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ON NODAL AND CONODAL IDEALS IN MV-ALGEBRAS

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In this paper, we introduce the notions of nodes and nodal ideals in an MV-algebra and we define the notions of conodes and conodal ideals in an MV-algebra. We state some examples and theorems. In addition, we investigate some relations between the nodal (conodal) ideals and some the other ideals of an MV-algebra. Also, we show that if I is a non principal nodal ideal, then A/I is an MV-chain. Moreover, we prove that if I is a conodal ideal of an MV-algebra A, then A/I is a semi-simple MV-algebra. Finally, we construct algorithm for studing the structure of the nodal (conodal) ideals in finite MV-algebras.

1. Introduction

MV-algebras were introduced in [4] as algebraic structures corresponding to the infinite calculus of Lukasiewicz, but their theory was also developed from an algebraic point of view. One can see [5] for an exhaustive study of MV-algebras. Then this class of algebras has been intensively studied by many researchers.

Ideal theory plays an important rule in studying these algebras. Chang [4] introduced the notions of ideals and prime ideals in MV-algebras.

L. P. Belluce et.al. [2] studied some properties of the prime ideals and primary ideals of MV-algebras. Also, some types of ideals in MV-algebras are introduced by many researchers [8, 9].

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In this paper, we introduce the notions of nodes (conodes) and nodal (conodal) ideals in MV-algebras and study some properties. In addition, we study relations between a node and molecule element of MV-algebras. Also, we study relationships between a nodal (conodal) ideals and some other ideals. In addition, we show that if I is a non principal nodal ideal, then A/I is an MV-chain. Finally, we show that if I is a conodal ideal of A, then A/I is a semi-simple MV-algebra.

Definition 1.1 ([4])**.** An MV-algebra is an algebra (M, $\oplus$, $*$, 0) of type $(2,1,0)$ satisfying the following equations, for any $a,b \in M$:
($MV1$) (M, $\oplus$, 0) is an abelian monoid,
($MV2$) $(a^*)^* = a$,
($MV3$) $0^* \oplus a = 0^*$,
($MV4$) $(a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a$.

Note that $1 = 0^*$ and the auxiliary operation $\odot$ as follow:

$$x \odot y = (x^* \oplus y^*)^*.$$

Lemma 1.2 ([4])**.** *Let M be an MV-algebra. For $x,y \in M$, the following conditions are equivalent:*
(1) $x^* \oplus y = 1$,
(2) $x \odot y^* = 0$,
(3) There is an element $z \in M$ such that $x \oplus z = y$,
(4) $y = x \oplus (y \ominus x)$.
For any two elements $x,y \in M$, $\quad x \leq y$ iff x and y satisfy the equivalent conditions (1)-(4) in the above lemma. So, $\leq$ is an order relation on M that is called the natural order on M.

We say that the element $x \in M$ has order n and we write $ord(x) = n$, if n is the smallest natural number such that $nx = 1$, where $nx := \underbrace{x \oplus x \oplus \cdots \oplus x}_{n \quad \text{times}}$. We say that the element x has a finite order, and write $ord(x) < \infty$. An MV-algebra M is locally finite if every non-zero element of M has finite order. We recall that the natural order determines a bounded distributive lattice structure such that

$$x \vee y = x \oplus (x^* \odot y) = y \oplus (x \odot y^*) \quad \text{and} \quad x \wedge y = x \odot (x^* \oplus y) = y \odot (y^* \oplus x).$$

Lemma 1.3 ([4, 5])**.** *In each MV-algebra, the following relations hold for all $x,y,z \in A$:*
(1) $x \leq y$ if and only if $y^* \leq x^*$,
(2) If $x \leq y$, then $x \oplus z \leq y \oplus z$ and $x \odot z \leq y \odot z$,
(3) $x,y \leq x \oplus y$ and $x \odot y \leq x,y$,

(4) $x \oplus x^* = 1$, $x \odot x^* = 0$, *and* $x \odot 0 = 0$,
(5) *If* $x \in B(A)$, *then* $x \wedge y = x \odot y$, *for any* $y \in A$.
Where $B(A)$ *is the set of all complemented elements of* $L(A)$, *where* $L(A)$ *is distributive lattice with* 0 *and* 1 *on* A.

Definition 1.4 ([4])**.** An ideal of an MV-algebra A is a nonempty subset I of A satisfying the following conditions:
($I1$) If $x \in I$, $y \in A$ and $y \leq x$ then $y \in I$,
($I2$) If $x, y \in I$, then $x \oplus y \in I$.
We denote by $Id(A)$ the set of ideals of an MV-algebra A.

Definition 1.5 ([5])**.** Let I be an ideal of an MV-algebra A. Then I is proper if $I \neq A$. Proper ideal I is a prime if and only if for all $x, y \in A$, $x \odot y^* \in I$ or $y \odot x^* \in I$. Equivalently a proper ideal I is a prime ideal, if $x \wedge y \in I$, then $x \in I$ or $y \in I$, for all $x, y \in A$.
• [2] An ideal I of an MV-algebra A is called a Boolean ideal if $x \wedge x^* \in I$, for all $x \in A$.
• [9] A proper ideal I of A is called an obstinate ideal of A if $x, y \notin I$ imply $x \odot y^* \in I$ and $y \odot x^* \in I$, for all $x, y \in A$.
• [10] An ideal I of an MV-algebra A is called an implicative if for any $x, y, z \in A$ such that $z \odot (y^* \odot x^*) \in I$ and $y \odot x^* \in I$, then $z \odot x^* \in I$.

Definition 1.6 ([5])**.** The intersection of all maximal ideals of A is called the radical of A and it is denoted by $Rad(A)$.

Definition 1.7 ([8])**.** Let I be a proper ideal of A. The intersection of all maximal ideals of A which contain I is called the radical of I and it is denoted by $Rad(I)$. Also, we proved that in [8],

$$Rad(I) = \{a \in A : na \odot a \in I, \quad \text{for all} \quad n \in N\}.$$

Definition 1.8 ([8])**.** Let I be a proper ideal of A. If $Rad(I) = I$, then I is called a semi-maximal ideal of A.

Definition 1.9 ([1])**.** A nonzero element m of a poset P with 0 is a molecule if whenever $0 < x, y \leq m$, then $\{x, y\}$ has a nonzero lower bound. Thus $m \in A$ is a molecule if and only if whenever $x, y \in A$ satisfy $0 < x, y \leq m$, then $x \wedge y > 0$. $Mol(A)$ denote the set of all molecules of A.

In an MV-algebra M, the distance function is
$d : M \times M \longrightarrow M, \qquad d(x, y) := (x \odot y^*) \oplus (y \odot x^*)$.
Suppose that I is an ideal of an MV-algebra A. Define $x \sim_I y$ if and only if $d(x, y) \in I$ if and only if $x \odot y^* \in I$ and $y \odot x^* \in I$. Then $\sim_I$ is a congruence relation on A. The set of all congruence classes is denoted by A/I then $A/I =$

$\{[x] : x \in A\}$, where $[x] = \{y \in A : x \sim_I y\}$. We can easily to see that $x \in I$ if and only if $x/I = 0/I$. The MV-algebra operations on A/I given by $x/I \oplus y/I = (x \oplus y)/I$ and $(x/I)^* = x^*/I$, are well defined. Hence $(A/I, \oplus, *, [0])$ becomes an MV-algebra [5, 11].

Definition 1.10 ([5, 11])**.** An MV-algebra A is simple, if A is nontrivial and $\{0\}$ is its only proper ideal.

We recall that for a nonempty subset $N \subseteq A$, the smallest ideal of A which contains N, i.e., $\bigcap\{I \in Id(A) : N \subseteq I\}$, is said to be the ideal of A generated by N and will be denoted by $(N]$ [4].

Theorem 1.11 ([4])**.** *If $N \subseteq A$ is a nonempty subset of an MV-algebra A, then we have:*
(i) $(N] = \{x \in A : x \leq x_1 \oplus \ldots \oplus x_n$ for some $x_1, \ldots, x_n \in N\}$. In particular, for $a \in A$:

$$(a] = \{x \in A : x \leq na \quad \text{for some integer} \quad n \geq 0\}.$$

(ii) If $I_1, I_2 \in Id(A)$, then $I_1 \vee I_2 = (I_1 \cup I_2] = \{a \in A : a \leq x_1 \oplus x_2$ for some $x_1 \in I_1, x_2 \in I_2\}$.
(iii) $[x \wedge y) = [x) \cap [y)$.

Definition 1.12 ([11])**.** A Heyting algebra is a lattice (L, $\vee$, $\wedge$) with 0 such that for every $a, b \in L$, there exists an element $a \to b \in L$ (called the pseudocomplement of a with respect to b) such that for every $x \in L$, $a \wedge x \leq b$ if and only if $x \leq a \to b$ (that is, $a \to b = sup\{x \in L : a \wedge x \leq b\}$).

Definition 1.13 ([5, 11])**.** An MV-algebra A is a semi-simple MV-algebra if and only if A is nontrivial and $Rad(A) = \{0\}$.

Lemma 1.14 ([8])**.** *Let I be an ideal of A. Then I is a semi-maximal ideal of A if and only if A/I is a semi-simple MV-algebra.*

2. Nodal ideals of MV-algebras

From now on $(A, \oplus, *, 0, 1)$ or simply A is an MV-algebra.

Definition 2.1. An element $a \in A$ is called a node of A if it is comparable with every element of A. It is clear that $0, 1$ are nodes in every MV-algebra.

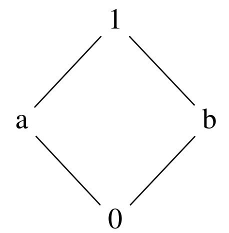
Note that an element $a \in A$ is a node if and only if for every $x \in A$ either $a \odot x^* = 0$ or $x \odot a^* = 0$.
Also, we can define a node of an arbitrary poset.

Definition 2.2. An element a of a poset P is called a node of P, if it is comparable with every element of P.

In the following examples, we show that the nodes exist and that an element is not in general node of A.

Example 2.3. Let $A = \{0,1,2\}$ be a linearly ordered set (chain $0 < 1 < 2$). A is an MV-algebra with operations $\wedge = min$, $x \oplus y = min\{2, x+y\}$, $x \odot y = max\{0, x+y-2\}$, for every $x, y \in A$. It is clear that the set of nodes of A is $\{0,1,2\}$.

Example 2.4. Let $A = \{0,a,b,1\}$, where $0 < a,b < 1$. Define $\odot$, $\oplus$ and $*$ as follows:



$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\oplus$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

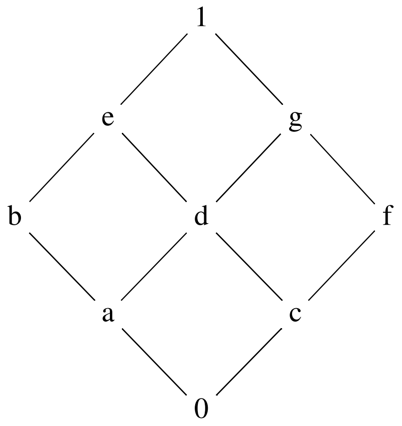
$*$	0	a	b	1
	1	b	a	0

Then $(A, \oplus, \odot, *, 0, 1)$ is an MV-algebra. $\{0,1\}$ is the set of all nodes of A.

Theorem 2.5. *In an MV-chain, all nodes of A are molecules.*

Proof. Let $x, y \in A$ such that $0 < x, y \leq m$. It is clear that m is a molecule, since x, y are nodes, so $x \wedge y = x > 0$ or $x \wedge y = y > 0$. Hence m is a molecule. $\square$

Example 2.6. Let $A = \{0,a,b,c,d,e,f,g,1\}$, where $0 < a,c < d < e,g < 1$, $0 < a < b < e < 1$ and $0 < c < f < g < 1$. Define $\oplus$ and $*$ as follows:



$\oplus$	0	a	b	c	d	e	f	g	1
0	0	a	b	c	d	e	f	g	1
a	a	b	b	d	e	e	g	1	1
b	b	b	b	e	e	e	1	1	1
c	c	d	e	f	g	1	f	g	1
d	d	e	e	g	1	1	g	1	1
e	e	e	e	1	1	1	1	1	1
f	f	g	1	f	g	1	b	a	1
g	g	1	1	g	1	1	a	1	1
1	1	1	1	1	1	1	1	1	1

$*$	0	a	b	c	d	e	f	g	1
	1	g	f	e	d	c	b	a	0

Then $(A,\oplus,*,0,1)$ is an MV-algebra and b,f are molecules but are not nodes of A.

Example 2.7. In Example 2.4, it is clear that 1 is a node but is not a molecule of A, since $0 < a,b \leq 1$, but $a \wedge b = 0$.

Theorem 2.8. *The following hold:*
(i) $node(A) \cap B(A) = \{0,1\}$,
(ii) $\{0,1\} \subseteq nod(A) \cap A$.

Proof. (i) We know that, $\{0,1\} \subseteq node(A) \cap B(A)$. Now let $0 \neq x \in node(A) \cap B(A)$. We show that $x = 1$. Since $x \in B(A)$, so there exists $y \in A$ such that

$x \wedge y = 0$, $x \vee y = 1$. On the other hand, since $0 \neq x \in node(A)$, for every $y \in A$, we get $x \leq y$ or $y \leq x$. Thus $x \wedge y = x$, $x \vee y = y$ or $x \wedge y = y$, $x \vee y = x$. Therefore $x = 1$.
(ii) it is an easy consequence of (i). □

Definition 2.9. An ideal I of A, is called a nodal ideal of A, if I is a node of $Id(A)$.

Example 2.10. (a) A and $\{0\}$ are trivial nodal ideal of every MV-algebra A.

(b) In Example 2.4, we have $Id(A) = \{\{0\}, \{0,a\}, \{0,b\}, A\}$. But only $\{0\}$ and A are nodal ideals of A. $Rad(A) = \{0\}$ is nodal ideal of A.

(c) Let $G = \oplus\{Z_i / i \in \mathbb{N}\}$ be the lexicographic product of denumerable infinite copies of the abelian l-group $\mathbb{Z}$ of the relative integers and $e^i \in G$ such that $e^i_k = 0$ if $k \neq i$ and $e^i_k = 1$ if $k = i$.
$u = (1,0,0,0,\ldots)$ is the strong unit of G, i.e., for every $x \in G$, there is an integer $n \geq 0$ such that $-nu \leq x \leq nu$, where $\leq$ is the lexicographic order on G. It is well-known that $A = \Gamma(G,u) = [0,u]_G$ [5], where Γ is a functor from the category of abelian l-groups to the category of MV-algebras, thus u is the top element of A.
If we set $P_i = \langle(0,e^i)\rangle$, then $P_i \subseteq P_j$, for $i > j$, hence the ideals of A is $Id(A) = \{P_i / i \in \mathbb{N}\}$ as in [7]. Hence $\langle(0,0)\rangle, \langle(0,e^1)\rangle, \cdots, \langle(0,e^n)\rangle$ are nodal ideals of A.

(d) We consider Chang's MV-algebra C. $C = \{0, c, 2c, 3c, \ldots, 1-2c, 1-c, 1\}$ be the MV-algebra defined in [11], Example 2.6 with operations as follows:
if $x = nc$ and $y = mc$, then $x \oplus y := (m+n)c$,
if $x = 1-nc$ and $y = 1-mc$, then $x \oplus y := 1$,
if $x = nc$ and $y = 1-mc$ and $m \leq n$, then $x \oplus y := 1$,
if $x = nc$ and $y = 1-mc$ and $n < m$, then $x \oplus y := 1-(m-n)c$,
if $x = 1-mc$ and $y = nc$ and $m \leq n$, then $x \oplus y := 1$,
if $x = 1-mc$ and $y = nc$ and $n < m$, then $x \oplus y := 1-(m-n)c$,
if $x = nc$, then $x^* := 1-nc$,
if $x = 1-nc$, then $x^* := nc$.

C is an MV-algebra. It has only three ideals: $\{0\}$, $M = \{0, c, 2c, 3c, \ldots\}$ and C. Every ideal of C is nodal ideal.

Lemma 2.11. *Let I be an ideal of A. If for all $x \in I$ and for all $y \notin I$, the relation $x < y$ is satisfied, then I is a nodal ideal of A.*

Proof. Let there exists an ideal J incomparable with I. Then there are elements $x, y \in A$ such that $x \in I - J$, $y \in J - I$ and $x \not< y$. Thus its contrary, so every ideal J of A is comparable with I, that is, I is a nodal ideal of A. □

Lemma 2.12. *Let I be a nodal ideal of A. For every $x \in I$ and $y \notin I$, if $y \in B(A)$, then the relation $x < y$ is satisfied.*

Proof. Since I is a nodal ideal of A, so for all $x,y \in A$, such that $x \in I$ and $y \notin I$, we have $[x) \subseteq I$ and $I \subseteq [y)$. Thus $[x) \subseteq I \subseteq [y)$, so $x \in [y)$, since $y \in B(A)$, we get $x < ny = y$. □

Theorem 2.13. *Let $A \neq \{0,1\}$ be simple MV-algebra. Then every ideal of A is a nodal ideal of A.*

Proof. Since simple MV-algebras are MV-chains, so every ideal of A is a nodal. □

Note. In chain A, every ideal is a nodal ideal.

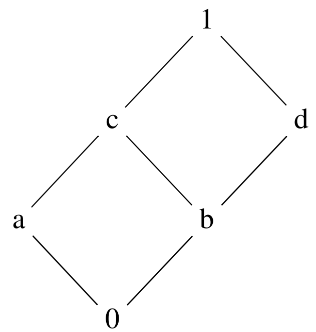
Corollary 2.14. *Let A be a chain. Then every ideal is a nodal ideal.*

Proposition 2.15. *If x is a node of A, then principal ideal $[x)$ is nodal ideal.*

Proof. Let x be a node in A and I be an ideal of A. If $x \in I$, then $[x) \subseteq I$. Let $x \notin I$. Now if $I \nsubseteq [x)$, then there exists $y \in I$ such that $y \notin [x)$, so for all $n \in \mathbb{N}$, $y \nleq nx$, so $y \nleq x$ and since x is a node, then $x < y$. Thus $[x) \subseteq [y) \subseteq I$. That is $x \in I$, its contrary, so if $x \notin I$, then $I \subseteq [x)$. □

In the following example, we show that the converse of the above proposition is not true in general.

Example 2.16. Let $A = \{0,a,b,c,d,1\}$, where $0 < a,b < c < 1$ and $0 < b < d < 1$. Define $\oplus$, $\odot$ and $*$ as follows:



$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	c	c	1	1
b	b	c	d	1	d	1
c	c	c	1	1	1	1
d	d	1	d	1	d	1
1	1	1	1	1	1	1

$*$	0	a	b	c	d	1
	1	d	c	b	a	0

Then $(A,\oplus,*,0,1)$ is an MV-algebra and $[c)=A$ is a nodal ideal of A, but c is not a node of A.

Theorem 2.17. *If A is a Boolean algebra and $[x)$ is nodal ideal of A, then x is a node of A.*

Proof. Let $[x)$ ba a nodal ideal of A and x be not a node of A. So there exists $y\in A$ such that x incomparable with y, thus $x\nleq y$ and $y\nleq x$, then $x\notin[y)$ and $y\notin[x)$, therefore $[x)\nsubseteq[y)$ and $[y)\nsubseteq[x)$ i.e, $[x)$ and $[y)$ are not comparable, which is a contradiction. Thus x is a node of A. □

Theorem 2.18. *Let I be a non principal nodal ideal of A. Then I is a prime ideal of A.*

Proof. Let I be a non principal nodal ideal of A, $x\wedge y\in I$ and $x\notin I$ and $y\notin I$. We have $[x\wedge y)\subseteq I$. On the other hand, since $x\notin I$ and $y\notin I$, then $[x)\nsubseteq I$ and $[y)\nsubseteq I$, so $I\subset[x)$ and $I\subset[y)$, thus $I\subseteq[x)\cap[y)=[x\wedge y)$. So $I=[x\wedge y)$, which is a contradiction, thus $x\in I$ or $y\in I$, so I is a prime ideal. □

Example 2.19. In Example 2.4, $\{0\}$ is a nodal ideal but is not prime ideal, since $a\wedge b=0$ but $a\neq 0$ and $b\neq 0$. Also $\{0,a\}$ is a prime ideal but is not nodal ideal of A.

Proposition 2.20. *If A is Boolean algebra and A has n node, then it has at least n nodal ideals.*

Proof. If x is a node of A, then $[x)$ is nodal ideal. Now, let x and y be two nodes of A. If $[x)=[y)$, then $x\in[y)$ and $y\in[x)$, since A is a Boolean algebra, so $x\leq y$ and $y\leq x$, thus $x=y$. Therefore, if A has n node, then it has at least n nodal ideals. □

Lemma 2.21. *The set $\mathcal{N}(A)$ of all nodal ideals of an MV-algebra A that is ordered by inclusion, is a chain whose greatest element is A and smallest element is $\{0\}$.*

Note: if I and J are two nodal ideals and $I\cap J=\{0\}$, then $I=\{0\}$ or $J=\{0\}$. We recall that a lattice $(L,\wedge,\vee)$ is called Brouwerian if it satisfies the identity $a\wedge(\vee_{i\in I}b_i)=\vee_{i\in I}(a\wedge b_i)$, whenever the arbitrary unions exists [3].

Theorem 2.22. *Let $I,J\in\mathcal{N}(A)$. We define $I\wedge J=I\cap J$ and $I\vee J=[I\cup J)$, then $(\mathcal{N}(A),\wedge,\vee,\{0\},A)$ is complete Brouwerian lattice.*

For $I,J\in\mathcal{N}(A)$, we define

$$I\to J=\{x\in A: I\cap[x)\subseteq J\}.$$

Theorem 2.23. *Let A be a Boolean algebra. If $I,J \in \mathcal{N}(A)$, and*
(a) all elements of A are node, then $I \to J \in \mathcal{N}(A)$.
(b) A is chain, then $I \to J \in \mathcal{N}(A)$.

Proof. (a) Let $x \in I \to J$ and $y \notin I \to J$. Then $I \cap [x) \subseteq J$ and $I \cap [y) \nsubseteq J$, so $I \cap [y) \nsubseteq I \cap [x)$, thus $[y) \nsubseteq [x)$. Since x and y are node, so $[x) \subset [y)$, therefore $x \in [y)$, that is $x < y$. Then $I \to J \in \mathcal{N}(A)$. □

Theorem 2.24. *Let A be a Boolean MV algebra and all elements are nodes. Then $(\mathcal{N}(A), \wedge, \vee, \to, \{0\})$ is a Heyting algebra.*

Proof. Let $I,J,K \in \mathcal{N}(A)$. We show that $I \wedge J \subseteq K$ if and only if $I \subseteq J \to K$. Let $x \in I$. Thus $J \cap [x) \subseteq J \cap I = J \wedge I \subseteq K$, so $x \in J \to K$.

Conversely, let $x \in I \wedge J$. So $x \in I$ and $x \in J$. Since $I \subseteq J \to K$, thus $x \in J \to K$, that is $[x) \wedge J \subseteq K$. Now since $x \in [x)$ and $x \in J$, so $x \in K$. Therefore $I \wedge J \subseteq K$. □

Corollary 2.25. *If A is a chain, then $(\mathcal{N}(A) = Id(A), \wedge, \vee, \to, \{0\}, A)$ is a Heyting algebra.*

The following example shows the extension theorem (If $I \subseteq J$ and I is a nodal ideal, then J is a nodal ideal of A) of nodal ideals does not hold.

Example 2.26. In Example 2.4, let $I = \{0\}$ and $J = \{0, a\}$ be ideals of A. Then $I \subseteq J$ and I is nodal ideal of A but J is not nodal ideal of A.

In the following example, we show that every nodal ideal is not a Boolean ideal and every Boolean ideal is not nodal ideal.

Example 2.27. Let A be an MV-algebra in Example 2.16, it is clear $I = \{0\}$ is a nodal ideal of A but it is not a Boolean ideal of A, because $b \wedge b^* = b \wedge c = b \notin I$.

Also, in Example 2.4, $I = \{0, a\}$ is a Boolean ideal but is not a nodal ideal.

The following example shows every nodal ideal is not an obstinate ideal and every obstinate ideal is not nodal ideal, in general.

Example 2.28. In Example 2.16, $I = \{0\}$ is nodal ideal but is not obstinate ideal, because $c \odot d^* = c \odot a = a \notin I$ and $d \odot c^* = b \notin I$.

Also, in Example 2.4, we have $I = \{0, a\}$ is an obstinate ideal but is not a nodal ideal of A.

The following example shows that a semi-maximal ideal may not be a nodal ideal and every nodal ideal is not semi-maximal ideal, in general.

Example 2.29. Let $\Omega = \{1,2\}$ and $\mathcal{A} = P(\Omega) = \{\{1\},\{2\},\{1,2\},\emptyset\}$, which is an MV-algebra with operations $\oplus = \cup$, $\odot = \cdot = \cap$ and $A^* = \Omega - A$, for any $A \in \mathcal{A}$. It is clear that $I = \{\emptyset,\{1\}\}$ is a semi-maximal ideal of $\mathcal{A}$ but is not nodal ideal.

Also, In Example 2.10 (d) $I = \{0\}$ is a nodal ideal but is not semi-maximal ideal, because $Rad(I) = M$.

We recall that $I(x) = [I \cup \{x\}) = I \vee [x)$.

Theorem 2.30. *If I is a nodal ideal of A and x is a node of A, then $I(x)$ is a nodal ideal of A.*

Proof. If $x \in I$, then $I(x) = I$, then $I(x)$ is a nodal ideal of A. Let $x \notin I$. Since x is a node of A, so $[x)$ is a nodal ideal of A. Also we have

$$I(x) = [I \cup \{x\}) = [I \cup [x)) = I \vee [x).$$

Thus $I(x)$ is a nodal ideal of A. □

By the following example, we show that if I is a nodal ideal of A and x is not a node of A, then $I(x)$ possible is not a nodal ideal of A.

Example 2.31. Consider Example 2.16, $I = \{0\}$ is nodal ideal and c, b are not nodes of A . We have $I(c) = I \cup [c) = A$ is a nodal ideal but $I(b) = \{0, b, d\}$ is not a nodal ideal of A.

Theorem 2.32. *Let A and B be two Boolean algebras and $f : A \to B$ a homomorphism. Then the following are satisfied:*
(a) If f is injective and $I \in \mathcal{N}(B)$, then $f^{-1}(I) \in \mathcal{N}(A)$,
(b) If f is surjective and $I \in \mathcal{N}(A)$, then $f(I) \in \mathcal{N}(B)$.

Proof. (a) Since $f(0_A) = 0_B \in I$, so $0_A \in f^{-1}(I)$, thus $f^{-1}(I) \neq \emptyset$. Let $x, y \in f^{-1}(I)$. Then $f(x), f(y) \in I$, so $f(x) \oplus f(y) \in I$, thus $f(x \oplus y) \in I$, that is $x \oplus y \in f^{-1}(I)$.

Now, let $x \in f^{-1}(I)$ and $y \notin f^{-1}(I)$. Then $f(x) \in I$ and $f(y) \notin I$. Since I is nodal ideal of Boolean algebra B, we get that $f(x) < f(y)$, thus $f(x)^* \oplus f(y) = 1_B = f(1_A)$. Hence $f(x^* \oplus y) = f(1_A)$, since f is injective, $x^* \oplus y = 1_A$, thus $x \leq y$, if $x = y$, so $f(x) = f(y)$, which is a contradiction. It follows from Lemma 2.11 that $f^{-1}(I) \in \mathcal{N}(A)$.

(b) Let $a \in f(I)$ and $b \notin f(I)$. Then there exists $x \in I$ such that $f(x) = a$ and there is not $y \in I$ such that $f(y) = b$. Since $y \notin I$ and I is a nodal ideal, by Lemma 2.12, we conclude that $x < y$, thus $x^* \oplus y = 1$, so $f(x)^* \oplus f(y) = f(x^* \oplus y) = 1$, thus $f(x) < f(y)$, that is, $a < b$. It follows from Lemma 2.11 that $f(I) \in \mathcal{N}(B)$. □

Theorem 2.33. *Let I be a non principal nodal ideal of A. Then $(A/I,\oplus,*,0/I,1/I)$ is an MV-chain.*

Proof. Let $a/I, b/I \in A/I$ and $a/I \nleq b/I$. Then $a \odot b^* \notin I$. Since I is a non principal nodal ideal, by Theorem 2.18, I is a prime ideal, so $b \odot a^* \in I$. Thus $b/I \leq a/I$. Therefore A/I is an MV-chain. □

3. Conodal ideals in MV-algebras

Definition 3.1. An element a of an MV-algebra A is called a conode of A, if there exist elements of A which are incomparable with a.

Note. An element $a \in A$ is called conode, if and only if for some $x \in A$, $a \odot x^* \neq 0$ and $x \odot a^* \neq 0$.
Also, we can define a conode of an arbitrary poset.

Definition 3.2. An element a of a poset P is called a conode of P, if there exist elements of P which are incomparable with a.

In the following examples, we show that the conodes exist and that an element is not in general conode of A.

Example 3.3. (i) Consider MV-algebra A in Example 2.4, $\{a,b\}$ is the set of all conodes of A.
(ii) Consider MV-algebra in Example 2.6, $\{a,c,b,d,f,e,g\}$ is the set of all conodes of A.

Definition 3.4. An ideal I of A, will be called a conodal ideal of A, if I is a conode of $Id(A)$.

Example 3.5. (i) In Example 2.4, $\{0,a\}$ and $\{0,b\}$ are conodal ideals of A.
(ii) In Example 2.16, $\{0,a\}$ and $\{0,b,d\}$ are conodal ideals of A.
(iii) Consider Chang's MV-algebra C in Example 2.10. Obviously, it has no conodal ideals.

Lemma 3.6. *Let I be an ideal of A. Then I is a conodal ideal of A if and only if for some $x \in I$ and for some $y \notin I$, the relation $x \nleq ny$, for all $n \in \mathbb{N}$ is satisfied.*

Proof. Let I be a conodal ideal of A. Then there exists an ideal J of A such that $I \nsubseteq J$ and $J \nsubseteq I$. Hence there exist $x \in I \setminus J$ and $y \in J \setminus I$. It follows that $x \nleq ny$, for all $n \in \mathbb{N}$.
Conversely, assume that I is not conodal ideal of A. Then $I \subseteq J$ or $J \subseteq I$, for all ideal J. Hence for some $x \in I$ and for some $y \notin I$, $I \subseteq (y]$ or $(y] \subseteq I$. Since $(y] \nsubseteq I$, so $I \subseteq (y]$. This results $x \in (y]$. It follows from Theorem 1.11, $x \leq ny$, for some $n \in \mathbb{N}$, which is a contradiction. Thus I is a conodal ideal of A. □

In the following examples , we study the relations between conodal ideals and the other types of ideals in MV-algebras.

Example 3.7. In Example 2.4, let $I = \{0, a\}$ and $J = A$ be ideals of A. Then $I \subseteq J$ and I is conodal ideal of A but A is not a conodal ideal of A.

In the following example, we show that every conodal ideal is not Boolean ideal and every Boolean ideal is not conodal ideal of A.

Example 3.8. (i) Let A be an MV-algebra in Example 2.16, it is clear that $I = \{0, a\}$ is a conodal ideal of A but it is not a Boolean ideal of A, because $b \wedge b^* = b \wedge c = b \notin I$.
(ii) In Example 2.4, $I = \{0\}$ is a Boolean ideal but it is not a conodal ideal of A.

The following example shows every conodal ideal is not an obstinate ideal and every obstinate ideal is not a conodal ideal of A.

Example 3.9. (i) In Example 2.16, $I = \{0, a\}$ is a conodal ideal of A. Since $d \odot c^* = b \notin I$, it is not an obstinate ideal of A.
(ii) Consider chang's MV-algebra in Example 2.10, $M = \{0, c, 2c, 3c, \dots\}$ is an obstinate ideal of A but is not a conodal ideal of A, in an MV chain no ideal is conodal because all ideals are nodal.

The following example shows every conodal ideal is not a maximal ideal and every maximal ideal is not a conodal ideal of A.

Example 3.10. (i) In Example 2.10 (d), consider the MV-algebra $C \times C$ and its prefect subalgebra

$$H = Rad(C \times C) \cup (Rad(C \times C)^*).$$

H has the following ideals: $I_0 = \{(0,0)\}$, $I_1 = \{(0, nc) : n \in N_0\}$, $I_2 = \{(nc, 0) : n \in \mathbb{N} \cup \{0\}\}$ (prime ideals) and $I_3 = \{(mc, nc) : m, n \in \mathbb{N} \cup \{0\}\}$ is a maximal ideal of A. I_1 and I_2 are conodal ideals but are not maximal ideals of H.
(ii) In chang's MV-algebra C, $M = \{0, c, 2c, 3c, \dots\}$ is a maximal ideal of C but is not conodal ideal of C.

Theorem 3.11. *If I is a conodal ideal of A, then I is a semi-maximal ideal of A.*

Proof. Let I be a conodal ideal of A. Assume that I is not a semi-maximal ideal of A. Then there exists an element $x \in A$ such that $nx \odot x \in I$ but $x \notin I$. It follows from Lemma 3.6 that $nx \odot x \nleq mx$, for all $m \in \mathbb{N}$. Hence $nx \odot x \odot x^* \neq 0$, which is a contradiction (since $x \odot x^* = 0$). Thus I is a semi-maximal ideal of A. □

The following example shows converse the above theorem is not true in general.

Example 3.12. Consider MV-algebra $\mathcal{A} = P(\Omega)$ in Example 2.29 and $I = \{\emptyset\}$. Since $Rad(I) = \{\emptyset, \{1\}\} \cap \{\emptyset, \{2\}\} = \{\emptyset\}$, so $I = \{\emptyset\}$ is a semi-maximal but is not conodal ideal of $\mathcal{A}$.

We recall that I is a semi-maximal ideal of A if and only if A/I is a semi-simple MV-algebra [8].

Corollary 3.13. *If I is a conodal ideal of A, then A/I is a semi-simple MV-algebra.*

Proof. It follows from Theorem 3.11 and Lemma 1.14. □

4. Conclusion and future research

MV-algebras were originally introduced by Chang in [5] in order to give an algebraic counterpart of the Łukasiewicz many valued logic.
In this paper, we introduced the notion of a node of MV-algebras.
Also, we introduced the notion of nodal ideals in MV-algebras and we have also presented a characterization and many important properties of nodal ideals in MV-algebras. In addition, we showed that if I is a non principal nodal ideal of A, then A/I is an MV-chain. Also, we defined the notions of conodes and conodal ideals in an MV-algebra. We studied some relations between the conodal ideals and some the other ideals of an MV-algebra.

Finally, we showed that if I is a conodal ideal of A, then A/I is a semi-simple MV-algebra.
In our future work, we would like to:
(i) develop the properties of nodal (conodal) ideals of MV-algebra,
(ii) construct the related logical properties of such structures.

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5. Appendix

Algorithmes

In the following, we construct some algorithms for studying ideals and conodal ideals in finite MV-algebras.

Algorithm for ideals in finite MV-algebras

Input (A: MV-algebra, I: subset of A);
Output ("I is an ideal or not")
Begin
If $I = \emptyset$ **then**
go to $(1.)$;
EndIf
If $0 \notin I$ **then**
go to $(1.)$;
EndIf
Stop:=false;
$i := 1$;
While $i \leq |A|$ **and not**(stop) **do**
$j := 1$;
While $j \leq |A|$ **and not**(stop) **do**
If $x_i \in I$ **and** $y_j \in I$ **then**
If $x_i \oplus y_j \notin I$ **then**
Stop:=true;
EndIf
EndIf
If $x_i^* \oplus y_j = 1$ **and** $y_j \in I$ **then**
If $x_i \notin I$ **then**
Stop:=true;
EndIf
EndIf
EndWhile
EndWhile
If Stop **then**
$(1.)$ **Output** ("I is not an ideal")
Else
Output ("I is an ideal")
EndIf
End.

Algorithm for conodal ideals in finite MV-algebras

Input (A: MV-algebra, I: ideal of A);
Output ("I is a conodal ideal or not")
Begin
If $I = \emptyset$ **then**
go to (1.);
EndIf
Stop:=false;
$i := 1$;
While $i \leq |A|$ **and not**(stop) **do**
$j := 1$;
While $j \leq |A|$ **and not**(stop) **do**
If $x_i \in I$ **and** $y_j \notin I$ **then**
If $\forall n \leq |A|, x_i^* \oplus (ny_j) \neq 1$ **then**
go to (2.)
EndIf
EndIf
EndWhile
EndWhile
If Stop **then**
(1.) **Output** ("I is not a conodal ideal")
Else
(2.) **Output** ("I is a conodal ideal")
EndIf
End.

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