T) \in V_{B,W}\). Then \((S(b),T(b)) \in W\) for all \(b \in B\). This shows that \((S(a),T(a)) \in W\), since \(a \in B\). Then \((\delta_a(S),\delta_a(T)) \in W\). This yields that \((\delta_a \times \delta_a)(S,T) \in W\).

Theorem 2.4. Let \((\mathcal{P}, \mathcal{U})\) be the inductive limit of the locally convex cones \((\mathcal{P}_\gamma, \mathcal{U}_\gamma)\) by the linear mappings \(u_\gamma, \gamma \in \Gamma\) and let \((\mathcal{Q}, \mathcal{W})\) be a locally convex cone. Let \(\mathcal{B}_\gamma\) be a class of bounded below subsets of \(\mathcal{P}_\gamma\) for every \(\gamma \in \Gamma\), which has property \((\mathcal{UW})\), and let \(\mathcal{B}\) be the class of all finite unions of the sets contained in \(\bigcup_{\gamma \in \Gamma} u_\gamma(B_\gamma)\). Then \((\mathcal{C}(\mathcal{P}, \mathcal{Q}), \mathcal{V}_{\mathcal{B},\mathcal{W}})\) is the projective limit of the locally convex cones \((\mathcal{C}(\mathcal{P}_\gamma, \mathcal{Q}), \mathcal{V}_{\mathcal{B}_\gamma,\mathcal{W}})\) by the linear mappings \(T_\gamma : \mathcal{C}(\mathcal{P}, \mathcal{Q}) \to \mathcal{C}(\mathcal{P}_\gamma, \mathcal{Q})\), \(T_\gamma(A) = A \circ u_\gamma\), for \(A \in \mathcal{C}(\mathcal{P}, \mathcal{Q})\).

Proof. Obviously, \(\mathcal{B}\) has property \((\mathcal{UW})\). Then \((\mathcal{C}(\mathcal{P}, \mathcal{Q}), \mathcal{V}_{\mathcal{B},\mathcal{W}})\) is a locally convex cone. Now, we prove that \(T_\gamma : (\mathcal{C}(\mathcal{P}, \mathcal{Q}), \mathcal{V}_{\mathcal{B},\mathcal{W}}) \to (\mathcal{C}(\mathcal{P}_\gamma, \mathcal{Q}), \mathcal{V}_{\mathcal{B}_\gamma,\mathcal{W}})\) is continuous for each \(\gamma \in \Gamma\). Let \(V_{B_\gamma,W} \in \mathcal{V}_{\mathcal{B}_\gamma,\mathcal{W}}\). We set \(B = u_\gamma(B_\gamma)\). Obviously, we have \(B \in \mathcal{B}\). We prove that \((T_\gamma \times T_\gamma)(V_{B,W}) \subseteq V_{B_\gamma,W}\). Let \((S,A) \in V_{B,W}\). Then \((S(b),A(b)) \in W\) for all \(b \in B\). For every \(b \in B\) there is \(b_\gamma \in B_\gamma\) such that \(b = u_\gamma(b_\gamma)\). This shows that \((S \circ u_\gamma(b_\gamma),A \circ u_\gamma(b_\gamma)) \in W\) and then \((T_\gamma(S),T_\gamma(A)) \in V_{B_\gamma,W}\).

Corollary 2.5. Let \((\mathcal{P}, \mathcal{U}) = \bigoplus_{\gamma \in \Gamma}(\mathcal{P}_\gamma, \mathcal{U}_\gamma)\) and let \((\mathcal{Q}, \mathcal{W})\) be a locally convex cone. Suppose \(\mathcal{B}_\gamma\) is a class of bounded below subsets of \(\mathcal{P}_\gamma\) for every \(\gamma \in \Gamma\), which has property \((\mathcal{UW})\) and \(\mathcal{B}\) is the class of all finite unions of the sets contained in \(\bigcup_{\gamma \in \Gamma} f_\gamma(B_\gamma)\). Then \((\mathcal{C}(\mathcal{P}, \mathcal{Q}), \mathcal{V}_{\mathcal{B},\mathcal{W}}) = \prod_{\gamma \in \Gamma}(\mathcal{C}(\mathcal{P}_\gamma, \mathcal{Q}), \mathcal{V}_{\mathcal{B}_\gamma,\mathcal{W}})\).

Corollary 2.6. Let \((\mathcal{P}, \mathcal{U}) = \bigoplus_{\gamma \in \Gamma}(\mathcal{P}_\gamma, \mathcal{U}_\gamma)\). Suppose \(\mathcal{B}_\gamma\) is a class of bounded below subsets of \(\mathcal{P}_\gamma\) for every \(\gamma \in \Gamma\), which has property \((\mathcal{UW})\) and \(\mathcal{B}\) is the class of all finite unions of the sets contained in \(\bigcup_{\gamma \in \Gamma} f_\gamma(B_\gamma)\). Then \((\mathcal{P}^*, \mathcal{V}_{\mathcal{B},\mathcal{W}}) = \prod_{\gamma \in \Gamma}(\mathcal{P}_\gamma^*, \mathcal{V}_{\mathcal{B}_\gamma,\mathcal{W}})\).

Example 2.7. Let \((\mathcal{P}, \mathcal{U})\) and \((\mathcal{Q}, \mathcal{W})\) be locally convex cones and \(\sim\) be an equivalence relation on \(\mathcal{P}\) which is compatible with the algebraic operations of \(\mathcal{P}\) (see
We denote the equivalence class of an element \( a \in \mathcal{P} \) by \([a]\) and set
\[
[\mathcal{P}] = \{[a] \mid a \in \mathcal{P}\}.
\]

The operations \([a] + [b] = [a + b]\) and \(\alpha [a] = [\alpha a]\) are well-defined for \(a, b \in \mathcal{P}\) and \(\alpha \geq 0\) and \([\mathcal{P}]\) becomes a cone with these operations, which had been called the quotient cone. On \([\mathcal{P}]\) we consider the finest convex quasiuniform structure \([\mathcal{U}]\), that makes the projection mapping \(\pi : \mathcal{P} \to [\mathcal{P}], \pi(a) = [a]\) continuous. In fact, \(([\mathcal{P}], [\mathcal{U}])\) is the inductive limit of \((\mathcal{P}, \mathcal{U})\) under the projection mapping.

Suppose that \(\mathcal{B}\) is a collection of bounded below subsets of \(\mathcal{P}\), which has property \((UW)\) and suppose \([\mathcal{B}]\) is the collection of all finite unions of the sets contained in \(\pi(\mathcal{B})\). Then \((\mathcal{C}([\mathcal{P}], \mathcal{Q}), \mathcal{V}_{[\mathcal{B}], \mathcal{W}})\) is the projective limit of the locally convex cone \((\mathcal{C}(\mathcal{P}, \mathcal{Q}), \mathcal{V}_{\mathcal{B}, \mathcal{W}})\) by the linear mapping \(T : \mathcal{C}([\mathcal{P}], \mathcal{Q}) \to \mathcal{C}(\mathcal{P}, \mathcal{Q}), T(A) = A \circ \pi\) by Theorem 2.4.

Let \((\mathcal{P}, \mathcal{U})\) be a locally convex cone. For a uniformly convex \(u\)-bounded subset \(H\) of \(\mathcal{P}^2\), we set
\[
\mathcal{P}_H = \{a \in \mathcal{P} : \exists \lambda > 0, (0, a) \in \lambda H\} \text{ and } \mathcal{U}_H = \{\alpha (H \cap \mathcal{P}^2_H) : \alpha > 0\}.
\]
Then \((\mathcal{P}_H, \mathcal{U}_H)\) is a \(uc\)-cone.

**Remark 2.8.** Suppose \((\mathcal{P}, \mathcal{U})\) is a bornological cone and \(\mathcal{H}\) is the collection of all uniformly convex \(u\)-bounded subsets of \(\mathcal{P}^2\), then it is proved in [1] that \((\mathcal{P}, \mathcal{U})\) is the inductive limit of \(uc\)-subcones \((\mathcal{P}_H, \mathcal{U}_H)_{H \in \mathcal{H}}\), with the inclusion mappings \(I_H : \mathcal{P}_H \to \mathcal{P}\). Now for every \(H \in \mathcal{H}\), suppose \(\mathcal{B}_H\) is a collection of bounded below subsets of \((\mathcal{P}_H, \mathcal{U}_H)\), which has property \((UW)\) and suppose \(\mathcal{B}\) is the class of all finite unions of the sets contained in \(\bigcup_{H \in \mathcal{H}} \mathcal{B}_H\). Then \((\mathcal{C}(\mathcal{P}, \mathcal{Q}), \mathcal{V}_{\mathcal{B}_H, \mathcal{W}})\) is the projective limit of the locally convex cones \((\mathcal{C}(\mathcal{P}_H, \mathcal{Q}), \mathcal{V}_{\mathcal{B}_H, \mathcal{W}})\) by the linear mappings \(T_H : \mathcal{C}(\mathcal{P}, \mathcal{Q}) \to \mathcal{C}(\mathcal{P}_H, \mathcal{Q}), T_H(A) = AoI_H\), by Theorem 2.4. If \((\mathcal{Q}, \mathcal{W})\) is a \(uc\)-cone, then for every \(H \in \mathcal{H}\), \((\mathcal{C}(\mathcal{P}_H, \mathcal{Q}), \mathcal{V}_{\mathcal{B}_H, \mathcal{W}})\) is a \(uc\)-cone by Proposition 2.1. Therefore \((\mathcal{C}(\mathcal{P}, \mathcal{Q}), \mathcal{V}_{\mathcal{B}_H, \mathcal{W}})\) is the projective limit of \(uc\)-cones in this case.

**Definition 2.9.** Let \(\mathcal{P}\) be a cone. We shall say that the subset \(B\) of \(\mathcal{P} \setminus \{0\}\) is a base for \(\mathcal{P}\) whenever
1. for every \(a \in \mathcal{P}\) there are \(n \in \mathbb{N}\), \(b_1, \ldots, b_n \in B\) and \(\alpha_1, \ldots, \alpha_n \geq 0\) such that
\[
a = \sum_{i=1}^n \alpha_i b_i,\] in the other words \(\mathcal{P} = \text{span}(B)\),
2. for every \(B' \subset B\), \(\mathcal{P} \neq \text{span}(B')\).

Let \(B\) be a base for the cone \(\mathcal{P}\). For \(b \in B\) we set \(\mathcal{P}_b = \{\alpha b : \alpha \geq 0\}\). Then we have \(\mathcal{P} = \bigoplus_{b \in B} \mathcal{P}_b\). Indeed, (1) shows that \(\mathcal{P} \subseteq \bigoplus_{b \in B} \mathcal{P}_b\). We prove that for \(b_1, b_2 \in B\), \(\mathcal{P}_{b_1} \cap \mathcal{P}_{b_2} = \{0\}\). If \(a \in \mathcal{P}_{b_1} \cap \mathcal{P}_{b_2}\) and \(a \neq 0\), then \(a = \alpha_1 b_1 = \alpha_2 b_2\) for some \(\alpha_1, \alpha_2 > 0\). Then \(b_2 = \frac{\alpha_1}{\alpha_2} b_1\). This shows that \(\mathcal{P} = \text{span}(B \setminus \{b_1\})\).
This is a contradiction by (2). Now, we suppose that \((P, \mathcal{U})\) is a locally convex cone and for \(b \in B\), \(\mathcal{U}_b\) is the convex quasiuniform structure on \(P_b\) induced by \(\mathcal{U}\). Then it is easy to see that \((P, \mathcal{U}) = \bigoplus_{b \in B} (P_b, \mathcal{U}_b)\).

**Example 2.10.** Let \(S\) be the cone of all sequences in \(\mathbb{R}\). For \(i \in \mathbb{N}\), we define the sequences \((a^i_n)_{n \in \mathbb{N}}, (b^i_n)_{n \in \mathbb{N}}\) and \((c^i_n)_{n \in \mathbb{N}}\) as following:

\[
da^i_n = \begin{cases} 1 & n = i \\ 0 & \text{else} \end{cases}, \quad b^i_n = \begin{cases} -1 & n = i \\ 0 & \text{else} \end{cases} \quad \text{and} \quad c^i_n = \begin{cases} +\infty & n = i \\ 0 & \text{else}. \end{cases}
\]

Then \(B = \{(a^i_n)_{n \in \mathbb{N}}, (b^i_n)_{n \in \mathbb{N}}, (c^i_n)_{n \in \mathbb{N}} : i \in \mathbb{N}\}\) is a base for \(S\). For \(\delta > 0\), we set

\[
\tilde{\delta} = \{(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}} : a_n \leq b_n + \delta, \forall n \in \mathbb{N}\}.
\]

Then \(\mathcal{U} = \{\tilde{\delta} : \delta > 0\}\) is a convex quasiuniform structure on \(S\). If \(P\) is the subcone of all bounded below elements of \(S\) with respect to \(\mathcal{U}\), then \((P, \mathcal{U})\) is a locally convex cone. The above discussion yields that \((P, \mathcal{U}) = \bigoplus_{b \in B} (P_b, \mathcal{U}_b)\).

Now, let \((Q, W)\) be a locally convex cone and \(B_b\) be a collection of bounded below subsets of \(P_b\) which have \((UW)\). If we assume that \(B\) is the collection of all the sets contained in \(\bigcup_{b \in B} B_b\), then we have

\[
(C(P, Q), V_{B, W}) = \prod_{b \in B} (C(P_b, Q), V_{B_b, W}),
\]

by Corollary 2.5. For \(i \in \mathbb{N}\) and \(b = (a^i_n)_{n \in \mathbb{N}}\) or \(b = (b^i_n)_{n \in \mathbb{N}}\) we have \((P_b, \mathcal{U}_b)^* = [0, +\infty)\). Also for \(b = (c^i_n)_{n \in \mathbb{N}}\) we have \((P_b, \mathcal{U}_b)^* = \{0, +\infty\}\). Now, formula (1) with \((Q, W) = (\mathbb{R}, \bar{V})\) implies that

\[
(P^*, V_{B, \bar{V}}) = \left( \prod_{i=1}^{\infty} ([0, +\infty), V_{B_b, \bar{V}}) \right) \times \left( \prod_{i=1}^{\infty} ([0, +\infty), V_{B_b, \bar{V}}) \right) \times \left( \prod_{i=1}^{\infty} \{0, +\infty\}, V_{B_b, \bar{V}} \right).
\]

**Lemma 2.11.** In a separated locally convex cone the only bounded subcone is \(\{0\}\).

**Proof.** Let \((P, \mathcal{U})\) be a separated locally convex cone and \(Q\) be a bounded subcone of \(P\). Then for every \(U \in \mathcal{U}\) there is \(\lambda > 0\) such that \((0, q) \in \lambda U\) and \((q, 0) \in \lambda U\) for all \(q \in Q\). Let \(q \in Q\) be a fixed element. We have \((0, nq) \in \lambda U\) and \((nq, 0) \in \lambda U\) for all \(n \in \mathbb{N}\), since \(Q\) is a subcone. This yields that

\[
q \in \bigcap_{n \in \mathbb{N}} \left( \frac{\lambda}{n} U \right) (0) \left( \frac{\lambda}{n} U \right).
\]

Therefore \(q = 0\), since the symmetric topology of \((P, \mathcal{U})\) is Hausdorff.
The situation is more telling if we assume \((P, \Omega)\) to be a projective limit locally convex cone. We suppose first that \((P, \Omega) = \prod_{\gamma \in \Gamma}(P_{\gamma}, \Omega_{\gamma})\) and \((Q, W)\) is a locally convex cone. Let \(S \in C(P, Q)\). If \(S_{\gamma}\) is the restriction of \(S\) to \(P_{\gamma}\) and \(p_{\gamma}\) is the projection mapping, then for \((a_{\gamma})_{\gamma \in \Gamma} \in P\) we have \(S_{\gamma}(a_{\gamma}) = S \circ p_{\gamma}((a_{\gamma})_{\gamma \in \Gamma})\) and \(S_{\gamma} \circ p_{\gamma} = S \circ p_{\gamma} \in C(P_{\gamma}, Q)\). If only finitely many \(S_{\gamma}\) are non zero, then \(\sum_{i=1}^{\infty} S_{\gamma} \in \bigoplus_{\gamma \in \Gamma} C(P_{\gamma}, Q)\) and \(S = \sum_{i=1}^{\infty} S_{\gamma} \circ p_{\gamma} \in C(P, Q)\). This shows that

\[
\bigoplus_{\gamma \in \Gamma} C(P_{\gamma}, Q) \subset C\left(\prod_{\gamma \in \Gamma} P_{\gamma}, Q\right).
\]

Generally \(\bigoplus_{\gamma \in \Gamma} C(P_{\gamma}, Q)\) is a proper subset of \(C(\prod_{\gamma \in \Gamma} P_{\gamma}, Q)\). For example consider the cone \(P = \prod_{i=1}^{\infty} P_{i}\), where \(P_{i} = \overline{\mathbb{R}}\) for all \(i \in \mathbb{N}\). Then the range of every linear operator \(T \in \bigoplus_{i=1}^{\infty} C(P_{i}, P)\) has a base with finite elements, but it is not true for the identity mapping \(I \in C(P, P)\).

Under an additional condition we have the equality in the above.

**Proposition 2.12.** Let \((P, \Omega) = \prod_{\gamma \in \Gamma}(P_{\gamma}, \Omega_{\gamma})\), where all elements of \(P_{\gamma}\) are bounded above for all \(\gamma \in \Gamma\). Also, let \((Q, W)\) be a separated locally convex cone with a sequence \(C_{1} \subset C_{2} \subset \ldots\) of bounded subsets such that every bounded subset of \(Q\) contained in some \(C_{i}, i \in \mathbb{N}\). Then

(a) Algebraically, we have

\[
C(P, Q) = \bigoplus_{\gamma \in \Gamma} C(P_{\gamma}, Q).
\]

(b) If for every \(\gamma \in \Gamma\), \(B_{\gamma}\) is a collection of bounded below subsets of \((P_{\gamma}, \Omega_{\gamma})\) and \(B\) is the collection of all sets \(\prod_{\gamma \in \Gamma} B_{\gamma}\), where \(B_{\gamma} \in B_{\gamma}\), then the inductive limit convex quasiform structure on \(C(P, Q)\) is finer than \(V_{B,W}\).

**Proof.** For (a) assume that there exists \(S \in C(P, Q)\) such that

\[
S \notin \bigoplus_{\gamma \in \Gamma} C(P_{\gamma}, \Omega_{\gamma}).
\]

Then there are infinitely many restrictions \(S_{\gamma}, n = 1, 2, \ldots\) such that \(S_{\gamma} \neq 0\). Then there is \(a_{\gamma} \in P_{\gamma}\) such that \(b_{\gamma} = S_{\gamma}(a_{\gamma}) \notin C_{n}\) for all \(n \in \mathbb{N}\), by Lemma 2.11. The net \((a_{\gamma})_{n \in \mathbb{N}}\) is bounded in \((P, \Omega)\), since all of its component are bounded by the assumption, but \(S((a_{\gamma})_{n \in \mathbb{N}}) = \sum_{n=1}^{\infty} S_{\gamma}(a_{\gamma}) = \sum_{n=1}^{\infty} b_{n}\) is unbounded in \((Q, W)\). This is a contradiction, because \(S\) is continuous. Then

\[
C(P, Q) \subseteq \bigoplus_{\gamma \in \Gamma} C(P_{\gamma}, \Omega_{\gamma}).
\]

For (b), let \(V_{B,W} \in V_{B,W}\), where \(B = \prod_{\gamma \in \Gamma} B_{\gamma}\), \(B_{\gamma} \in B_{\gamma}\) and \(W \in W\). It is enough to show that \(\bigcup_{\gamma \in \Gamma}(j_{\gamma} \times j_{\gamma})(V_{B_{\gamma}, W}) \subseteq V_{B,W}\). For \(\gamma \in \Gamma\), let \((S_{\gamma}, T_{\gamma}) \in\)
Firstly, we prove that for every \( \gamma \in \Gamma \), we have \( (S_{\gamma}(b_{\gamma}), T_{\gamma}(b_{\gamma})) \in W \). Now, since for \((b_{\gamma})_{\gamma \in \Gamma} \in B\) we have \((j_{\gamma}(S_{\gamma}))(b_{\gamma})_{\gamma \in \Gamma} = S_{\gamma}(b_{\gamma})\) and \((j_{\gamma}(T_{\gamma}))(b_{\gamma})_{\gamma \in \Gamma} = T_{\gamma}(b_{\gamma})\) by (a), we conclude that

\[
(j_{\gamma} \times j_{\gamma})(S_{\gamma}, T_{\gamma}) \in V_{B,W}.
\]

**Theorem 2.13.** Let \((P, \Omega)\) be a locally convex cone and let \((Q, W)\) be the projective limit of the locally convex cones \((Q_{\gamma}, W_{\gamma})\) by the linear mappings \(v_{\gamma}, \gamma \in \Gamma\). If \(B\) is a collection of bounded below subsets of \((P, \Omega)\) which has property \((UW)\), then the locally convex cone \((C(P, Q), V_{B,W})\) is the projective limit of the locally convex cones \((C(P, Q_{\gamma}), V_{B,W_{\gamma}}), \gamma \in \Gamma,\) by the linear mappings \(T_{\gamma}: C(P, Q) \rightarrow C(P, Q_{\gamma}), T_{\gamma}(A) = v_{\gamma} \circ A\).

**Proof.** Firstly, we prove that for every \( \gamma, T_{\gamma} \) is continuous. Let \( V_{B,W_{\gamma}} \in V_{B,W_{\gamma}}\). Since \(v_{\gamma}\) is continuous, there is \( W \in \mathcal{W} \) such that \((v_{\gamma} \times v_{\gamma})(W) \subseteq W_{\gamma}\). We show \((T_{\gamma} \times T_{\gamma})(V_{B,W}) \subseteq V_{B,W_{\gamma}}\). If \((S, A) \in V_{B,W}\), then \((S(b), A(b)) \in W\) for all \(b \in B\). Therefore \((v_{\gamma} \circ S(b), v_{\gamma} \circ A(b)) \in W_{\gamma}\) and then \((T_{\gamma} \times T_{\gamma})(S, A) = (v_{\gamma} \circ S, v_{\gamma} \circ A) \in V_{B,W_{\gamma}}\). Now, we prove that \(V_{B,W}\) is the coarsest convex quasiuniform structure on \(\mathcal{L}(P, Q_{\gamma})\) that makes all \(T_{\gamma}, \gamma \in \Gamma\) continuous. For this aim let \(\mathcal{H}\) be another convex quasiuniform structure on \(\mathcal{C}(P, Q)\) that makes all \(T_{\gamma}, \gamma \in \Gamma\) continuous.

We shall prove that \(\mathcal{H}\) is finer than \(V_{B,W}\). There are \(n \in \mathbb{N}\) and \(\gamma_{1}, \ldots, \gamma_{n} \in \Gamma\) such that \(\bigcap_{i=1}^{n}(v_{\gamma_{i}} \times v_{\gamma_{i}})^{-1}(W_{\gamma}) \subseteq W\), since \((Q, W)\) is the projective limit of \((Q_{\gamma}, W_{\gamma}), \gamma \in \Gamma\). For every \(i = 1, \ldots, n\) there is \(H_{i} \in \mathcal{H}\) such that \((T_{\gamma} \times T_{\gamma})(H_{i}) \subseteq V_{B,W_{\gamma}}\). Since \(\mathcal{H}\) is a convex quasiuniform structure, there is \(H \in \mathcal{H}\) such that \(H \subseteq \bigcap_{i=1}^{n}H_{i}\). We claim that \(H \subseteq V_{B,W}\). Let \((S, A) \in H\). Then for every \(i = 1, \ldots, n,\) we have \((S, A) \in H_{i}\). This shows that

\[
(T_{\gamma}(S), T_{\gamma}(A)) = (v_{\gamma} \circ S, v_{\gamma} \circ A) \in V_{B,W_{\gamma}}.
\]

Then for every \(i = 1, \ldots, n,\) \((v_{\gamma} \circ S(b), v_{\gamma} \circ A(b)) \in W_{\gamma}\) for all \(b \in B\). Therefore

\[
(S(b), A(b)) \in \bigcap_{i=1}^{n}V_{\gamma}^{-1}(W_{\gamma}) \subseteq W,
\]

for all \(b \in B\). This yields that \((S, A) \in V_{B,W}\). \(\square\)

**Corollary 2.14.** Let \((P, \Omega)\) be a locally convex cone and let

\[
(Q, W) = \prod_{\gamma \in \Gamma}(Q_{\gamma}, W_{\gamma}).
\]

If \(B\) is a collection of bounded below subsets of \(P\) which has property \((UW)\), then \((C(P, Q), V_{B,W}) = \prod_{\gamma \in \Gamma}(C(P, Q_{\gamma}), V_{B,W_{\gamma}}).\)
Example 2.15. Let $(\mathcal{P}, \mathcal{U})$ and $(\mathcal{Q}, \mathcal{W})$ be locally convex cones. We consider the locally convex cone $(\mathcal{Q}, \mathcal{W}_\sigma(\mathcal{Q}, \mathcal{Q}^*))$. We note that $(\mathcal{Q}, \mathcal{W}_\sigma(\mathcal{Q}, \mathcal{Q}^*))$ is the projective limit of $(\mathbb{R}, \hat{\mathcal{V}})$ under the functionals $\mu \in \mathcal{Q}^*$. If $B$ is a collection of bounded below subsets of $(\mathcal{P}, \mathcal{U})$ which has property $(\mathcal{UW})$, then the locally convex cone $(\mathcal{C}(\mathcal{P}, \mathcal{Q}), \mathcal{V}_{B, W_\sigma(\mathcal{Q}, \mathcal{Q}^*)})$ is the projective limit of the locally convex cone $(\mathcal{P}^*, \mathcal{V}_{B, \hat{\mathcal{V}}})$ by the linear mappings $T_\mu : \mathcal{C}(\mathcal{P}, \mathcal{Q}) \to \mathcal{P}^*$, $T_\mu(A) = \mu o A$, $\mu \in \mathcal{Q}^*$, by Theorem 2.13.

In the following proposition we present some conditions under which the locally convex cone $(\mathcal{C}(\mathcal{P}, \mathcal{Q}), \mathcal{V}_{B, \hat{\mathcal{V}}})$ is separated.

**Proposition 2.16.** Let $(\mathcal{P}, \mathcal{U})$ and $(\mathcal{Q}, \mathcal{W})$ be locally convex cones and $B$ be a collection of bounded below subsets of $\mathcal{P}$, which have property $(\mathcal{UW})$. If $(\mathcal{Q}, \mathcal{W})$ is separated and $\mathcal{P} = \bigcup_{B \in B} B$, then $(\mathcal{C}(\mathcal{P}, \mathcal{Q}), \mathcal{V}_{B, \hat{\mathcal{V}}})$ is separated.

**Proof.** It is sufficient to show that the symmetric topology of $(\mathcal{C}(\mathcal{P}, \mathcal{Q}), \mathcal{V}_{B, \hat{\mathcal{V}}})$ is Hausdorff. Let $S, T \in \mathcal{C}(\mathcal{P}, \mathcal{Q})$ and $S \neq T$. There is $a \in \mathcal{P}$ such that $S(a) \neq T(a)$. Since $(\mathcal{Q}, \mathcal{W})$ is separated, there are $W, W' \in \mathcal{W}$ such that $W(S(a))W \cap W'(T(a))W' = \emptyset$.

We have $a \in B$ for some $B \in B$, since $\mathcal{Q} = \bigcup_{B \in B} B$. Now, we claim that

$$\mathcal{V}_{B, W}(S)\mathcal{V}_{B, W} \cap \mathcal{V}_{B, W'}(T)\mathcal{V}_{B, W'} = \emptyset.$$ 

If $K \in \mathcal{V}_{B, W}(S)\mathcal{V}_{B, W} \cap \mathcal{V}_{B, W'}(T)\mathcal{V}_{B, W'}$, then

$$K(a) \in W(S(a))W \cap W'(T(a))W',$$

and this is a contradiction. \qed

**Example 2.17.** Let $(\mathcal{P}, \mathcal{U})$ be a locally convex cone and $B$ be the collection of all finite subsets of $\mathcal{P}$. If we set $(\mathcal{Q}, \mathcal{W}) = (\mathbb{R}, \hat{\mathcal{V}})$, then $\mathcal{P}^* = \mathcal{C}(\mathcal{P}, \mathcal{Q})$, endowed with the convex quasiuniform structure $\mathcal{V}_{B, \hat{\mathcal{V}}}$ is a separated locally convex cone by Proposition 2.16. We note that the convex quasiuniform structure $\mathcal{V}_{B, \hat{\mathcal{V}}}$ is equivalent with $\mathcal{U}_\sigma(\mathcal{P}^*, \mathcal{P})$ on $\mathcal{P}^*$. Then the locally convex cone $(\mathcal{P}^*, \mathcal{U}_\sigma(\mathcal{P}^*, \mathcal{P}))$ is separated.
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