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BIHARMONIC GREEN FUNCTIONS

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The harmonic Green and Neumann function and a particular Robin function are used to construct bi-harmonic Green, Neumann and particular Robin functions. Moreover hybrid bi-harmonic Green functions are given. They all are constructed via a convolution of the mentioned harmonic particular fundamental solutions. In case of the unit disc they are explicitly expressed. Besides these 9 bi-harmonic Green functions there is another bi-harmonic Green function in explicit form for the unit disc not defined by convolution.

Related boundary value problems are not all well posed. In case they are, the unique solutions are given. For the other cases solvability conditions are determined and the unique solutions found. There are all together 12 Dirichlet kind boundary value problems for the inhomogeneous bi-harmonic equation treated.

The investigation is restricted to the two dimensional case and complex notation is used.

1. Introduction.

Fundamental solutions for higher order differential operators can be constructed from ones for lower order operators by an integration process.

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Proper primitives have to be found. Having got a fundamental solution then desires proper modification to adjust it to boundary conditions. What boundary conditions however are appropriate for a given differential operator is not right away obvious. If the differential operator is factorizable and boundary value problems available for each factor then an iteration process produces proper boundary conditions [4, 6, 7, 8, 9, 10, 13, 14, 15, 16, 17]. The classical Dirichlet problem for the bi-Laplacian is the problem

$$w = \gamma_0, \partial_\nu w = \gamma_1 \text{on} \partial D$$

for a regular (bounded smooth) domain where ∂_ν denotes the outward normal derivative [9, 18, 21]. Although the bi-Laplacian is a product of the Laplacian with itself this Dirichlet problem can not be decomposed in a bi-vector - valued Dirichlet problem for the Laplace operator.

But the system

$$\begin{aligned} w_{z\bar{z}} &= \omega \text{ in } D, \quad w = \gamma_0 \text{ on } \partial D \\ \omega_{z\bar{z}} &= f \text{ in } D, \quad \omega = \gamma_2 \text{ on } \partial D \end{aligned}$$

leads to the well posed bi-harmonic Dirichlet problem

$$(\partial_z \partial_{\bar{z}})^2 w = f \text{in } D, w = \gamma_0, w_{z\bar{z}} = \gamma_2 \text{ on } \partial D.$$

Using also Neumann and a Robin condition several proper boundary value problems occur [5, 9, 12]. They are treated by contructing related bi-harmonic Green functions. It turns out that these problems differ essentially from the one stated at first.

For this exceptional case see [5, 18] and also [11, 20, 21] for the particular case of the upper half plane. Part of this work is attained in [9]. Higher order Green functions, i.e. polyharmonic Green functions are given in [1, 9, 20] and used in [2, 3, 21] for orthogonal decompositions of $L_2(\mathbb{D}; \mathbb{C})$.

2. Harmonic Green functions.

Three harmonic Green functions are available. The first is characterized by the properties

- $G_1(\cdot, \zeta)$ is harmonic in $D \backslash \{\zeta\}$ for any $\zeta \in D$
- $G_1(z, \zeta) + \log |\zeta - z|^2$ is harmonic in $z \in D$ for any $\zeta \in D$
- $G_1(\cdot, \zeta) = 0$ on $\partial \mathbb{D}$ for any $\zeta \in D$.

is a symmetric function $G_2(z,\zeta) = G_1(\zeta, z)$.

Remark. The outward normal derivative $\partial_{\nu_z} G_1(z,\zeta)$ for $z \in \partial D, \zeta \in D$ is the -2 fold of the Poisson kernel function.

These second Green function also called Neumann function is determined by

- $N_1(\cdot,\zeta)$ is harmonic in $D\backslash\{\zeta\}$ for any $\zeta \in D$
- $N_1(z,\zeta) + \log|\zeta - z|^2$ is harmonic in $z \in D$ for any $\zeta \in D$
- $\partial_{\nu_z} N_1(z,\zeta) = -2$ on ∂D for any $\zeta \in D$
- $\dfrac{1}{2\pi}\displaystyle\int_{\partial D} N_1(z,\zeta) ds_z = 0$ for any $\zeta \in D$.

It is a symmetric function $N_1(z,\zeta) = N_1(\zeta, z)$.

The third Green function is the Robin function. In a particular case it is given by

- $R_1(\cdot,\zeta)$ is harmonic in $D\backslash\{\zeta\}$ for any $\zeta \in D$
- $R_1(z,\zeta) + \log|\zeta - z|^2$ is harmonic in $z \in D$ for any $\zeta \in D$
- $R_1(z,\zeta) + \partial_{\nu_z} R_1(z,\zeta) = 0$ on ∂D for any $\zeta \in D$.

It turns out to be symmetric $R_1(z,\zeta) = R_1(\zeta, z)$.

Remark. Often just 1/2 of the functions above arre taken as Green, Neumann, Robin function for the Laplace operator denoted by G, N, R. In the case of the unit disc $\mathbb{D}$ they are

$$G_1(z,\zeta) = \log\left|\frac{1 - z\bar{\zeta}}{\zeta - z}\right|^2$$

$$N_1(z,\zeta) = -\log|(1 - z\zeta)(\zeta - z)|^2$$

$$R_1(z,\zeta) = \log\left|\frac{1 - z\bar{z}}{\zeta - z}\right|^2 - 2\Big[\frac{\log(1 - z\bar{z})}{z\bar{z}} + \frac{\log(1 - \bar{z}\zeta)}{\bar{z}\zeta} + 1\Big].$$

Green functions are related to Dirichlet problems.

Theorem 1. *The Dirichlet problem*

$$w_{z\bar{z}} = f \quad \textit{in } \mathbb{D}, w = \gamma \textit{ on } \partial\mathbb{D}, f \in L_1(\mathbb{D};\mathbb{C}), \gamma \in C(\partial\mathbb{D};\mathbb{C})$$

is uniquely solvably. The solution is

$$w(z) = -\frac{1}{4\pi i} \int\limits_{\mathbb{D}} \partial_{\nu_\zeta} G_1(z,\zeta)\gamma(\zeta)\frac{d\zeta}{\zeta} - \frac{1}{\pi}\int\limits_{\mathbb{D}} G_1(z,\zeta) f(\zeta) d\xi d\eta.$$

Theorem 2. *The Neumann problem*

$$w_{z\bar{z}} = f \text{in} \mathbb{D}, \partial_\nu w = \gamma \text{on } \mathbb{D}, \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} w(z)\frac{dz}{z} = c,$$

$$f \in L_1(\mathbb{D};\mathbb{C}), \gamma \in C(\partial\mathbb{D};\mathbb{C}), c \in \mathbb{C}$$

is uniquely solvable if and only if

$$\frac{1}{2\pi}\int\limits_{\partial\mathbb{D}} \gamma(\zeta) ds = \frac{2}{\pi}\int_{\mathbb{D}} f(\zeta) d\xi d\eta.$$

The solution is

$$w(z) = c + \frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}} N_1(z,\zeta)\gamma(\zeta)\frac{d\zeta}{\zeta} - \frac{1}{\pi}\int\limits_{\mathbb{D}} N_1(z,\zeta) f(\zeta) d\xi d\eta.$$

Theorem 3. *The Robin problem*

$$w_{z\bar{z}} = f \text{ in } \mathbb{D}, w + \partial_\nu w = \gamma \text{ on } \partial\mathbb{D}, f \in L_1(\mathbb{D};\mathbb{C}), \gamma \in C(\partial\mathbb{D};\mathbb{C})$$

is uniquely solvable. The solution is

$$w(z) = \frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}} R_1(z,\zeta)\gamma(\zeta)\frac{d\zeta}{\zeta} - \frac{1}{\pi}\int\limits_{\mathbb{D}} R_1(z,\zeta) f(\zeta) d\xi d\eta.$$

3. Biharmonic Green functions.

Biharmonic Green functions can be determined by convoluting harmonic ones. They are

$$G_2(z,\zeta) = -\frac{1}{\pi}\int_D G_1(z,\tilde{z})G_1(\tilde{z},\zeta)d\tilde{\xi}d\tilde{\eta},$$

$$H_2(z,\zeta) = -\frac{1}{\pi}\int_D G_1(z,\tilde{z})N_1(\tilde{z},\zeta)d\tilde{\xi}d\tilde{\eta},$$

$$N_2(z,\zeta) = -\frac{1}{\pi}\int_D N_1(z,\tilde{z})N_1(\tilde{z},\zeta)d\tilde{\xi}d\tilde{\eta},$$

$$I_2(z,\zeta) = -\frac{1}{\pi}\int_D G_1(z,\tilde{z})R_1(\tilde{z},\zeta)d\tilde{\xi}d\tilde{\eta},$$

$$R_2(z,\zeta) = -\frac{1}{\pi}\int_D R_1(z,\tilde{z})R_1(\tilde{z},\zeta)d\tilde{\xi}d\tilde{\eta},$$

$$J_2(z,\zeta) = -\frac{1}{\pi}\int_D N_1(z,\tilde{z})R_1(\tilde{z},\zeta)d\tilde{\xi}d\tilde{\eta}.$$

A better notation is

$$G_{11}:=G_1,\ \ G_{12}:=N_1,\ \ G_{13}:=R_1$$

$$G_{21}:=G_2,\ \ G_{22}:=H_2,\ \ G_{23}:=I_2, G_{24}:=N_2,\ \ G_{25}:=J_2,\ \ G_{26}:=R_2.$$

The properties of these six bi-harmonic Green functions follow from those of G_{1k}, $1 \le k \le 3$, together with Theorems 1, 2, 3. In particular

$$\partial_z\partial_{\bar{z}}G_{21} = G_{11} \in D,\ \ G_{21}=0 \text{ on } \partial D \text{ for } \zeta \in D$$

$$\partial_z\partial_{\bar{z}}G_{22} = G_{12} \text{ in } D,\ \ G_{22}=0 \text{ on } \partial D \text{ for } \zeta \in D$$

$$\partial_z\partial_{\bar{z}}G_{23} = G_{13} \text{ in } D,\ \ G_{23}=0 \text{ on } \partial D \text{ for } \zeta \in D$$

$$\partial_z\partial_{\bar{z}}G_{24} = G_{12} \text{ in } D,\ \ \partial_{\nu_z}G_{24} = 2(1-|\zeta|^2) \text{ on } \partial D,$$
$$\frac{1}{2\pi}\int_{\partial D} G_{24}(z,\zeta)ds_z = 0 \text{ for } \zeta \in D$$

$$\partial_z\partial_{\bar z}G_{25} = G_{13} \text{ in } D,\ \ \partial_{\nu_z}G_{25} = 2(1-|\zeta|^2) \text{ on } \partial D,$$
$$\frac{1}{2\pi}\int_{\partial D} G_{25}(z,\zeta)ds_z = 0 \text{ for } \zeta\in D$$
$$\partial_z\partial_{\bar z}G_{26} = G_{13} \text{ in } D, G_{26}+\partial_{\nu_z}G_{26} = 0 \text{ on } \partial D \text{ for } \zeta\in D.$$

For the unit disc they are

$$G_{21}(z,\zeta) = |\zeta - z|^2\log\left|\frac{1-z\bar z}{\zeta - z}\right|^2 + (1-|z|^2)(1-|\zeta|^2)\left[\frac{\log(1-z\bar z)}{z\bar z} + \frac{\log(1-z\bar z)}{\bar z\zeta}\right]$$

$$G_{22}(z,\zeta) = -|\zeta-z|^2\log|\zeta-z|^2 - (1-|z|^2)\left[4+\frac{1-z\bar z}{z\bar z}\log(1-z\bar z)+\frac{1-\bar z\zeta}{\bar z\zeta}\log(1-\bar z\zeta)\right] - \frac{(\zeta-z)(1-z\bar z)}{z}\log(1-z\bar z) - \frac{(\overline{\zeta-z}(1-\bar z\zeta)}{\bar z}\log(1-\bar z\zeta)$$

$$G_{23}(z,\zeta) \ = G_{21}(z,\zeta) - 2(1-|z|^2)\left[\sum_{k=1}^{\infty}\frac{1}{k^2}\big[(z\bar z)^{k-1}+(\bar z\zeta)^{k-1}\big]-1\right]$$

$$G_{24}(z,\zeta) \ = |\zeta-z|^2\left[4-\log|(\zeta-z)(1-z\bar z)|^2 - 4\sum_{k=2}^{\infty}\frac{1}{k^2}\big[(z\bar z)^k+(\bar z\zeta)^k\big]\right] - 2(z\bar z+\bar z\zeta)\log(1-z\bar z)^2 - (1+|z|^2)(1+|\zeta|^2) \times\left[\frac{\log(1-z\bar z)}{z\bar z}+\frac{\log(1-\bar z\zeta)}{\bar z\zeta}\right]$$

$$
\begin{aligned}
G_{25}(z,\zeta) = &-|\zeta - z|^2 \log|\zeta - z|^2 - (1-|\zeta|^2) \\
&\times \left[4 + \frac{1-z\bar{z}}{z\bar{z}}\log(1-z\bar{z}) + \frac{1-\bar{z}\zeta}{\bar{z}\zeta}\log(1-\bar{z}\zeta)\right] \\
&+ \frac{(\zeta - z)(1-\bar{z}\zeta)}{\zeta}\log(1-\bar{z}\zeta) + \frac{(\overline{\zeta - z})(1-z\bar{z})}{\bar{z}}\log(1-z\bar{z}) \\
&- 4\log|1-\bar{z}\zeta|^2 + 4\left[\frac{\log(1-z\bar{z})}{z\bar{z}} + \frac{\log(1-\bar{z}\zeta)}{\bar{z}\zeta}\right] \\
&- 2(1+|z|^2)\sum_{k=1}^{\infty}\frac{(z\bar{z})^{k-1} + (\bar{z}\zeta)^{k-1}}{k^2} + 16 - 2(1-|z|^2)
\end{aligned}
$$

$$
\begin{aligned}
G_{26}(,z\zeta) = &|\zeta - z|^2 \log\left|\frac{1-z\bar{z}}{\zeta - z}\right|^2 \\
&+ (1-|z|^2)(1-|\zeta|^2)\left[\frac{\log(1-z\bar{z})}{z\bar{z}} + \frac{\log(1-\bar{z}\zeta)}{\bar{z}\zeta}\right] \\
&-2(2-|z|^2-|\zeta|^2)\sum_{k=1}^{\infty}\frac{(z\bar{z})^k+(\bar{z}\zeta)^k}{(k+1)^2} - 4 - 4\sum_{k=1}^{\infty}\frac{(z\bar{z})^k+(\bar{z}\zeta)^k}{(k+1)^3}.
\end{aligned}
$$

Another bi-harmonic Green function for $\mathbb{D}$ is, see [1],

$$
G_2(z,\zeta) = |\zeta - z|^2 \log\left|\frac{1-z\bar{z}}{\zeta - z}\right|^2 - (1-|\zeta|^2(1-|\zeta|^2)
$$

satisfying

$$
\begin{aligned}
&\partial_z\partial_{\bar{z}}G_2(z,\zeta) = G_1(z,\zeta) - g_1(z,\zeta)(1-|\zeta|^2) \text{ in } \mathbb{D} \text{ for } \zeta \in \mathbb{D} \\
&G_2(z,\zeta) = 0, \partial_z G_2(z,\zeta) = 0, \partial_{\bar{z}} G_2(z,\zeta) = 0 \text{ on } \partial\mathbb{D} \text{ for } \zeta \in \mathbb{D}.
\end{aligned}
$$

While G_{21}, G_{24}, G_{26} and G_2 are symmetric in z and ζ G_{22}, G_{23}, G_{25} are not. From

$$
G_{22}(z,\zeta) = -\frac{1}{\pi}\int_D N_1(\zeta,\tilde{z})G_1(\tilde{z},z)d\tilde{\xi}d\tilde{\eta}
$$

$$G_{23}(z,\zeta) = -\frac{1}{\pi}\int_D R_1(\zeta,\tilde{z})G_1(\tilde{z},z)d\tilde{\xi}d\tilde{\eta}$$

$$G_{25}(z,\zeta) = -\frac{1}{\pi}\int_D R_1(\zeta,\tilde{z})N_1(\tilde{z},z)d\tilde{\xi}d\tilde{\eta}$$

the properties

$$\partial_\zeta\partial_{\bar{z}}G_{22}(z,\zeta) = G_1(z,\zeta) \text{ in } D, \partial_{\nu_\zeta}G_{22}(z,\zeta) = 2(1-|z|^2) \text{ on } \partial\mathbb{D},$$
$$\frac{1}{2\pi}\int_{\partial D} G_{22}(z,\zeta)ds_\zeta = 0 \text{ for } z\in D$$
$$\partial_\zeta\partial_{\bar{z}}G_{23}(z,\zeta) = G_1(z,\zeta) \text{ in} D,\ \ G_{23}(z,\zeta) + \partial_{\nu_\zeta}G_{23}(z,\zeta) = 0$$
$$\text{on } \partial D \text{ for } z\in D$$
$$\partial_\zeta\partial_{\bar{z}}G_{25}(z,\zeta)=N_1(z,\zeta), G_{25}(z,\zeta)+\partial_{\nu_\zeta}G_{25}(z,\zeta)=0 \text{ on } \partial D \text{ for } z\in D$$

follow. Hence, there are alltogether 10 bi-harmonic Green functions listed. More are available by replacing the particular Robin boundary operator $id + \partial_\nu$ by other linear combinations of both operators appearing.

Green functions are always related to Dirichlet boundary value problems.

4. Bi-harmonic Dirichlet problems.

Some of the boundary value problems for the inhomogeneous bi-harmonic equation can be decomposed into two problems for the harmonic operator. The solutions then can be attained by iterating the results from Theorems 1, 2 and 3, respectively. Others may not be decomposed. They can be solved by evaluating the area integral of the product of the bi-Laplacian applied to the unknown function and the respective Green function, see [19]. Moreover, not all problems are wellposed and hence, occasionally solvability conditions become involved, see also Theorem 2 above. For simplicity $D = \mathbb{D}$ is considered. The simplest way of proof is by verification plus some argumentation for uniqueness.

Theorem 4. *The Dirichlet-2 problem*

$$(\partial_z\partial_{\bar{z}})^2 w = f \ \textit{in}\ \mathbb{D}, w = \gamma_0, w_{z\bar{z}} = \gamma_2 \ \textit{on}\ \partial\mathbb{D}$$

is uniquely solvable for $f \in L_1(\mathbb{D};\mathbb{C}), \gamma_0, \gamma_2 \in (\partial\mathbb{D};\mathbb{C})$ *by*

$$w(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} [g_1(z,\zeta)\gamma_0(\zeta) + \hat{g}_2(z,\zeta)\gamma_2(\zeta)]\frac{d\zeta}{\zeta} - \frac{1}{\pi}\int_{\mathbb{D}} G_{21}(z,\zeta)f(\zeta)d\xi d\eta.$$

Here

$$\hat{g}_2(z,\zeta) = -\frac{1}{\pi}\int_{\mathbb{D}} G_1(z,\tilde{z})g_1(\tilde{z},\zeta)d\tilde{\xi}d\tilde{\eta} = (1-|z|^2) \Big[\frac{\log(1-z\tilde{z})}{z\tilde{z}} + \frac{\log(1-\bar{z}\zeta)}{\bar{z}\zeta} + 1\Big]$$

is satisfying

$$\partial_z\partial_{\bar{z}}\hat{g}_2(z,\zeta) = g_1(z,\zeta) \text{ for } z,\zeta \in \mathbb{D}, \hat{g}_2(z,\zeta) = 0 \text{ for } z \in \partial\mathbb{D}, \zeta \in \mathbb{D}.$$

Theorem 5. *The Dirichlet-Neumann problem*

$$(\partial_z\partial_{\bar{z}})^2 w = f \text{ in } \mathbb{D},\ \ w = \gamma_0, \partial_\nu w_{z\bar{z}} = \gamma_3 \text{ on } \partial\mathbb{D}, c_2 = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} w_{z\bar{z}}(z)\frac{dz}{z}$$

is uniquely solvable for $f \in L_1(\mathbb{D};\mathbb{C}), \gamma_0, \gamma_3 \in (\partial\mathbb{D};\mathbb{C}), c_2 \in \mathbb{C}$ *if and only if*

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma_3(\zeta)\frac{d\zeta}{\zeta} = \frac{2}{\pi}\int_{\mathbb{D}} f(\zeta)d\xi d\eta.$$

The solution is

$$w(z) = (|z|^2-1)c_2 + \frac{1}{2\pi i}\int_{\partial\mathbb{D}} g_1(z,\zeta)\gamma_0(\zeta)\frac{d\zeta}{\zeta} + \frac{1}{4\pi i}\int_{\partial\mathbb{D}} G_{22}(z,\zeta)\gamma_3(\zeta)\frac{d\zeta}{\zeta} - \frac{1}{\pi}\int_{\mathbb{D}} G_{22}(z,\zeta)f(\zeta)d\xi d\eta.$$

Theorem 6. *Neumann-Dirichlet problem*

$$(\partial_z\partial_{\bar{z}})^2 w = f \text{ in } \mathbb{D}, \partial_\nu w = \gamma_1, w_{z\bar{z}} = \gamma_2 \text{ on } \partial\mathbb{D}, c_0 = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} w(z)\frac{dz}{z}$$

is uniquely solvable for $f \in L_1(\mathbb{D};\mathbb{C}), \gamma_1, \gamma_2 \in C(\partial\mathbb{D};\mathbb{C}), c_0 \in \mathbb{C}$ *if and only if*

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}}(\gamma_1(\zeta)+2\gamma_2(\zeta))\frac{d\zeta}{\zeta} = \frac{2}{\pi}\int_{\mathbb{D}} f(\zeta)(1-|\zeta|^2)d\xi d\eta.$$

The solution is

$$w(z) = c_0 + \frac{1}{4\pi i}\int_{\partial\mathbb{D}} N_1(z,\zeta)\gamma_1(\zeta)\frac{d\zeta}{\zeta} - \frac{1}{4\pi i}\int_{\partial\mathbb{D}} \gamma_2(\zeta)\partial_{\nu_\zeta} G_{22}(\zeta,z)\frac{d\zeta}{\zeta}$$
$$-\frac{1}{\pi}\int_{\mathbb{D}} f(\zeta)G_{22}(\zeta,z)d\xi d\eta.$$

Theorem 7. *The Dirichlet-Robin problem*

$$(\partial_z\partial_{\bar z})^2 w = f \quad in\ \mathbb{D},\ w=\gamma_0,\ w_{z\bar z}+\partial_\nu w_{z\bar z}=\gamma_3\ on\ \partial\mathbb{D}$$

is uniquely solvable for $f \in L_1(\mathbb{D};\mathbb{C}), \gamma_0, \gamma_3 \in C(\partial\mathbb{D};\mathbb{C})$ *by*

$$w(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} g_1(z,\zeta)\gamma_0(\zeta)\frac{d\zeta}{\zeta} + \frac{1}{4\pi i}\int_{\partial\mathbb{D}} G_{23}(z,\zeta)\gamma_3(\zeta)\frac{d\zeta}{\zeta}$$
$$-\frac{1}{\pi}\int_{\mathbb{D}} G_{23}(z,\zeta)f(\zeta)d\xi d\eta.$$

Theorem 8. *The Robin-Dirichlet problem*

$$(\partial_z\partial_{\bar z})w = f \quad in\ \mathbb{D},\ w+\partial_\nu w=\gamma_1,\ \partial_z\partial_{\bar z}w=\gamma_2\ on\ \ \partial\mathbb{D}$$

is uniquely solvable for $f \in L_1(\mathbb{D};\mathbb{C}), \gamma_1, \gamma_2 \in C(\partial\mathbb{D};\mathbb{C})$ *by*

$$w(z) = \frac{1}{4\pi i}\int_{\partial\mathbb{D}} R_1(z,\zeta)\gamma_1(\zeta)\frac{d\zeta}{\zeta} - \frac{1}{4\pi i}\int_{\partial\mathbb{D}} \gamma_2(\zeta)\partial_{\nu_\zeta} G_{23}(\zeta,z)\frac{d\zeta}{\zeta}$$
$$-\frac{1}{\pi}\int_{\mathbb{D}} f(\zeta)G_{23}(\zeta,z)d\xi d\eta.$$

Theorem 9. *The Neumann-2 problem*

$$(\partial_z\partial_{\bar z})^2 w = f\ \ in\ \mathbb{D},\ \partial_\nu w=\gamma_1,\ \partial_\nu w_{z\bar z}=\gamma_3\ on\ \partial\mathbb{D},$$
$$c_0 = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} w(z)\frac{dz}{z},\ c_2 = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} w_{z\bar z}(z)\frac{dz}{z}$$

is uniquely solvable for $f \in L_1(\mathbb{D};\mathbb{C}), \gamma_1, \gamma_3 \in C(\partial\mathbb{D};\mathbb{C}), c_0, c_2 \in \mathbb{C}$ *if and only if*

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma_1(\zeta)\frac{d\zeta}{\zeta} = 2c_2 + \frac{2}{\pi}\int_{\mathbb{D}} f(\zeta)(1-|\zeta|^2)d\xi d\eta$$

and

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma_2(\zeta)\frac{d\zeta}{\zeta} = \frac{2}{\pi}\int_{\mathbb{D}} f(\zeta)d\xi d\eta.$$

The solution is

$$w(z) = \; c_0 - (1-|z|^2)c_2 + \frac{1}{4\pi i}\int_{\partial\mathbb{D}} \big[N_1(z,\zeta)\gamma_1(\zeta) - N_2(z,\zeta)\gamma_3(\zeta)\big]\frac{d\zeta}{\zeta}$$
$$-\frac{1}{\pi}\int_{\mathbb{D}} N_2(z,\zeta)f(\zeta)d\xi d\eta.$$

Theorem 10. *The Neumann-Robin problem*

$$(\partial_z\partial_{\bar z})^2 w = f \text{ in } \mathbb{D}, \partial_\nu w = \gamma_1, \partial_z\partial_{\bar z}w + \partial_\nu\partial_z\partial_{\bar z}w$$
$$= \gamma_3 \text{ on } \partial\mathbb{D}, c_0 = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} w(z)\frac{dz}{z}$$

is uniquely solvable for $f \in L_1(\mathbb{D};\mathbb{C}), \gamma_1, \gamma_3 \in C(\partial\mathbb{D};\mathbb{C}), c_0 \in \mathbb{C}$ *if and only if*

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \big[\gamma_1(\zeta) - (3-|\zeta|^2)\gamma_3(\zeta)\big]\frac{d\zeta}{\zeta} + \frac{2}{\pi}\int_{\mathbb{D}} (3-|\zeta|^2)f(\zeta)d\xi d\eta.$$

The solution is

$$w(z) = c_0 + \frac{1}{4\pi i}\int_{\partial\mathbb{D}} \big[N_1(z,\zeta)\gamma_1(\zeta) + G_{25}(z,\zeta)\gamma_3(\zeta)\big]$$
$$\frac{d\zeta}{\zeta} - \frac{1}{\pi}\int_{\mathbb{D}} G_{25}(z,\zeta)f(\zeta)d\xi d\eta.$$

Theorem 11. *The Robin-Neumann problem*

$$(\partial_z\partial_{\bar z})^2 w = f \text{ in } \mathbb{D}, w+\partial_\nu w = \gamma_1, \; \partial_\nu\partial_z\partial_{\bar z} = \gamma_3 \text{ on } \partial\mathbb{D}, c_2 = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} w_{z\bar z}(z)\frac{dz}{z}$$

is uniquely solvable for $f \in L_1(\mathbb{D};\mathbb{C}), \gamma_1, \gamma_3 \in C(\partial\mathbb{D};\mathbb{C}), c_2 \in \mathbb{C}$ *if and only if*

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma_3(\zeta)\frac{d\zeta}{\zeta} = \frac{2}{\pi}\int_{\mathbb{D}} f(\zeta)d\xi d\eta.$$

The solution is

$$w(z) = (|z|^2-1)c_2+\frac{1}{4\pi i}\int_{\partial\mathbb{D}} R_1(z,\zeta)\gamma_1(\zeta)\frac{d\zeta}{\zeta}+\frac{1}{4\pi i}\int_{\partial\mathbb{D}} \gamma_3(\zeta)G_{25}(\zeta,z)\frac{d\zeta}{\zeta}$$

$$-\frac{1}{\pi}\int_{\mathbb{D}} f(\zeta)G_{25}(\zeta,z)d\xi d\eta.$$

Theorem 12. *The Robin-2 problem*

$$(\partial_z\partial_{\bar z})^2 w = f \text{in } \mathbb{D},\, w+\partial_\nu w = \gamma_1,\, w_{z\bar z}+\partial_\nu w_{z\bar z} = \gamma_3 \text{ on } \partial\mathbb{D}$$

is uniquely solvable for $f \in L_1(\mathbb{D};\mathbb{C}), \gamma_1, \gamma_3 \in C(\mathbb{D};\mathbb{C})$ *by*

$$w(z) = \frac{1}{4\pi i}\int_{\partial\mathbb{D}} \big[R_1(z,\zeta)\gamma_1(\zeta) - R_2(z,\zeta)\gamma_3(\zeta)\big]\frac{d\zeta}{\zeta}$$

$$-\frac{1}{\pi}\int_{\mathbb{D}} R_2(z,\zeta)f(\zeta)d\xi d\eta.$$

The three Dirichlet problems related to G_2 are of different nature. They can not be decomposed into boundary value problems for the Poisson equation and the boundary data need more smoothness properties. For the basic Cauchy-Pompeiu representation see [4].

Theorem 13. *The Dirichlet problem*

$$(\partial_z\partial_{\bar z})^2 w = f \text{in } \mathbb{D},\, w = \gamma_0,\, w_{\bar z} = \gamma_1 \text{on } \partial\mathbb{D}$$

is uniquely solvable for $f \in L_1(\mathbb{D};\mathbb{C}), \gamma_0 \in C^2(\partial\mathbb{D};\mathbb{C}), \gamma_1 \in C^1(\partial\mathbb{D};\mathbb{C})$ *by*

$$w(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \Big\{\Big[g_1(z,\zeta) + \frac{\bar z\zeta}{(1-\bar z\zeta)^2}\Big]\gamma_0(\zeta)\frac{d\zeta}{\zeta}$$

$$+\frac{1}{2\pi i}\int_{\partial\mathbb{D}} g_1(z,\zeta)(1-|z|^2)\gamma_1(\zeta)d\bar z - \frac{1}{\pi}\int_{\mathbb{D}} G_2(z,\zeta)f(\zeta)d\xi d\eta.$$

Theorem 14. *The Dirichlet problem*

$$(\partial_z \partial_{\bar z})^2 w = f \text{ in } \mathbb{D}, \quad w = \gamma_0, w_z = \gamma_1 \text{ on } \partial\mathbb{D}$$

is uniquely solvable for $f \in L_2(\mathbb{D}; \mathbb{C}), \gamma_0 \in C^2(\partial\mathbb{D}; \mathbb{C}), \gamma_1 \in C^1(\partial\mathbb{D}; \mathbb{C})$ *by*

$$w(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \Big[g_1(z,\zeta) + \frac{z\bar z}{(1 - z\bar z)^2}(1 - |z|^2) \Big] \gamma_0(\zeta) \frac{d\zeta}{\zeta}$$
$$- \frac{1}{2\pi i} \int_{\partial\mathbb{D}} g_1(z,\zeta)(1 - |z|^2)\gamma_1(\zeta) d\zeta - \frac{1}{\pi} \int_{\mathbb{D}} G_2(z,\zeta) f(\zeta) d\xi d\eta.$$

Theorem 15. *The Dirichlet-Neumann problem*

$$(\partial_z \partial_{\bar z})^2 w = f \textit{ in } \mathbb{D}, w = \gamma_0, \partial_\nu w = \gamma_1$$

is uniquely solvable for $f \in L_1(\mathbb{D}; \mathbb{C}), \gamma_0 \in C^2(\partial\mathbb{D}; \mathbb{C}), \gamma_1 \in C^1(\partial\mathbb{D}; \mathbb{C})$ *by*

$$w(z) = \frac{1}{4\pi i} \int_{\partial\mathbb{D}} [g_1(z,\zeta(1 + |z|^2) + g_2(z,\zeta)(1 - |z|^2)]\gamma_0(\zeta) \frac{d\zeta}{\zeta}$$
$$- \frac{1}{4\pi i} \int_{\partial\mathbb{D}} g_1(z,\zeta)(1 - |z|^2)\gamma_1(\zeta) \frac{d\zeta}{\zeta} - \frac{1}{\pi} \int_{\mathbb{D}} G_2(z,\zeta) f(\zeta) d\xi d\eta.$$

Here for $z, \zeta \in \mathbb{D}$

$$g_2(z,\zeta) = \frac{1}{(1 - z\bar z)^2} + \frac{1}{(1 - \bar z \zeta)^2} - 1.$$

In case the boundary data in the last theorems are just taken from $C(\partial\mathbb{D}; \mathbb{C})$ as in the theorems before solvability conditions appear. They are e.g.

$$\lim_{|z| \to 1} \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{\bar z}{(1 - \bar z \zeta)^2}(1 - |z|^2)\gamma_0(\zeta) d\zeta = 0$$
$$\lim_{|z| \to 1} \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{\bar z}{(1 - \bar z \zeta)^3}(1 - |z|^2)\gamma_0(\zeta) d\zeta = 0$$
$$\lim_{|z| \to 1} \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{1}{(1 - \bar z \zeta)^2} \gamma_1(\zeta) \frac{d\zeta}{\zeta} = 0$$

in case of the problem in Theorem 13.

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