SOME PROPERTIES FOR CERTAIN CLASS OF ANALYTIC FUNCTIONS DEFINED BY CONVOLUTION

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In this paper, we introduce a new class $H_T(f, g; \alpha, k)$ of analytic functions in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ defined by convolution. The object of the present paper is to determine coefficient estimates, extreme points, distortion theorems, partial sums and integral means for functions belonging to the class $H_T(f, g; \alpha, k)$. We also obtain several results for the neighborhood of functions belonging to this class.

1. Introduction

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$.

For functions $f$ given by (1) and $g \in A$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (b_n > 0),$$

the Hadamard product (or convolution) of $f$ and $g$ is defined by

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\[(f \ast g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g \ast f)(z).\]

**Definition 1.1** ([9], [10], [13] and [16]). For \( k \geq 0, 0 \leq \alpha < 1 \) and \( z \in U \), let \( S(k, \alpha) \) denote the subclass of functions \( f \in A \) and satisfying the condition:

\[
\text{Re} \left( \frac{z f'(z)}{f(z)} + k \frac{z^2 f''(z)}{f(z)} \right) > \alpha.
\]

**Definition 1.2.** For \( 0 \leq \alpha < 1, k \geq 0 \) and for all \( z \in U \), let \( H(f, g; \alpha, k) \) denote the subclass of \( A \) consisting of functions \( f(z), g(z) \in A \) and satisfying the analytic criterion:

\[
\text{Re} \left\{ \frac{z f'(z)}{f(z)} + k \frac{z^2 (f \ast g)''(z)}{(f \ast g)(z)} \right\} > \alpha.
\]  

We note that for suitable choice of \( g \), we obtain the following subclasses.

1. If we take \( g(z) = \frac{z - 1}{z} \), then the class \( H(f, \frac{z - 1}{z}; \alpha, k) \) reduces to the class \( S(k, \alpha) \) (see [13]);
2. If we take \( g(z) = z + \sum_{n=2}^{\infty} \sigma_n z^n \) (or \( b_n = \sigma_n \)), where

\[
\sigma_n = \frac{\Theta \Gamma(\alpha_1 + A_1(n - 1)) \cdots \Gamma(\alpha_q + A_q(n - 1))}{(n - 1)! \Gamma(\beta_1 + B_1(n - 1)) \cdots \Gamma(\beta_s + B_s(n - 1))}
\]

\((\alpha_i, A_i > 0, i = 1, \ldots, q; \beta_j, B_j > 0, j = 1, \ldots, s; q \leq s + 1; q, s \in \mathbb{N}, \mathbb{N} = \{1, 2, \ldots\})\)

and

\[
\Theta = \frac{\prod_{j=0}^{s} \Gamma(\beta_j)}{(q \prod_{i=0}^{q} \Gamma(\alpha_i)}
\]

then the class \( H(f, z + \sum_{n=2}^{\infty} \sigma_n z^n; \alpha, k) \) reduces to the class \( W_s^q(\alpha, k) \) (see [5])

\[
\left\{ f \in A : \text{Re} \left\{ \frac{z (W_s^q f(z))'}{W_s^q f(z)} + k \frac{z^2 (W_s^q f(z))''}{W_s^q f(z)} \right\} > \alpha, \right\}
\]
where $W^q_s f(z)$ is the Wright’s generalized hypergeometric function (see [6] and [22]) which contains well known operators such as the Dziok-Srivastava operator (see [7]), the Carlson-Shaffer linear operator (see [1]), the Bernardi-Libera-Livingston operator (see [11]), Owa-Srivastava fractional derivative operator (see [15]), the Choi-Saigo-Srivastava operator (see [4]), the Cho-Kwon-Libera-Livingston operator (see [11]), Owa-Srivastava fractional derivative operator (see [17]) and the Noor integral operator of n-th order (see [14];

(3) If we take

$$g(z) = z + \sum_{n=2}^{\infty} \left( \frac{l+1+\mu(n-1)}{l+1} \right)^m z^n$$

(or $b_n = \left( \frac{l+1+\mu(n-1)}{l+1} \right)^m$, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mu \geq 0, l \geq 0$), then the class $H(f, z + \sum_{n=2}^{\infty} \left( \frac{l+1+\mu(n-1)}{l+1} \right)^m z^n; \alpha, k)$ reduces to the class $\mathcal{L}_m(\mu, l, \alpha, k)$:

$$\{ f \in A : \text{Re} \left\{ \frac{z(l^m(\mu,l)f(z))'}{l^m(\mu,l)f(z)} + k^{\frac{2(l^m(\mu,l)f(z))''}{l^m(\mu,l)f(z)}} \right\} > \alpha, $$

$$0 \leq \alpha < 1; k \geq 0; \mu, l \geq 0, m \in \mathbb{N}_0; z \in U \}$$

where the operator $l^m(\mu,l)$ was introduced and studied by Cătaş et al. (see [2]).

Denote by $T$ the subclass of $A$ consisting of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0),$$

which are analytic in $U$. We define the class $H_T(f, g; \alpha, k)$ by:

$$H_T(f, g; \alpha, k) = H(f, g; \alpha, k) \cap T.$$  

Also we note that:

(1) $H_T(f, z + \sum_{n=2}^{\infty} \sigma_n z^n; \alpha, k) = TW^q_s(\alpha, k)$ ($q, s \in \mathbb{N}, k \geq 0, 0 \leq \alpha < 1$), where $\sigma_n$ given by (5) (see [5]);

(2) $H_T(f, \frac{1}{1-z}; \alpha, k) = S_T(k, \alpha)$ ($k \geq 0, 0 \leq \alpha < 1$);

(3) $H_T(f, z + \sum_{n=2}^{\infty} \left( \frac{l+1+\mu(n-1)}{l+1} \right)^m z^n; \alpha, k) = T\mathcal{L}_m(\mu, l, \alpha, k)$ ($0 \leq \alpha < 1, k \geq 0, \mu, l \geq 0, m \in \mathbb{N}_0$).
2. Coefficient estimates

Unless otherwise mentioned, we shall assume in the reminder of this paper that, \(0 \leq \alpha < 1, k \geq 0, b_n > 0, n \geq 2, z \in U\) and \(g(z)\) is defined by (2).

**Theorem 2.1.** A function \(f(z)\) of the form (1) is in the class \(H(f, g; \alpha, k)\) if

\[
\sum_{n=2}^{\infty} \left( kn^2 + n - kn - \alpha \right) b_n |a_n| \leq 1 - \alpha. \tag{12}
\]

**Proof.** Assume that the inequality (12) holds true. Then we have

\[
\left| \frac{z(f * g)'(z)}{(f * g)(z)} + k z^2 (f * g)''(z) - 1 \right| \leq \frac{\sum_{n=2}^{\infty} |n + kn(n - 1) - 1| b_n |a_n| |z|^{n-1}}{1 + \sum_{n=2}^{\infty} b_n |a_n| |z|^{n-1}} \leq \frac{\sum_{n=2}^{\infty} |n + kn(n - 1) - 1| b_n |a_n|}{1 - \sum_{n=2}^{\infty} b_n |a_n|} \leq 1 - \alpha.
\]

This shows that the values of the function

\[
\Phi(z) = \left( \frac{z(f * g)'(z) + k z^2 (f * g)''(z)}{(f * g)(z)} \right)
\]

lie in a circle centered at \(w = 1\) and whose radius is \(1 - \alpha\). Hence \(f(z)\) satisfies the condition (12). This completes the proof of Theorem 2.1.

**Theorem 2.2.** A necessary and sufficient condition for \(f(z)\) of the form (10) to be in the class \(H_T(f, g; \alpha, k)\) is that

\[
\sum_{n=2}^{\infty} \left( kn^2 + n - kn - \alpha \right) b_n a_n \leq 1 - \alpha. \tag{14}
\]

**Proof.** In view of Theorem 2.1, we need only to show that \(f(z) \in H_T(f, g; \alpha, k)\) satisfies the coefficient inequality (12). If \(f(z) \in H_T(f, g; \alpha, k)\) then the function \(\Phi(z)\) given by (13) satisfies \(\text{Re} \{\Phi(z)\} > \alpha\). This implies that

\[
(f * g)(z) = z - \sum_{n=2}^{\infty} b_n a_n z^n \neq 0 (z \in U \setminus \{0\}),
\]

Noting that \(\frac{(f * g)(r)}{r}\) is the real continuous function in the open interval \((0, 1)\) with \(f(0) = 1\), we have

\[
1 - \sum_{n=2}^{\infty} b_n a_n r^{n-1} > 0 (0 < r < 1). \tag{15}
\]
Now
\[ \Phi(r) = \frac{1 - \sum_{n=2}^{\infty} nb_n a_n r^{n-1} - k \sum_{n=2}^{\infty} n(n-1) b_n a_n r^{n-1}}{1 - \sum_{n=2}^{\infty} b_n a_n r^{n-1}} > \alpha, \]
and consequently by (15) we obtain
\[ \sum_{n=2}^{\infty} (kn^2 + n - kn - \alpha) b_n a_n r^{n-1} \leq 1 - \alpha. \]  \(16\)

Letting \( r \to 1^- \) in (16), we get (14). This completes the proof of Theorem 2.2.

**Corollary 2.3.** Let the function \( f \) defined by (10) be in the class \( H_T(f, g; \alpha, k) \), then
\[ a_n \leq \frac{(1 - \alpha)}{(kn^2 + n - kn - \alpha) b_n} (n \geq 2). \]  \(17\)
The result is sharp for the function
\[ f(z) = z - \frac{(1 - \alpha)}{(kn^2 + n - kn - \alpha) b_n} z^n (n \geq 2). \]  \(18\)

### 3. Distortion theorems

**Theorem 3.1.** Let the function \( f(z) \) defined by (10) belong to the class \( H_T(f, g; \alpha, k) \). Then for \( |z| = r < 1 \), we have
\[ r - \frac{(1 - \alpha)}{(2k + 2 - \alpha)b_2} r^2 \leq |f(z)| \leq r + \frac{(1 - \alpha)}{(2k + 2 - \alpha)b_2} r^2, \]  \(19\)
provided \( b_n \geq b_2 \) \((n \geq 2)\). The result is sharp with equality for the function \( f(z) \) defined by
\[ f(z) = z - \frac{(1 - \alpha)}{(2k + 2 - \alpha)b_2} z^2 \]  \(20\)
at \( z = r \) and \( z = re^{i(2n+1)\pi} \) \((n \in \mathbb{N})\).

**Proof.** We have
\[ |f(z)| \leq r + \sum_{n=2}^{\infty} a_n r^n \leq r + r^2 \sum_{n=2}^{\infty} a_n. \]  \(21\)
Since for \( n \geq 2 \), we have

\[(2k + 2 - \alpha)b_2 \leq (kn^2 + n - kn - \alpha) b_n,\]

then (14) yields

\[(2k + 2 - \alpha)b_2 \sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} (kn^2 + n - kn - \alpha) b_n a_n \leq (1 - \alpha) \quad (22)\]

or

\[\sum_{n=2}^{\infty} a_n \leq \frac{(1 - \alpha)}{(2k + 2 - \alpha)b_2}. \quad (23)\]

From (23) and (21) we have

\[|f(z)| \leq r + \frac{(1 - \alpha)}{(2k + 2 - \alpha)b_2} r^2\]

and similarly, we have

\[|f(z)| \geq r - \frac{(1 - \alpha)}{(2k + 2 - \alpha)b_2} r^2.\]

This completes the proof of Theorem 3.1.

**Theorem 3.2.** Let the function \( f(z) \) defined by (10) belong to the class \( H_T(f, g; \alpha, k) \). Then for \(|z| = r < 1\), we have

\[1 - \frac{2(1 - \alpha)}{(2k + 2 - \alpha)b_2} r \leq |f'(z)| \leq 1 + \frac{2(1 - \alpha)}{(2k + 2 - \alpha)b_2} r, \quad (24)\]

provided \( b_n \geq b_2 \) (\( n \geq 2 \)). The result is sharp for the function \( f(z) \) given by (20) at \( z = r \) and \( z = re^{i(2n+1)\pi} \) (\( n \in \mathbb{N} \)).

**Proof.** For a function \( f(z) \in H_T(f, g; \alpha, k) \), it follows from (14) and (23) that

\[\sum_{n=2}^{\infty} na_n \leq \frac{2(1 - \alpha)}{(2k + 2 - \alpha)b_2}. \quad (25)\]

Since the remaining part of the proof is similar to the proof of Theorem 3.1, we omit the details.
4. Extreme points

**Theorem 4.1.** The class $H_T(f,g;\alpha,k)$ is closed under convex linear combinations.

**Proof.** Let $f_j(z) \in H_T(f,g;\alpha,k)$ ($j = 1, 2$), where

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j}z^n \quad (a_{n,j} \geq 0; \quad j = 1, 2). \quad (26)$$

Then it is sufficient to prove that the function $h(z)$ given by

$$h(z) = \mu f_1(z) + (1 - \mu) f_2(z) \quad (0 \leq \mu \leq 1)$$

is also in the class $H_T(f,g;\alpha,k)$. For $0 \leq \mu \leq 1$

$$h(z) = z - \sum_{n=2}^{\infty} [\mu a_{n,1} + (1 - \mu) a_{n,2}]z^n,$$

and with the aid of Theorem 2.2, we have

$$\sum_{n=2}^{\infty} (kn^2 + n - kn - \alpha) b_n \cdot [\mu a_{n,1} + (1 - \mu) a_{n,2}]z^n \leq \mu (1 - \alpha) + (1 - \mu)(1 - \alpha) = 1 - \alpha,$$

which implies that $h(z) \in H_T(f,g;\alpha,k)$. This completes the proof of Theorem 4.1. \qed

As a consequence of Theorem 4.1, there exist extreme points of the class $H_T(f,g;\alpha,k)$, which are given by:

**Theorem 4.2.** Let $f_1(z) = z$ and

$$f_n(z) = z - \frac{(1 - \alpha)}{(kn^2 + n - kn - \alpha) b_n} z^n.$$

Then $f(z)$ is in the class $H_T(f,g;\alpha,k)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z), \quad (27)$$

where $\mu_n \geq 0$ ($n \geq 1$) and $\sum_{n=1}^{\infty} \mu_n = 1$. 
Proof. Assume that
\[ f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z) = z - \sum_{n=2}^{\infty} \frac{(1 - \alpha)}{(kn^2 + n - kn - \alpha) b_n} \mu_n z^n. \]

Then it follows that
\[
\sum_{n=2}^{\infty} \frac{(kn^2 + n - kn - \alpha) b_n}{(1 - \alpha) (kn^2 + n - kn - \alpha) b_n} \mu_n = \sum_{n=2}^{\infty} \mu_n = (1 - \mu_1) \leq 1. \tag{28}
\]

So, by Theorem 2.2, we have \( f(z) \in H_T(f, g; \alpha, k) \).
Conversely, assume that the function \( f(z) \) defined by (4) belongs to the class \( H_T(f, g; \alpha, k) \). Then \( a_n \) are given by (14). Setting
\[
\mu_n = \frac{(kn^2 + n - kn - \alpha) b_n}{(1 - \alpha) a_n},
\]
and
\[
\mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n,
\]
we can see that \( f(z) \) can be expressed in the form (27). This completes the proof of Theorem 4.2.

**Corollary 4.3.** The extreme points of the class \( H_T(f, g; \alpha, k) \) are the functions \( f_1(z) = z \) and
\[
f_n(z) = z - \frac{(1 - \alpha)}{(kn^2 + n - kn - \alpha) b_n} z^n (n \geq 2).\]

### 5. Partial sums

In this section, applying methods used by Silverman [21], we investigate the ratio of a function of the form (1) to its sequence of partial sums \( f_m(z) = z + \sum_{n=2}^{m} a_n z^n \). More precisely, we will determine sharp lower bounds for \( \text{Re} \left\{ \frac{f(z)}{f_m(z)} \right\} \), \( \text{Re} \left\{ \frac{f'(z)}{f_m'(z)} \right\} \) and \( \text{Re} \left\{ \frac{f''(z)}{f_m''(z)} \right\} \). In the sequel, we will make use of the well-known result that \( \text{Re} \left\{ \frac{1 + w(z)}{1 - w(z)} \right\} > 0 (z \in U) \) if and only if \( w(z) = \sum_{n=1}^{\infty} c_n z^n \) satisfies the inequality \( |w(z)| \leq |z| \).
Theorem 5.1. If \( f(z) \) is of the form (1) and satisfies the condition (12) and \( \frac{f(z)}{z} \neq 0 \) \( (0 < |z| < 1) \), then

\[
\text{Re} \left\{ \frac{f(z)}{f_m(z)} \right\} \geq 1 - \frac{1}{C_{m+1}} \tag{30}
\]

and

\[
C_n \geq \begin{cases} 
1 & n = 2, 3, \ldots, m \\
C_{m+1} & n = m+1, m+2, \ldots
\end{cases}, \tag{31}
\]

where

\[
C_n = \frac{(kn^2 + n - kn - \alpha) b_n}{(1 - \alpha)}.
\tag{32}
\]

The result in (30) is sharp for every \( m \), with the extremely function

\[
f(z) = z + z^{m+1}. \tag{33}
\]

Proof. We may write

\[
\frac{1 + w(z)}{1 - w(z)} = C_{m+1} \left\{ \frac{f(z)}{f_m(z)} - \left(1 - \frac{1}{C_{m+1}} \right) \right\}
\]

\[
= \left\{ \frac{1 + \sum_{n=2}^{m} a_n z^{n-1} + C_{m+1} \sum_{n=m+1}^{\infty} a_n z^{n-1}}{1 + \sum_{n=2}^{m} a_n z^{n-1}} \right\}.
\tag{34}
\]

Then

\[
w(z) = \frac{C_{m+1} \sum_{n=m+1}^{\infty} a_n z^{n-1}}{2 + 2 \sum_{n=2}^{m} a_n z^{n-1} + C_{m+1} \sum_{n=m+1}^{\infty} a_n z^{n-1}}
\]

and

\[
|w(z)| \leq \frac{C_{m+1} \sum_{n=m+1}^{\infty} |a_n|}{2 - 2 \sum_{n=2}^{m} |a_n| - C_{m+1} \sum_{n=m+1}^{\infty} |a_n|}.
\]

Now \( |w(z)| \leq 1 \) if

\[
2C_{m+1} \sum_{n=m+1}^{\infty} |a_n| \leq 2 - 2 \sum_{n=2}^{m} |a_n|,
\]
which is equivalent to
\[
\sum_{n=2}^{m} |a_n| + C_{m+1} \sum_{n=m+1}^{\infty} |a_n| \leq 1. 
\] (35)

It is suffices to show that the left hand side of (35) is bounded above by
\[
\sum_{n=2}^{\infty} C_n |a_n|, 
\]
which is equivalent to
\[
\sum_{n=2}^{m} (C_n - 1) |a_n| + \sum_{n=m+1}^{\infty} (C_n - C_{m+1}) |a_n| \geq 0.
\]

To see that the function \( f \) given by (33) gives the sharp result, we observe for \( z = re^{i\pi/n} \) that
\[
\frac{f(z)}{f_m(z)} = 1 + \frac{z^m}{C_{m+1}}.
\] (36)

Letting \( z \to 1^- \), we have
\[
\frac{f(z)}{f_m(z)} = 1 - \frac{1}{C_{m+1}}.
\]

This completes the proof of Theorem 5.1.

**Theorem 5.2.** If \( f(z) \) is of the form (1) and satisfies the condition (12) and \( \frac{f(z)}{z} \neq 0 \) (0 < \( |z| < 1 \), then
\[
Re \left\{ \frac{f_m(z)}{f(z)} \right\} \geq \frac{C_{m+1}}{1 + C_{m+1}}.
\]
The result is sharp for every \( m \), with the extremely function \( f(z) \) given by (33).

**Proof.** We may write
\[
\frac{1 + w(z)}{1 - w(z)} = (1 + C_{m+1}) \left\{ \frac{f_m(z)}{f(z)} - \frac{C_{m+1}}{1 + C_{m+1}} \right\}
\]
\[
= \left\{ \frac{1 + \sum_{n=2}^{m} a_n z^{n-1} - C_{m+1} \sum_{n=m+1}^{\infty} a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n z^{n-1}} \right\},
\] (37)
where
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\[ w(z) = \frac{(1 + C_{m+1}) \sum_{n=m+1}^{\infty} a_n z^{n-1}}{2 + 2 \sum_{n=2}^{m} a_n z^{n-1} + (C_{m+1} - 1) \sum_{n=m+1}^{\infty} a_n z^{n-1}}, \]

and

\[ |w(z)| \leq \frac{(1 + C_{m+1}) \sum_{n=m+1}^{\infty} |a_n|}{2 - 2 \sum_{n=2}^{m} |a_n| - (C_{m+1} - 1) \sum_{n=m+1}^{\infty} |a_n|}. \]

Now \(|w(z)| \leq 1\) if and only if

\[ 2C_{m+1} \sum_{n=m+1}^{\infty} |a_n| \leq 2 - 2 \sum_{n=2}^{m} |a_n|, \]

which is equivalent to

\[ \sum_{n=2}^{m} |a_n| + C_{m+1} \sum_{n=m+1}^{\infty} |a_n| \leq 1. \] (38)

It is suffices to show that the left hand side of (38) is bounded above by \(\sum_{n=2}^{\infty} C_n |a_n|\), which is equivalent to

\[ \sum_{n=2}^{m} (C_n - 1) |a_n| + \sum_{n=m+1}^{\infty} (C_n - C_{m+1}) |a_n| \geq 0. \]

This completes the proof of Theorem 5.2.

**Theorem 5.3.** If \(f(z)\) is of the form (1) and satisfies the condition (12) and \(\frac{f(z)}{z} \neq 0\) \((0 < |z| < 1)\), then

\[ (a) \quad \text{Re} \left\{ \frac{f'(z)}{f_m(z)} \right\} \geq 1 - \frac{m + 1}{C_{m+1}} \] (39)

and

\[ (b) \quad \text{Re} \left\{ \frac{f_m'(z)}{f'(z)} \right\} \geq \frac{C_{m+1}}{1 + m + C_{m+1}}, \] (40)

where

\[ C_n \geq \begin{cases} 
1 & n = 1, 2, 3, \ldots, m \\
\frac{C_{m+1}}{m + 1} & n = m + 1, m + 2, \ldots
\end{cases} \]
and $C_n$ is defined by (32). The estimates in (39) and (40) are sharp with the extremely function given by (33).

**Proof.** We prove only (a), which is similar in spirit of the proof of Theorem 5.1. The proof of (b) follows the pattern of that in Theorem 5.2. We write

$$\frac{1 + w(z)}{1 - w(z)} = C_{m+1} \left\{ \frac{f'(z)}{f'_m(z)} - \left( 1 - \frac{1 + m}{C_{m+1}} \right) \right\}$$

$$= \left\{ \frac{1 + \sum_{n=2}^{m} na_n z^{n-1} + \frac{C_{m+1}}{m+1} \sum_{n=m+1}^{\infty} na_n z^{n-1}}{1 + \sum_{n=2}^{m} na_n z^{n-1}} \right\},$$

where

$$w(z) = \frac{\frac{C_{m+1}}{m+1} \sum_{n=m+1}^{\infty} na_n z^{n-1}}{2 + 2 \sum_{n=2}^{m} na_n z^{n-1} + \frac{C_{m+1}}{m+1} \sum_{n=m+1}^{\infty} na_n z^{n-1}}$$

and

$$|w(z)| \leq \frac{\frac{C_{m+1}}{m+1} \sum_{n=m+1}^{\infty} n |a_n|}{2 - 2 \sum_{n=2}^{m} n |a_n| - \frac{C_{m+1}}{m+1} \sum_{n=m+1}^{\infty} n |a_n|}.$$ 

Now $|w(z)| \leq 1$ if and only if

$$\sum_{n=2}^{m} n |a_n| + \frac{C_{m+1}}{m+1} \sum_{n=m+1}^{\infty} n |a_n| \leq 1, \tag{41}$$

since the left hand side of (41) is bounded above by $\sum_{n=2}^{\infty} C_n |a_n|$, this completes the proof of Theorem 5.3.

**6. Integral means**

In [19] Silverman found that the function $f_2 = z - \frac{z^2}{2}$ is often extremal over the family $T$. He applied this function to resolve his integral means inequality, conjectured and settled in [20]:

[41]
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\[
\int_0^{2\pi} |f(re^{i\theta})|^\delta d\theta \leq \int_0^{2\pi} |f_2(re^{i\theta})|^\delta d\theta,
\]

for all \( f \in T, \delta > 0 \) and \( 0 < r < 1 \). In [19], he also proved his conjecture for the subclasses \( T^*(\alpha) \) and \( C(\alpha) \) of \( T \), where \( C(\alpha) \) and \( T^*(\alpha) \) are the classes of convex and starlike functions of order \( \alpha, 0 \leq \alpha < 1 \), respectively.

In this section, we prove Silverman’s conjecture for functions in the class \( H_T(f, g; \alpha, k) \).

**Lemma 6.1 ([12]).** If the functions \( f \) and \( g \) are analytic in \( U \) with \( g \prec f \), then for \( \delta > 0 \) and \( 0 < r < 1 \),

\[
\int_0^{2\pi} |g(re^{i\theta})|^\delta d\theta \leq \int_0^{2\pi} |f(re^{i\theta})|^\delta d\theta.
\]

Applying Theorems 2.1, 2.2 and Lemma 6.1 we prove the following theorem.

**Theorem 6.2.** Suppose \( f(z) \in H_T(f, g; \alpha, k), \delta > 0 \), the sequence \( \{b_n\} \ (n \geq 2) \) is non-decreasing and \( f_2(z) \) is defined by:

\[
f_2(z) = z - \frac{1 - \alpha}{(2k + 2 - \alpha)b_2}z^2,
\]

then for \( z = re^{i\theta}, 0 < r < 1 \), we have

\[
\int_0^{2\pi} |f(re^{i\theta})|^\delta d\theta \leq \int_0^{2\pi} |f_2(re^{i\theta})|^\delta d\theta.
\]

**Proof.** For \( f(z) \) of the form (10) (43) is equivalent to prove that

\[
\int_0^{2\pi} \left| 1 - \sum_{n=2}^{\infty} a_n z^{n-1} \right|^\delta d\theta \leq \int_0^{2\pi} \left| 1 - \frac{(1 - \alpha)}{(2k + 2 - \alpha)b_2}z \right|^\delta d\theta.
\]

By using Lemma 6.1, it suffices to show that

\[
1 - \sum_{n=2}^{\infty} a_n z^{n-1} \prec 1 - \frac{(1 - \alpha)}{(2k + 2 - \alpha)b_2}z.
\]

Setting

\[
1 - \sum_{n=2}^{\infty} a_n z^{n-1} = 1 - \frac{(1 - \alpha)}{(2k + 2 - \alpha)b_2}w(z),
\]

and using (14) and the hypothesis \( \{b_n\} \ (n \geq 2) \) is non-decreasing, we obtain
\[
|w(z)| = \left| \frac{(2k + 2 - \alpha)b_2}{(1 - \alpha)} \sum_{n=2}^{\infty} a_n z^{n-1} \right|
\leq |z| \sum_{n=2}^{\infty} \frac{(2k + 2 - \alpha)b_2}{(1 - \alpha)} a_n
\leq |z| \sum_{n=2}^{\infty} \frac{(kn^2 + n - kn - \alpha)b_n}{(1 - \alpha)} a_n
\leq |z|.
\]

This completes the proof of Theorem 6.2.

\section{Neighborhood for the class $H_T(f, g; \alpha, k$)}

In [8], Goodman and in [18], Ruscheweyh defined the $\delta$- neighborhood of function $T$ by

\[ N_\delta(f) = \left\{ h \in T : h(z) = z - \sum_{n=2}^{\infty} c_n z^n, \sum_{n=2}^{\infty} n|a_n - c_n| \leq \delta \right\}. \]

(46)

In particular, if

\[ e(z) = z, \]

(47)

we immediately have

\[ N_\delta(e) = \left\{ h \in T : h(z) = z - \sum_{n=2}^{\infty} c_n z^n, \sum_{n=2}^{\infty} n|c_n| \leq \delta \right\}. \]

(48)

\textbf{Theorem 7.1.} If $b_n \geq b_2 \ (n \geq 2)$ and

\[ \delta = \frac{2(1 - \alpha)}{(2k + 2 - \alpha)b_2}, \]

(49)

then

\[ H_T(f, g; \alpha, k) \subset N_\delta(e) \]

(50)

\textbf{Proof.} Let $f \in H_T(f, g; \alpha, k)$. Then, in view of the assertion (14) of Theorem 2.2 and the given condition that $b_n \geq b_2 \ (n \geq 2)$, we have
\[
(2k + 2 - \alpha) b_2 \sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} \left( kn^2 + n - kn - \alpha \right) b_n a_n \\
\leq (1 - \alpha),
\]
so that
\[
\sum_{n=2}^{\infty} a_n \leq \frac{(1 - \alpha)}{(2k + 2 - \alpha) b_2}. \tag{51}
\]

Making use of (14) again, in conjunction with (51), we get
\[
b_2 \sum_{n=2}^{\infty} na_n \leq (1 - \alpha) + (\alpha - 2k) b_2 \sum_{n=2}^{\infty} a_n \\
\leq (1 - \alpha) + (\alpha - 2k) b_2 \frac{(1 - \alpha)}{(2k + 2 - \alpha) b_2} \\
\leq \frac{2(1 - \alpha)}{(2k + 2 - \alpha)}.
\]

Hence
\[
\sum_{n=2}^{\infty} na_n \leq \frac{2(1 - \alpha)}{(2k + 2 - \alpha) b_2} = \delta, \tag{52}
\]
which, by means of the definition (48). This completes the proof of Theorem 7.1.

Now we determine the neighborhood for the class \( H_T^{(\gamma)}(f, g; \alpha, k) \), which we define as follows. A function \( f(z) \in T \) is said to the class \( H_T^{(\gamma)}(f, g; \alpha, k) \) if there exists a function \( \zeta(z) \in H_T(f, g; \alpha, k) \) such that
\[
\left| \frac{f(z)}{\zeta(z)} - 1 \right| < 1 - \gamma \quad (0 \leq \gamma < 1) \tag{53}
\]

**Theorem 7.2.** If \( \zeta(z) \in H_T(f, g; \alpha, k) \) and
\[
\gamma = 1 - \frac{\delta (2k + 2 - \alpha) b_2}{2 [(2k + 2 - \alpha) b_2 - (1 - \alpha)]} \tag{54}
\]

then
\[
N_\delta(\zeta) \subset H_T^{(\gamma)}(f, g; \alpha, k) \tag{55}
\]
where
\[
\delta \leq 2 - 2 (1 - \alpha) [(2k + 2 - \alpha) b_2]^{-1}. \tag{56}
\]
Proof. Suppose that $\zeta(z) \in N_\delta(\zeta)$. We find from (46) that

$$\sum_{n=2}^{\infty} |a_n - c_n| \leq \delta,$$

(57)

which readily implies that

$$\sum_{n=2}^{\infty} |a_n - c_n| \leq \frac{\delta}{2}.$$  

(58)

Next, since $\zeta(z) \in H_T(f, g; \alpha, k)$, we have [cf. equation (51)] that

$$\sum_{n=2}^{\infty} c_n \leq \frac{(1 - \alpha)}{(2k + 2 - \alpha)b_2},$$

(59)

so that

$$\left| \frac{f(z)}{\zeta(z)} - 1 \right| \leq \frac{\sum_{n=2}^{\infty} |a_n - c_n|}{1 - \sum_{n=2}^{\infty} c_n} \leq \frac{\delta}{2} \frac{(2 + 2k - \alpha)}{(2k + 2 - \alpha)b_2 - (1 - \alpha)} \leq 1 - \gamma,$$

thus, by the above definition, $f(z) \in H_T^{(g)}(f, g; \alpha, k)$ for $\gamma$ given by (54). This completes the proof of Theorem 7.2. \qed

Remark 7.3. (i) Taking $g(z) = z + \sum_{n=2}^{\infty} \sigma_n z^n \ (q, s \in \mathbb{N}, k \geq 0, 0 \leq \alpha < 1)$, where $\sigma_n$ given by (5), in the above results we obtain the corresponding results for the class $TW_{s,q}^\alpha(\alpha, k)$, we obtain the results obtained by Dziok and Murugusundaramoorthy (see [5]);
(ii) Taking $g(z) = \frac{z}{1-z}$ and $g(z) = z + \sum_{n=2}^{\infty} \left( \frac{l+1+\mu(n-1)}{l+1} \right)^m z^n$, respectively, in the above results we obtain the corresponding results for the classes $S_T(k, \alpha)$ and $T \Sigma_m(\mu, l, \alpha, k)$, respectively.

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