QUALITATIVE AND QUANTITATIVE UNCERTAINTY PRINCIPLES FOR THE GENERALIZED FOURIER TRANSFORM ASSOCIATED WITH THE RIEMANN-LIOUVILLE OPERATOR

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The aim of this paper is to establish an extension of qualitative and quantitative uncertainty principles for the Fourier transform connected with the Riemann-Liouville operator.

1. Introduction

Classical uncertainty principles give us information about a function and its Fourier transform. If we try to limit the behavior of one we lose control of the other. Uncertainty principles have implications in two main areas: quantum physics and signal analysis. In quantum physics they tell us that a particle's speed and position cannot both be measured with infinite precision. In signal analysis they tell us that if we observe a signal only for a finite period of time, we will lose information about the frequencies the signal consists of. The mathematical equivalent is that a function and its Fourier transform cannot both be

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There are two categories of uncertainty principles: Quantitative uncertainty principles and Qualitative uncertainty principles.

Quantitative uncertainty principles is just another name for some special inequalities. These inequalities give us information about how a function and its Fourier transform relate. They are called uncertainty principles since they are similar to the classical Heisenberg Uncertainty Principle, which has had a big part to play in the development and understanding of quantum physics. For example: Benedicks [3], Slepian and Pollak [29], Landau and Pollak [18], and Donoho and Stark [10] paid attention to the supports of functions and gave qualitative uncertainty principles for the Fourier transforms.

Qualitative uncertainty principles are not inequalities, but are theorems that tell us how a function (and its Fourier transform) behave under certain circumstances. For example: Hardy [14], Morgan [23], Cowling and Price [8], Beurling [4], Miyachi [22] theorems enter within the framework of the quantitative uncertainty principles.

The quantitative and qualitative uncertainty principles has been studied by many authors for various Fourier transforms, for examples (cf. [6, 7, 12, 13, 19, 20, 30]).

In [2], the authors considered the singular partial differential operators defined by

$$\Delta_1 = \frac{\partial}{\partial x},$$

$$\Delta_2 = \frac{\partial^2}{\partial r^2} + \frac{2\alpha + 1}{r} \frac{\partial}{\partial r} - \frac{\partial^2}{\partial x^2}, \quad (r,x) \in (0,\infty) \times \mathbb{R}, \quad \alpha \geq 0$$

and they associated to $\Delta_1$ and $\Delta_2$ the following integral transform, called the Riemann-Liouville operator, defined on $C^\infty_\ast(\mathbb{R}^2)$ by

$$R_\alpha(f)(r,x) = \begin{cases} \frac{\alpha}{\pi} \int_{-1}^{1} \int_{-1}^{1} f(rs\sqrt{1-t^2}, x+rt)(1-t^2)^{\alpha-\frac{1}{2}}(1-s^2)^{\alpha-1} dt ds, & \text{if } \alpha > 0 \\ \frac{1}{\pi} \int_{-1}^{1} f(rs\sqrt{1-t^2}, x+rt)(1-t^2)^{-\frac{1}{2}} dt, & \text{if } \alpha = 0 \end{cases}$$

In addition, a convolution product and a Fourier transform $F_\alpha$ connected with the mapping $R_\alpha$ have been studied and many harmonic analysis results have been established for the Fourier transform $F_\alpha$ (Inversion formula, Plancherel formula, Paley-Winer and Plancherel theorems, ...). Our purpose in this work is to study the uncertainty principles for the Fourier transform $F_\alpha$ connected with $R_\alpha$.

Our aim here is to consider quantitative and qualitative uncertainty principles when the transform under consideration is the Fourier transform connected with the Riemann-Liouville operator.

The remaining part of the paper is organized as follows. In §2, we recall the main results about the Riemann-Liouville operator. §3 is devoted to generalize Cowling-Price’s theorem for the generalized Fourier transform $F_\alpha$. In
§4 we generalize Miyachi’s theorem and in §5 Beurling’s theorem for $F_\alpha$. §6 is devoted to Donoho-Stark’s uncertainty principle and variants of Heisenberg’s inequalities for $F_\alpha$.

2. Riemann-Liouville operator

In this section, we define and recall some properties of the Riemann-Liouville operator. For more details see ([2, 21]). We denote by

- $C_s(\mathbb{R}^2)$ the space of continuous functions on $\mathbb{R}^2$, even with respect to the first variable.
- $C_{s,\mathcal{C}}(\mathbb{R}^2)$ the subspace of $C_s(\mathbb{R}^2)$ formed by functions with compact support.
- $\mathcal{E}_s(\mathbb{R}^2)$ the space of infinitely differentiable functions on $\mathbb{R}^2$, even with respect to the first variable.
- $S_s(\mathbb{R}^2)$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}^2$, even with respect to the first variable.
- $S^1$ the unit sphere in $\mathbb{R}^2$,
  $$S^1 = \left\{ (\eta, \xi) \in \mathbb{R}^2 : \eta^2 + \xi^2 = 1 \right\}.$$
- $\mathbb{R}_+^2 = \left\{ (r, x) \in \mathbb{R}^2 : r > 0 \right\}$.

It is well known [2] that for all $(\mu, \lambda) \in \mathbb{C}^2$, the system

$$\begin{cases}
\Delta_1 u(r, x) = -i\lambda u(r, x), \\
\Delta_2 u(r, x) = -\mu^2 u(r, x) \\
u(0, 0) = 1, \quad \partial u / \partial r (0, x) = 0, \forall x \in \mathbb{R},
\end{cases}$$

admits a unique solution $\varphi_{\mu, \lambda}$, given by

$$\varphi_{\mu, \lambda}(r, x) = j_\alpha(r\sqrt{\mu^2 + \lambda^2})e^{-i\lambda x},$$

where $j_\alpha$ is the normalized Bessel function defined by

$$\forall z \in \mathbb{C}, \quad j_\alpha(z) = \Gamma(\alpha + 1)\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+1+\alpha)}(z/2)^{2k}.$$
Definition 1. The Riemann-Liouville operator is defined on $C^\infty(\mathbb{R}^2)$ by: $\forall (r, x) \in \mathbb{R}^2_+$

$$
\mathcal{R}_\alpha f(r, x) = \begin{cases} 
\frac{\alpha}{\pi} \int_{-1}^{1} \int_{-1}^{1} f(rs\sqrt{1-t^2}, x+rt)(1-t^2)^{\alpha-\frac{1}{2}}(1-s^2)^{\alpha-1} dt ds & \text{if } \alpha > 0 \\
\frac{1}{\pi} \int_{-1}^{1} f(r\sqrt{1-t^2}, x+rt)(1-t^2)^{-\frac{1}{2}} dt & \text{if } \alpha = 0.
\end{cases}
$$

Remark 1. (i) The function $\varphi_{\mu, \lambda}, (\mu, \lambda) \in \mathbb{C}^2$, can be written as $\forall (r, x) \in \mathbb{R}^2_+$, $\varphi_{\mu, \lambda}(r, x) = \mathcal{R}_\alpha(\cos(\mu \cdot) e^{-i\lambda \cdot})(r, x)$.

(ii) For all $\nu \in \mathbb{N}^2, (r, x) \in \mathbb{R}^2$ and $z = (\mu, \lambda) \in \mathbb{C}^2$,

$$
|D^\nu_z \varphi_{\mu, \lambda}(r, x)| \leq ||(r, x)||^{|\nu|} \exp(2||(r, x)|| ||\text{Im}z||),
$$

where $D^\nu_z = \frac{\partial^{|\nu|}}{\partial z_1^{\nu_1} \partial z_2^{\nu_2}}$ and $|\nu| = \nu_1 + \nu_2$.

Now let $\Gamma$ be the set

$$
\Gamma = \mathbb{R}^2 \cup \left\{(it, x); (t, x) \in \mathbb{R}^2, |t| \leq |x| \right\}.
$$

$\Gamma_+$ the subset of $\Gamma$, given by

$$
\Gamma_+ = \mathbb{R}^2 \cup \left\{(it, x); (t, x) \in \mathbb{R}^2, 0 \leq t \leq |x| \right\}.
$$

We have for all $(\mu, \lambda) \in \Gamma$,

$$
\sup_{(r,x) \in \mathbb{R}^2} |\varphi_{\mu, \lambda}(r, x)| = 1.
$$

In the following, we denote by

- $d\nu_\alpha(r, x)$ the measure defined on $\mathbb{R}^2_+$ by

$$
d\nu_\alpha(r, x) = k_\alpha r^{2\alpha+1} dr \otimes dx,
$$

with

$$
k_\alpha = \frac{1}{2^{\alpha} \Gamma(\alpha + 1)(2\pi)^{1/2}}.
$$

- $L^p(d\nu_\alpha), 1 \leq p \leq \infty$, the space of measurable functions on $\mathbb{R}^2_+$, satisfying

$$
\|f\|_{L^p(d\nu_\alpha)} = \left( \int_{\mathbb{R}^2_+} |f(r, x)|^p d\nu_\alpha(r, x) \right)^{1/p} < \infty, \ 1 \leq p < \infty,
$$

$$
\|f\|_{L^\infty(d\nu_\alpha)} = \text{ess sup}_{(r,x) \in \mathbb{R}^2_+} |f(r, x)| < \infty, \ p = \infty.
$$
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• $\mathcal{B}_{\Gamma_+}$ the $\sigma$-algebra defined on $\Gamma_+$ by

$$\mathcal{B}_{\Gamma_+} = \left\{ \theta^{-1}(B) : \ B \in \mathcal{B}_{\text{Bor}(\mathbb{R}^2_+)} \right\},$$

where $\theta$ defined on the set $\Gamma_+$ by $\theta(\lambda, \mu) = (\sqrt{\mu^2 + \lambda^2}, \lambda)$.

• $d\gamma_\alpha$ the measure defined on $\mathcal{B}_{\Gamma_+}$ by

$$\forall A \subset \mathcal{B}_{\Gamma_+}, \quad \gamma_\alpha(A) = \nu_\alpha(\theta(A)).$$

• $L^p(d\gamma_\alpha), 1 \leq p \leq \infty$, the space of measurable functions on $\Gamma_+$, satisfying

$$\|f\|_{L^p(d\gamma_\alpha)} = \left( \int_{\Gamma_+} |f(\mu, \lambda)|^pd\gamma_\alpha(\mu, \lambda) \right)^{1/p} < \infty, \quad 1 \leq p < \infty,$$

$$\|f\|_{L^\infty(d\gamma_\alpha)} = \text{ess sup}_{(\mu, \lambda) \in \Gamma_+} |f(\mu, \lambda)| < \infty, \quad p = \infty.$$

We have the following properties.

**Proposition 1.** i) For every nonnegative measurable function $g$ on $\Gamma_+$, we have

$$\int_{\Gamma_+} f(\mu, \lambda)d\gamma_\alpha(\mu, \lambda) = k_\alpha \int_{\mathbb{R}^2_+} f(\mu, \lambda)(\mu^2 + \lambda^2)^{\alpha} \mu d\mu d\lambda$$

$$+ \int_{\mathbb{R}^2_+} f(i\mu, \lambda)(\lambda^2 - \mu^2)^{\alpha} \mu d\mu d\lambda.$$

ii) For every nonnegative measurable function $f$ on $\mathbb{R}^2_+$ (resp. integrable on $\mathbb{R}^2_+$ with respect to the measure $d\nu_\alpha$), $f \circ \theta$ is a measurable nonnegative function on $\Gamma_+$, (resp. integrable on $\Gamma_+$ with respect to the measure $d\gamma_\alpha$) and we have

$$\int_{\Gamma_+} f \circ \theta(\mu, \lambda)d\gamma_\alpha(\mu, \lambda) = \int_{\mathbb{R}^2_+} f(r, x)d\nu_\alpha(r, x). \quad (2.2)$$

In the following we recall some results on the dual of the Riemann-Liouville operator $R_\alpha$.

**Definition 2.** The dual $^t\mathcal{R}_\alpha$ of the Riemann-Liouville operator $\mathcal{R}_\alpha$ is defined by

: $\forall (s, y) \in \mathbb{R}^2,$

$$^t\mathcal{R}_\alpha(f)(s, y) = \begin{cases} \frac{\alpha}{\pi} \int_{1}^{\infty} \int_{\sqrt{u^2 - r^2}}^{\infty} f(u, x + v)(u^2 - v^2 - r^2)^{\alpha - 1}(1 - s^2)^{\alpha - 1} u du dv & \text{if } \alpha > 0 \\
\frac{1}{\pi} \int_{\mathbb{R}} f(r^2 + (x - y)^2), y) dy, & \text{if } \alpha = 0 \end{cases} \quad (2.3)$$
Example 1. Let \( p \in [1, \infty) \). For all \( a > 0, \beta > 0 \) we have
\[
\forall (s, y) \in \mathbb{R}^2, \quad {^tR}_\alpha(E_{a, \beta}^{p})(s, y) = C(a, \beta, p)E_{\frac{a \beta}{1 + p}, 1 + \beta}^{p}(s, y), \tag{2.4}
\]
with \( E_{a, \beta} \) is the Gauss kernel associated with the Riemann-Liouville operator \( R_\alpha \) defined by
\[
\forall (r, x) \in \mathbb{R}^2, \quad E_{a, \beta}(r, x) = k(a, \beta)e^{-a(\beta r^2 + x^2)}, \tag{2.5}
\]
where
\[
k(a, \beta) = \frac{2\sqrt{\pi}a^{\alpha + \frac{1}{2}}(\beta)^{\alpha + 1}}{\Gamma(\alpha + 1)}, \quad \text{and} \quad C(a, \beta, p) = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi}} \left[ \frac{(1 + \beta)^{p - 1}}{a\beta^p} \right]^{\frac{2\alpha + 1}{2}}.
\]

Proposition 2. The function \( {^tR}_\alpha(f) \) defined almost everywhere on \( \mathbb{R}^2_+ \), by relation (2.3), is Lebesgue integrable on \( \mathbb{R}^2_+ \). Moreover for all bounded function \( g \in C_\ast(\mathbb{R}^2) \), we have the formula
\[
\int_{\mathbb{R}^2_+} {^tR}_\alpha(f)(s, y)g(s, y)dsdy = \int_{\mathbb{R}^2_+} R_\alpha(g)(r, x)f(r, x)r^{2\alpha + 1}drdx. \tag{2.6}
\]

Remark 2. Let \( f \) be in \( L^1(d\nu_\alpha) \). By taking \( g \equiv 1 \) in the relation (2.6) we deduce that
\[
\int_{\mathbb{R}^2_+} {^tR}_\alpha(f)(s, y)dsdy = \int_{\mathbb{R}^2_+} f(r, x)r^{2\alpha + 1}drdx. \tag{2.7}
\]

We consider the generalized Fourier transform \( F_\alpha \) associated with the Riemann Liouville operator \( R_\alpha \) and we recall its main properties.

Definition 3. The Fourier transform associated with the Riemann Liouville mean operator is defined on \( L^1(d\nu_\alpha) \) by
\[
\forall (\mu, \lambda) \in \Gamma, F_\alpha(f)(\mu, \lambda) = \int_{\mathbb{R}^2_+} f(r, x)\varphi_{\mu, \lambda}(r, x)d\nu_\alpha(r, x). \tag{2.8}
\]

Example 2. Let \( a, \beta > 0 \). The Fourier transform of Gauss kernel associated with Riemann-Liouville operator is given by
\[
\forall (\mu, \lambda) \in \Gamma, F_\alpha(E_{a, \beta})(\mu, \lambda) = C(a, \beta, \alpha)E_{\frac{a \beta}{1 + p}, 1 + \beta}^{p}(\mu, \lambda),
\]
where
\[
C(a, \beta, \alpha) = 2^{4\alpha + 2}\Gamma(\alpha + 1)(a\beta)^{2\alpha + \frac{1}{2}} \left( \frac{\pi}{1 + \beta} \right)^{\frac{2\alpha + 1}{2}}.
\]
Proposition 3. For all \( f \in L^1(d\nu_\alpha) \), we have the relation
\[
\forall (\mu, \lambda) \in \Gamma, \mathcal{F}_\alpha(f)(\mu, \lambda) = \mathcal{F}_0 \circ \mathcal{R}_\alpha(f)(\mu, \lambda),
\] (2.9)
where \( \mathcal{F}_0 \) is the Fourier-cosine transform on \( \mathbb{R}^2 \) defined for \( f \) in \( \mathcal{S}_s(\mathbb{R}^2) \) by
\[
\forall (\mu, \lambda) \in \mathbb{R}^2, \mathcal{F}_0(f)(\mu, \lambda) = \int_{\mathbb{R}_+^2} f(r, x)e^{-i\lambda x} \cos(r\mu)drdx.
\]
In the follow we recall some properties on the Fourier transform \( \mathcal{F}_\alpha \).
For all \( f \in L^1(d\nu_\alpha) \),
\[
\|\mathcal{F}_\alpha(f)\|_{L^\infty(d\gamma_\alpha)} \leq \|f\|_{L^1(d\nu_\alpha)}. \tag{2.10}
\]
For \( f \in L^1(d\nu_\alpha) \) such that \( \mathcal{F}_\alpha(f) \in L^1(d\gamma_\alpha) \), we have the inversion formula for \( \mathcal{F}_\alpha \): for almost every \((r, x) \in \mathbb{R}^2_+\),
\[
f(r, x) = \int_{\Gamma_+} \mathcal{F}_\alpha(f)(\mu, \lambda) \overline{\phi_{\mu, \lambda}(r, x)} d\gamma_\alpha(\mu, \lambda). \tag{2.11}
\]

Theorem 1. (Plancherel formula). For every \( f \) in \( \mathcal{S}_s(\mathbb{R}^2) \), we have
\[
\int_{\Gamma} |\mathcal{F}_\alpha(f)(\lambda, \mu)|^2 d\gamma_\alpha(\lambda, \mu) = \int_{\mathbb{R}^2_+} |f(r, x)|^2 d\nu_\alpha(r, x). \tag{2.12}
\]
In particular, the Fourier transform \( \mathcal{F} \) can be extended to an isometric isomorphism from \( L^2(d\nu_\alpha) \) onto \( L^2(d\gamma_\alpha) \).

Proposition 4. Let \( f \) be in \( L^p(d\nu_\alpha) \), \( p \in [1, 2] \). Then \( \mathcal{F}_\alpha(f) \) belongs to \( L^{p'}(d\gamma_\alpha) \) with \( \frac{1}{p} + \frac{1}{p'} = 1 \), and we have
\[
\|\mathcal{F}_\alpha(f)\|_{L^{p'}(d\gamma_\alpha)} \leq \|f\|_{L^p(d\nu_\alpha)}.
\]
For \((r, x) \in \mathbb{R}^2, s > 0 \), we note \( N_s(r, x) \), by
\[
N_s(r, x) := e^{-s(r^2+x^2)}.
\tag{2.13}
\]
We have
\[
\mathcal{F}_\alpha(N_s(r, x))(t, y) = C(s)e^{-\frac{(t^2+y^2)}{2s}}.
\]
We define the following functions \( W_l^s, \tilde{W}_l^s, l \in \mathbb{N}^2, s > 0 \) by
\[
\forall (r, x) \in \mathbb{R}^2, W_l^s(r, x) = r^{2k}x^m e^{-s(r^2+x^2)}, \quad l = (k, m), \tag{2.14}
\]
and
\[
\forall (r, x) \in \mathbb{R}^2, \tilde{W}_l^s(r, x) = \mathcal{F}_\alpha^{-1}(\lambda^{2k} \mu^m e^{-s(\lambda^2+\mu^2)})(r, x), \quad l = (k, m), \tag{2.15}
\]
Notation. We denote by \( \mathcal{P}_m(\mathbb{R}^2) \) the set of homogeneous polynomials of degree \( m \).
\textbf{Proposition 5. ([26])}. Let \( l \in \mathbb{N}^2 \). For all \( s>0 \), there exists a homogeneous \( Q \in \mathcal{P}_l(\mathbb{R}^2) \) such that
\[
\forall (r,x) \in \mathbb{R}^2, \quad F_\alpha(W_l^\alpha)(r,x) = Q(r) e^{-\frac{1}{4}(r^2+2x^2)}. \tag{2.16}
\]

3. Generalized Cowling-Price theorem for the Generalized Fourier transform

\textbf{Theorem 2.} Let \( f \) be a measurable function on \( \mathbb{R}^2_+ \) such that
\[
\int_{\mathbb{R}^2_+} e^{ap||r(x)||^2} |f(r,x)|^n (1 + ||(r,x)||)^m \, d\nu_\alpha(r,x) < \infty \tag{3.17}
\]
and
\[
\int_{\mathbb{R}^2_+} e^{4bq||\theta(\mu,\xi)||^2} |\mathcal{F}_\alpha(f)(\mu,\xi)|^q (1 + ||(\mu,\xi)||)^m \, d\mu d\xi < \infty, \tag{3.18}
\]
for some constants \( a>0, b>0, 1 \leq p, q < \infty \), and for any \( n \in (2\alpha + 3, 2\alpha + 3 + p] \) and \( m \in (2, 2 + q] \). Then
\begin{enumerate}[(i)]
  \item If \( ab > \frac{1}{4} \), we have \( f = 0 \) almost everywhere.
  \item If \( ab = \frac{1}{4} \), we have \( f = CN_b \).
  \item If \( ab < \frac{1}{4} \), for all \( \delta \in ]b, \frac{1}{4\alpha}[, \) the functions of the form \( f(r,x) = N_\delta(r,x) \), where \( P \in \mathcal{P} \), satisfy (3.17) and (3.18).
\end{enumerate}

\textbf{Proof.} We shall show that \( \mathcal{F}_\alpha(f)(z) \) exists and is an entire function in \( z \in \mathbb{C}^2 \) and
\[
|\mathcal{F}_\alpha(f)(z)| \leq C e^{\frac{1}{2}||\theta(lmz)||^2} (1 + ||lmz||^2)^s, \quad \text{for all } z \in \mathbb{C}^2, \quad \text{for some } s > 0. \tag{3.19}
\]

The first assertion follows from the hypothesis on the function \( f \) and H"older's inequality using (3.17) and the derivation theorem under the integral sign. We want to prove (3.19). Actually, it follows from (2.8) and (2.1) that for all \( z = (z_1, z_2) = (\mu + i\lambda, \xi + i\eta) \in \mathbb{C}^2 \),
\[
|\mathcal{F}_\alpha(f)(\mu + i\lambda, \xi + i\eta)| \leq \int_{\mathbb{R}^2_+} |f(r,x)||\varphi_{(\mu+i\lambda,\xi+i\eta)}(r,x)| \, d\nu_\alpha(r,x)
\leq e^{\frac{||\langle \lambda, \eta \rangle \rangle^2}{a}} \int_{\mathbb{R}^2_+} e^{ap||r(x)||^2} (1 + ||(r,x)||)^n \, e^{-a||\theta(r,x)||^2 - ||\langle \lambda, \eta \rangle \rangle^2} \, d\nu_\alpha(r,x)
\]
Then by using the H"older inequality, (3.17) we can obtain that
\[
|\mathcal{F}_\alpha(f)(\mu + i\lambda, \xi + i\eta)| \leq C e^{\frac{2a^2n^2}{a}} \left( \int_{\mathbb{R}^2_+} (1 + ||(r,x)||)^n \, e^{-ap||r(x)||^2 - ||\langle \lambda, \eta \rangle \rangle^2} \, d\nu_\alpha(r,x) \right)^{\frac{1}{p}}
\leq C e^{\frac{2a^2n^2}{a} \left( \int_0^\infty (1 + t)^{n+2\alpha} e^{-ap(t^2 - ||\langle \lambda, \eta \rangle \rangle^2)} \, dt \right)^{\frac{1}{p}}}
\leq C e^{\frac{||\langle \lambda, \eta \rangle \rangle^2}{a}} \left( 1 + ||\langle \lambda, \eta \rangle \rangle^2 \right)^{\frac{n+2\alpha+2}{p}}
\leq C e^{\frac{1}{2}||\theta(lmz)||^2} (1 + ||lmz||^2)^s. 
\]
Thus (3.19) is proved.

- If $ab = \frac{1}{4}$, then

$$|F_\alpha(f)(z)| \leq Ce^{4\theta|\text{Im}z|}\left(1 + ||\text{Im}z||\right)^{\frac{n + 2\alpha + 2}{p} + \frac{2\alpha + 2}{p'}}.$$ 

Therefore, if we let $g(z) = e^{4b(z_1^2 + z_2^2)}F_\alpha(f)(z)$, then

$$|g(z)| \leq Ce^{4\theta|\text{Re}z|}\left(1 + ||\text{Re}z||\right)^{\frac{n + 2\alpha + 2}{p} + \frac{2\alpha + 2}{p'}}.$$ 

Hence it follows from (3.18) that

$$\int_{\mathbb{R}^2} \frac{|g(\mu, \xi)|^q}{(1 + ||(\mu, \xi)||)^s}d\mu d\xi < \infty.$$ 

Here we use the following lemma.

**Lemma 1.** ([28]) Let $h$ be an entire function on $\mathbb{C}^2$ such that

$$|h(z)| \leq Ce^{a||\text{Re}z||}\left(1 + ||\text{Im}z||\right)^m$$

for some $m > 0$, $a > 0$ and

$$\int_{\mathbb{R}^2} \frac{|h(x)|^q}{(1 + ||(r, x)||)^s}|Q(x)|dx < \infty$$

for some $q \geq 1$, $s > 1$ and $Q \in \mathcal{P}_{M}(\mathbb{R}^2)$. Then $h$ is a polynomial with $\text{deg} h \leq \min\{m, \frac{s-M-2}{q}\}$ and, if $s \leq q + M + 2$, then $h$ is a constant.

Hence by this lemma $g$ is a polynomial, we say $P_b$, with $\text{deg} P_b := d \leq \min\{\frac{n}{p} + \frac{2\alpha + 2}{p'}, \frac{m-2}{q}\}$. Then

$$F_\alpha(f)(\lambda, \mu) = P_b(\lambda, \mu)e^{-4b(\lambda^2 + 2\mu^2)}.$$ 

Thus, by using (2.16), we can find constants $c_i^d$ such that

$$f(r, x) = \sum_{|l| \leq d} c_i^d W_i^d(r, x) \quad \text{for all } (r, x) \in \mathbb{R}^2.$$ 

Therefore, nonzero $f$ satisfies (3.17) provided that

$$n > 2\alpha + 3 + p \min\left\{\frac{n}{p} + \frac{2\alpha + 2}{p'}, \frac{m-2}{q}\right\}.$$ 

Furthermore, if $m \leq q + 2$, then $g$ is a constant by the Lemma 1 and thus

$$F_\alpha(f)(\lambda, \mu) = Ce^{-4b(\lambda^2 + 2\mu^2)} \quad \text{and} \quad f(r, x) = C_b e^{-a||r, x||^2}.$$
When \( n > 2\alpha + 3 \) and \( m > 2 \), these functions satisfy (3.18) and (3.17) respectively. This proves ii).

- If \( ab > \frac{1}{4} \), then we can choose positive constants, \( a_1, b_1 \) such that \( a > a_1 = \frac{1}{4b_1} > \frac{1}{b} \). Then \( f \) and \( \mathcal{F}_\alpha (f) \) also satisfy (3.17) and (3.18) with \( a \) and \( b \) replaced by \( a_1 \) and \( b_1 \) respectively. Therefore, it follows that \( \mathcal{F}_\alpha (f) \) cannot satisfy (3.18) unless \( P_{b_1} \equiv 0 \), which implies \( f \equiv 0 \). This proves i).

- If \( ab < \frac{1}{4} \), then for all \( \delta \in (b, \frac{1}{4a}) \), the functions of the form \( f(r, x) = W_{\delta}^\beta (r, x) \), where \( P \in \mathcal{P} \), satisfy (3.17) and (3.18). This proves iii).

The following is an immediate consequence of Theorem 2.

**Corollary 1.** Let \( f \) be a measurable function on \( \mathbb{R}_+^2 \) such that

\[
|f(r, x)| \leq Me^{-a|| (r, x) ||^2} (1 + || (r, x) ||)^m \text{ a.e.} \tag{3.20}
\]

and for all \( (\mu, \xi) \in \mathbb{R}_+^2 \),

\[
|\mathcal{F}_\alpha (f)(\mu, \xi)| \leq Me^{-4b|| \theta(\mu, \xi) ||^2} \tag{3.21}
\]

for some constants \( a, b > 0, r \geq 0 \) and \( M > 0 \).

i) If \( ab > \frac{1}{4} \), then \( f = 0 \) almost everywhere.

ii) If \( ab = \frac{1}{4} \), then \( f \) is of the form \( f(r, x) = CEb(r, x) \).

iii) If \( ab < \frac{1}{4} \), then there are infinity many nonzero \( f \) satisfying (3.20) and (3.21).

### 4. Miyachi’s theorem for the Generalized Fourier transform

**Theorem 3.** Let \( f \) be a measurable function on \( \mathbb{R}_+^2 \) even with respect to the first variable such that

\[
E_{a, \beta}^{-1} f \in L^p (d\nu_\alpha) + L^q (d\nu_\alpha) \tag{4.22}
\]

and

\[
\int_{\mathbb{R}^2} \log^+ \frac{E_{b(1+\beta)}^{-1} (\mu, \xi) |\mathcal{F}_\alpha (f)(\mu, \xi)|}{\lambda} d\mu d\xi < \infty, \tag{4.23}
\]

for some constants \( a > 0, b > 0 \lambda > 0, 1 \leq p, q \leq \infty \).

- If \( ab > \frac{1}{4} \), we have \( f = 0 \) almost everywhere.
- If \( ab = \frac{1}{4} \), we have \( f = CEb, \beta \) with \( |C| \leq \lambda \).
- If \( ab < \frac{1}{4} \), for all \( \delta \in (b, \frac{1}{4a}) \), the functions of the form \( f(x) = CE\delta, \beta \), satisfy (4.22) and (4.23).

To prove this result we need the following lemmas.
Lemma 2. ([20]). Let $h$ be an entire on $\mathbb{C}^2$ function such that
\[ |h(z)| \leq Ae^{B|Rez|^2} \quad \text{and} \quad \int_{\mathbb{R}^2} \log^+ |h(y)|dy < \infty, \tag{4.24} \]
for some positive constants $A, B$. Then $h$ is a constant on $\mathbb{C}^2$.

Lemma 3. Let $r$ be in $[1, \infty]$. We consider a function $g$ in $L'(d\nu_{\alpha})$. Then there exists a positive constant $C$ such that:
\[ ||E_{a, \beta}^{-1}(s, y)||_{L'(\mathbb{R}^2_+)} \leq C ||g||_{L'(d\nu_{\alpha})}, \]
where $|| \cdot ||_{L'(\mathbb{R}^2_+)}$ is the norm of the usual Lebesgue space $L'(\mathbb{R}^2_+)$ and $a > 0$.

Proof. From the hypothesis it follows that $E_{a, \beta}^{-1}g$ belongs to $L^1(d\nu_{\alpha})$. Then by Proposition 2, the function $E_{a, \beta}^{-1}g$ is defined almost everywhere on $\mathbb{R}^2$. Now we consider two cases.

i) If $r \in [1, \infty)$, we have
\[ ||E_{a, \beta}^{-1}(s, y)||_{L'(\mathbb{R}^2_+)} = \int_{\mathbb{R}^2_+} E_{a, \beta}^{-r}(s, y) ||E_{a, \beta}^{-1}(s, y)||_{L'(\mathbb{R}^2_+)} dsdy. \]
By applying Hölder’s inequality we obtain
\[ ||E_{a, \beta}^{-1}(s, y)||_{L'(\mathbb{R}^2_+)} \leq \int_{\mathbb{R}^2_+} E_{a, \beta}^{-r}(s, y) \left( ||E_{a, \beta}^{-r}(s, y)||_{L'(\mathbb{R}^2_+)} \right)^{r'/r} dsdy, \]
where $r'$ is the conjugate exponent of $r$. But from (2.4) we deduce that
\[ ||E_{a, \beta}^{-1}(s, y)||_{L'(\mathbb{R}^2_+)} \leq C \int_{\mathbb{R}^2_+} ||E_{a, \beta}^{-1}(s, y)||_{L'(\mathbb{R}^2_+)} dsdy. \]
Thus using the relation (2.7) we obtain
\[ ||E_{a, \beta}^{-1}(s, y)||_{L'(\mathbb{R}^2_+)} \leq C \int_{\mathbb{R}^2_+} |g(s, y)|^{r'}\alpha|ds\alpha(s, y) < \infty. \]

ii) If $r = \infty$, we have
\[ |E_{a, \beta}^{-1}(s, y)||E_{a, \beta}^{-1}(s, y)||_{L'(\mathbb{R}^2_+)} \leq E_{a, \beta}^{-1}(s, y)||E_{a, \beta}^{-1}(s, y)||_{L'(\mathbb{R}^2_+)} dsdy, \]
and from (2.4) we deduce that
\[ ||E_{a, \beta}^{-1}(s, y)||_{L'(\mathbb{R}^2_+)} \leq C ||g||_{L'(d\nu_{\alpha})} < \infty. \]
This completes the proof. \qed
**Lemma 4.** Let $p, q$ in $[1, \infty]$ and $f$ a measurable function on $\mathbb{R}^2_+$ such that

$$E_{a, \beta}^{-1} f \in L^p(d\nu_\alpha) + L^q(d\nu_\alpha),$$  \hfill (4.25)

for some $a > 0$, $\beta > 0$. Then the function defined on $\mathbb{C}^2$ by

$$F_\alpha(f)(\mu, \lambda) = \int_{\mathbb{R}^2_+} f(r, \xi) e^{-i\lambda x} \, d\nu_\alpha(r, \xi),$$  \hfill (4.26)

is well defined and entire on $\mathbb{C}^2$. Moreover there exists a positive constant $C$ such that for all $\xi, \eta, \mu, \theta \in \mathbb{R}$ we have

$$|F_\alpha(f)(\mu + i\theta, \xi + i\eta)| \leq Ce^{\frac{(1+\beta)\eta^2 + \theta^2}{4a\beta}}.$$  \hfill (4.27)

**Proof.** The first assertion follows from the hypothesis on the function $f$ and H"older’s inequality using (4.25) and the derivation theorem under the integral sign. We want to prove (4.27).

The condition (4.25) implies that the function $f$ belongs to $L^1(d\nu_\alpha)$. Hence we deduce from (2.9) that for all $\xi, \eta, \alpha, \theta \in \mathbb{R}$, we have

$$|F_\alpha(f)(\mu + i\theta, \xi + i\eta)| = |\int_{\mathbb{R}^2_+} \mathcal{R}_\alpha(f)(s, \xi + i\eta) \cos(s(\mu + i\theta)) \, dsdy|.$$

The integral of the second member can also be written in the form

$$c_0 E_{\frac{1}{1+\beta}, \frac{1}{1+\beta} \alpha}^{-1} (\theta, \eta) \int_{\mathbb{R}^2_+} E_{\frac{1}{1+\beta}, \frac{1}{1+\beta} \alpha}^{-1} (s, \xi + i\eta) \mathcal{R}_\alpha(|f|(s, \xi + i\eta)) \, dsdy,$$

where $c_0$ is a positive constant. On the follow we will to estimate

$$\int_{\mathbb{R}^2_+} E_{\frac{1}{1+\beta}, \frac{1}{1+\beta} \alpha}^{-1} (s, \xi + i\eta) \mathcal{R}_\alpha(|f|(s, \xi + i\eta)) \, dsdy.$$

Indeed from (4.25) there exists $u$ in $L^p(d\nu_\alpha)$ and $v$ in $L^q(d\nu_\alpha)$ such that

$$f = E_{a, \beta} (u + v).$$

Thus using the Lemma 3 and H"older inequality we obtain

$$\int_{\mathbb{R}^2_+} E_{\frac{1}{1+\beta}, \frac{1}{1+\beta} \alpha}^{-1} (s, \xi + i\eta) \mathcal{R}_\alpha(|f|(s, \xi + i\eta)) \, dsdy \leq C(||u||_{L^p(d\nu_\alpha)} + ||v||_{L^q(d\nu_\alpha)}) < \infty.$$  

Hence there exists a positive constant $C$ such that

$$|F(f)(\mu + i\theta, \xi + i\eta)| \leq Ce^{\frac{(1+\beta)\eta^2 + \theta^2}{4a\beta}}.$$

□
Proof. of Theorem 3.

We will divide the proof in several cases.

1st case \( ab > \frac{1}{4} \).

Consider the function \( h \) defined on \( \mathbb{C}^2 \) by

\[
h(\gamma, \zeta) = E^{-1}_{\frac{1+\beta}{4a\beta}, \frac{1}{1+\beta}} (\gamma, \zeta) \mathcal{F}_\alpha(f)(\gamma, \zeta),
\]

with \( \gamma = \mu + i\theta \in \mathbb{C} \) and \( \zeta = \xi + i\eta \in \mathbb{C} \). This function is entire on \( \mathbb{C}^2 \) and using (4.27) we obtain:

\[
|h(\gamma, \zeta)| \leq CE^{-1}_{\frac{1+\beta}{4a\beta}, \frac{1}{1+\beta}} (\mu, \xi),
\]

for all \( \zeta, \gamma \in \mathbb{C} \). On the other hand we have

\[
\int_{\mathbb{R}^2_+} \log^+ |h(\mu, \xi)| d\mu d\xi = \int_{\mathbb{R}^2_+} \log^+ |E^{-1}_{\frac{1+\beta}{4a\beta}, \frac{1}{1+\beta}} (\mu, \xi) \mathcal{F}_\alpha(f)(\mu, \xi)| d\mu d\xi,
\]

\[
= \int_{\mathbb{R}^2_+} \log^+ \left[ \frac{E_{\frac{1+\beta}{4a\beta}, \frac{1}{1+\beta}} (\mu, \xi) \mathcal{F}_\alpha(f)(\mu, \xi)}{\lambda E_{\frac{1+\beta}{4a\beta}, \frac{1}{1+\beta}} (\mu, \xi)} \right] d\mu d\xi
\]

\[
\leq \int_{\mathbb{R}^2_+} \log^+ \left[ \frac{E_{\frac{1+\beta}{4a\beta}, \frac{1}{1+\beta}} (\mu, \xi) \mathcal{F}_\alpha(f)(\mu, \xi)}{\lambda} \right] d\mu d\xi + \int_{\mathbb{R}^2_+} \frac{\lambda E_{\frac{1+\beta}{4a\beta}, \frac{1}{1+\beta}} (\mu, \xi) d\mu d\xi},
\]

because \( \log^+ (cd) \leq \log^+ (c) + d \) for all \( c, d > 0 \). Since \( ab > \frac{1}{4} \), (4.23) implies that

\[
\int_{\mathbb{R}^2_+} \log^+ |h(\mu, \xi)| d\mu d\xi < \infty. \tag{4.30}
\]

From the relations (4.29) and (4.30), it follows from Lemma 2 that there exists a constant \( C \) such that

\[
h(\mu, \xi) = C, \ (\mu, \xi) \in \mathbb{C}^2.
\]

Thus

\[
\mathcal{F}_\alpha(f) = CE_{\frac{1+\beta}{4a\beta}, \frac{1}{1+\beta}}.
\]

Using now the condition (4.23) and that \( ab > \frac{1}{4} \), we deduce that \( C = 0 \) and hence we obtain

\[
\forall (\mu, \xi) \in \Gamma, \ \mathcal{F}_\alpha(f)(\mu, \xi) = 0.
\]
Then the injectivity of $F$ implies the result of the theorem.

**Second case** $ab = \frac{1}{4}$.

The same proof as for the first step give that

$$F_\alpha(f) = CE^{\frac{1}{4a^2} \frac{1}{1+b}}_t,$$

with $|C| \leq \lambda$. Thus

$$f = CE^{\frac{1}{4a^2} \frac{1}{1+b}}_t.$$

**Third case** $ab < \frac{1}{4}$

In the sequel we will construct a family of nonzero functions which satisfy the conditions (4.22),(4.23). By considering the family of functions $cE^{\delta, \beta}_\alpha$, we see that

$$F_\alpha(f) = cE^{\frac{1}{4a^2} \frac{1}{1+b}}_t.$$

These functions clearly satisfy the conditions (4.22),(4.23) for all $\delta \in (b, \frac{1}{4a})$. The proof of the Theorem is complete.

The following is an immediate corollary of Theorem 3.

**Corollary 2.** Let $f$ be a measurable function on $\mathbb{R}^2_+$ such that

$$E^{-1}_{a, \beta} f \in L^p(d\nu_\alpha) + L^q(d\nu_\alpha)$$

and

$$\int_{\mathbb{R}^2_+} E^{r}_{b(1+b)} \frac{1}{\beta} \frac{1}{r^\beta} (\mu, \xi) |F_\alpha(f)(\mu, \xi)| d\mu d\xi < \infty,$$

for some constants $a > 0, b > 0, 1 \leq p, q \leq \infty, 0 < r \leq \infty$. Then

If $ab \geq \frac{1}{4}$, we have $f = 0$ almost everywhere.

If $ab < \frac{1}{4}$, for all $\delta \in (b, \frac{1}{4a})$, the functions of the form $CE^{\delta, \beta}_\alpha$ satisfy (4.31) and (4.32).

5. **Beurling’s theorem for the Generalized Fourier transform**

Beurling’s theorem and Bonami, Demange, and Jaming’s extension are generalized for the generalized Fourier transform as follows.

**Theorem 4.** Let $N \in \mathbb{N}$, $\delta > 0$ and $f \in L^2(d\nu_\alpha)$ satisfy

$$\int_{\mathbb{R}^2_+} \int_{\mathbb{R}^2_+} \frac{|f(r, x)||F_\alpha(f)(t, y)||R(t, y)|^\delta e^{-\frac{|(r, x)||(|(t, y)||)|}}}{(1 + |(r, x)| + |(t, y)|)^N} d\nu_\alpha(r, x) dt dy < \infty, \quad (5.33)$$
where $R$ is a polynomial of degree $m$. If $N \geq m\delta + 4$, then
\[
f(r,x) = \sum_{|l| < \frac{N-m\delta-2}{2}} a_i^s \tilde{W}_i^s(r,x) \text{ a.e.,}
\]
(5.34)

where $s > 0$, $a_i^s \in \mathbb{C}$ and $\tilde{W}_i^s$ is given by (2.15). Otherwise, $f(r,x) = 0$ almost everywhere.

**Proof.** We start the following lemma.

**Lemma 5.** We suppose that $f \in L^2(d\nu_\alpha)$ satisfies (5.33). Then $f \in L^1(d\nu_\alpha)$.

**Proof.** We may suppose that $f$ is not negligible. (5.33) and the Fubini theorem imply that for almost every $(t,y) \in \mathbb{R}^2_+$,
\[
\frac{\|F_\alpha(f)(y)||R(t,y)||\delta}{(1 + ||(t,y)||)^N} \int_{\mathbb{R}^2_+} \frac{|f(r,x)|}{(1 + ||(r,x)||)^N} e^{||r,x)||||(t,y)||} d\nu_\alpha(r,x) < \infty.
\]

Since $f$ and thus, $F_\alpha(f)$ are not negligible, there exist $(t_0,y_0) \in \mathbb{R}^2_+$, $(t_0,y_0) \neq (0,0)$, such that
\[
F_\alpha(f)(t_0,y_0)R(t_0,y_0) \neq 0.
\]

Therefore,
\[
\int_{\mathbb{R}^2_+} \frac{|f(r,x)|}{(1 + ||(r,x)||)^N} e^{||r,x)||||(t_0,y_0)||} d\nu_\alpha(r,x) < \infty.
\]

Since $e^{||r,x)||||(t_0,y_0)||}$ is well-defined almost everywhere on $\mathbb{R}^2_+$. By the same techniques used in [7], we can deduce that
\[
\int_{\mathbb{R}^2_+} \int_{\mathbb{R}^2_+} e^{||r,x)||||(t,y)||} \frac{\|\mathcal{R}_\alpha(f)(r,x)||\mathcal{F}_0(\mathcal{R}_\alpha)(f)(t,y)||R(t,y)||^\delta}{(1 + ||(r,x)|| + ||(t,y)||)^N} d\nu_\alpha(r,x) dtdy < \infty.
\]

According to Theorem 2.3 in [25], we conclude that for all $(r,x) \in \mathbb{R}^2_+$,
\[
\mathcal{R}_\alpha(f)(r,x) = P(r,x)e^{-\frac{||r,x||^2}{4\nu}},
\]
where $s > 0$ and $P$ a polynomial of degree strictly lower than $\frac{N-m\delta-2}{2}$. Then by (2.9),
\[
F_\alpha(f)(t,y) = \mathcal{F}_0 \circ \mathcal{R}_\alpha(f)(t,y) = \mathcal{F}_0 \left( P(r,x)e^{-\frac{||r,x||^2}{4\nu}} \right)(t,y) = Q(t,y)e^{-s||t,y||^2},
\]
where $Q$ is a polynomial of degree $\deg P$. Then by using (2.15), we can find constants $a_i^j$ such that

$$F_\alpha(f)(t,y) = F_\alpha\left(\sum_{|l|<\frac{N-\alpha}{2}} a_i^j \tilde{W}_i^j\right)(t,y).$$

By the injectivity of $F_\alpha$ the desired result follows. \qed

As an application of Theorem 4, by using the same techniques in [19], we can deduce the following Gelfand-Shilov type theorem for the generalized Fourier transform.

**Corollary 3.** Let $N,m \in \mathbb{N}$, $\delta > 0$, $a,b > 0$ with $ab \geq \frac{1}{4}$, and $1 < p,q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $f \in L^2(d\nu_\alpha)$ satisfy

$$\int_{\mathbb{R}^2} |f(r,x)| e^{\frac{2a}{p}||r,x||^p} (1 + ||(r,x)||)^N \, d\nu_\alpha(r,x) < \infty \quad (5.35)$$

and

$$\int_{\mathbb{R}^2} \left|\frac{F_\alpha(f)(t,y)}{e^{\frac{2b}{q}||t,y||^q}}\right| |R(t,y)|^\delta \, dt \, dy < \infty \quad (5.36)$$

for some $R \in \mathcal{P}_m$.

i) If $ab > \frac{1}{4}$ or $(p,q) \neq (2,2)$, then $f(r,x) = 0$ almost everywhere.

ii) If $ab = \frac{1}{4}$ and $(p,q) = (2,2)$, then $f$ is of the form (5.34) whenever $N \geq \frac{m\delta}{2} + 2$ and $r = 2b^2$. Otherwise, $f(x) = 0$ almost everywhere.

**Proof.** Since

$$4ab||r,x||||t,y|| \leq \frac{(2a)^p}{p}||r,x||^p + \frac{(2b)^q}{q}||t,y||^q,$$

it follows from (5.35) and (5.36) that

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |f(r,x)||F_\alpha(f)(t,y)||R(t,y)|^\delta \frac{e^{4ab||(r,x)||||t,y||}}{(1 + ||(r,x)|| + ||(t,y)||)^{2N}} \, d\nu_\alpha(r,x) \, dt \, dy < \infty.$$ 

Then (5.33) is satisfied, because $4ab \geq 1$. Therefore, according to the proof of Theorem 4, we can deduce that

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{4ab||(r,x)||||t,y||} |\mathcal{R}_\alpha(f)(r,x)||F_\alpha(f)(t,y)||R(t,y)|^\delta \frac{e^{4ab||(r,x)||||t,y||}}{(1 + ||(r,x)|| + ||(t,y)||)^{2N}} \, d\nu_\alpha(r,x) \, dt \, dy < \infty,$$
and \( \mathcal{R}_\alpha(f) \) and \( f \) are of the forms
\[
\mathcal{R}_\alpha(f)(r,x) = P(r,x)e^{-\frac{(r,x)^2}{4s}} \quad \text{and} \quad F_\alpha(f)(t,y) = Q(t,y)e^{-s|t,y|^2},
\]
where \( s > 0 \) and \( P,Q \) are polynomials of the same degree strictly lower than \( \frac{2N-m\delta-2}{2} \). Therefore, substituting these from, we can deduce that
\[
\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-(\sqrt{3}(r,x)) - \frac{1}{2s}((r,x))^2} e^{(4ab-1)(r,x)|t,y|^2} |P(r,x)||Q(r,x)||R(t,y)|^\delta d\nu_\alpha(r,x)dtdy < \infty.
\]
When \( 4ab > 1 \), this integral is not finite unless \( f = 0 \) almost everywhere. Moreover, it follows from (5.35) and (5.36) that
\[
\int_{\mathbb{R}^2} |P(r,x)|e^{-\frac{1}{4s}((r,x))^2} e^{\frac{2ab}{p}(r,x)|t,y|^2} d\nu_\alpha(r,x) < \infty
\]
and
\[
\int_{\mathbb{R}^2} |Q(t,y)|e^{-s|t,y|^2} e^{\frac{2b}{a}|t,y|^2} |R(t,y)|^\delta dtdy < \infty.
\]
Hence, one of these integrals is not finite unless \( (p,q) = (2,2) \). When \( 4ab = 1 \) and \( (p,q) = (2,2) \), the finiteness of above integrals implies that \( r = 2b^2 \) and the rest follows from Theorem 4.

6. Quantitative Uncertainty Principle For the generalized Fourier transform

We shall investigate the case where \( f \) and \( F_\alpha(f) \) are close to zero outside measurable sets. Here the notion of ”close to zero” is formulated as follows. If \( f \in L^p(d\nu_\alpha) \), \( 1 \leq p \leq 2 \), is \( \varepsilon \)-concentrated on a measurable set \( E \subset \mathbb{R}^2 \), if there is a measurable function \( g \) vanishing outside \( E \) such that \( \|f-g\|_{L^p(d\nu_\alpha)} \leq \varepsilon \|f\|_{L^p(d\nu_\alpha)} \). Therefore, if we introduce a projection operator \( P_E \) as
\[
P_E f(r,x) = \begin{cases} f(r,x) & \text{if } (r,x) \in E \\ 0 & \text{if } (r,x) \notin E, \end{cases}
\]
then \( f \) is \( \varepsilon \)-concentrated on \( E \) if and only if \( \|f-P_E f\|_{L^p(d\nu_\alpha)} \leq \varepsilon \|f\|_{L^p(d\nu_\alpha)} \).

We define a projection operator \( Q_W \) as
\[
Q_W f(r,x) = F_\alpha^{-1}(P_w(F_\alpha(f)))(r,x).
\]
Similarly, we say that $\mathcal{F}_\alpha(f)$ is $\varepsilon_W$-concentrated to $W$ in $L^{p'}(d\gamma_\lambda)$ if and only if
\[
\|\mathcal{F}_\alpha(f) - \mathcal{F}_\alpha(Q_W f)\|_{L^{p'}(d\gamma_\lambda)} \leq \varepsilon_W \|\mathcal{F}_\alpha(f)\|_{L^{p'}(d\gamma_\lambda)}.
\] (6.37)

We note that, for measurable set $E \subset \mathbb{R}^2_+$ and $W \subset \Gamma$,
\[
Q_W P_E f(r, x) = \int_{\mathbb{R}^2_+} q(t, y; r, x) f(t, y) d\nu_\alpha(t, y),
\]
where
\[
q(t, y; r, x) = \begin{cases} 
\int_W \varphi_{\mu, \lambda}(t, y) \varphi_{\mu, \lambda}(r, x) d\gamma_\alpha(\mu, \lambda) & \text{if } (t, y) \in E \\
0 & \text{if } (t, y) \notin E.
\end{cases}
\]

Indeed, by the Fubini’s theorem we see that
\[
Q_W P_E f(r, x) = \int_W \mathcal{F}_\alpha(P_E f)(\mu, \lambda) \overline{\varphi_{\mu, \lambda}(r, x)} d\gamma_\alpha(\mu, \lambda) \\
= \int_W \left( \int_E f(t, y) \varphi_{\mu, \lambda}(t, y) d\nu_\alpha(t, y) \right) \overline{\varphi_{\mu, \lambda}(r, x)} d\gamma_\alpha(\mu, \lambda) \\
= \int_E f(t, y) \left( \int_W \varphi_{\mu, \lambda}(t, y) \overline{\varphi_{\mu, \lambda}(r, x)} d\gamma_\alpha(\mu, \lambda) \right) d\nu_\alpha(t, y).
\]

The Hilbert-Schmidt norm $\|Q_W P_E\|_{HS}$ is given by
\[
\|Q_W P_E\|_{HS} = \left( \int_{\mathbb{R}^2_+} \int_{\mathbb{R}^2_+} |q(t, y; r, x)|^2 d\nu_\alpha(t, y) d\nu_\alpha(r, x) \right)^{\frac{1}{2}}.
\]

We denote by $\|T\|_2$ the operator norm on $L^2(d\nu_\alpha)$. Since $P_E$ and $Q_W$ are projections, it is clear that $\|P_E\|_2 = \|Q_W\|_2 = 1$. Moreover, it follows that
\[
\|Q_W P_E\|_{HS} \geq \|Q_W P_E\|_2.
\] (6.38)

**Lemma 6.** If $E$ and $W$ are sets of finite measure, then
\[
\|Q_W P_E\|_{HS} \leq \sqrt{mes_{\nu_\alpha}(E) mes_{\gamma_\alpha}(W)},
\]
where
\[
mes_{\nu_\alpha}(E) := \int_E d\nu_\alpha(r, x), \quad mes_{\gamma_\alpha}(W) := \int_W d\gamma_\alpha(\mu, \lambda).
\]

**Proof.** For $(t, y) \in E$, let $g_{t, y}(r, x) = q(t, y; r, x)$. (2.11) implies that
\[
\mathcal{F}_\alpha(g_{t, y})(\mu, \lambda) = P_W(\varphi_{\mu, \lambda}(t, y)).
\]
Hence, integrating over 
\[ \int_{\mathbb{R}_+^2} |q(t,y; r,x)|^2 d\nu(r,x) = \int_{\mathbb{R}_+^2} |g(t,y; r,x)|^2 d\nu(r,x) \]

\[ = \int_{\mathbb{T}} |\mathcal{F}_{\nu}(g(t,y))(\mu, \lambda)|^2 d\nu(\mu, \lambda) \leq \text{mes}_{\nu}(W) \]

Hence, integrating over \((t,y) \in E\), we see that 
\[ ||Q_W P_E||_{hs}^2 \leq \text{mes}_{\nu}(E) \text{mes}_{\nu}(W). \]

\[ \square \]

**Proposition 6.** Let \( E \) and \( W \) be measurable sets and suppose that 
\[ ||f||_{L^2(d\nu)} = ||\mathcal{F}_{\nu}(f)||_{L^2(d\nu)} = 1. \]

Assume that \( \varepsilon_E + \varepsilon_W < 1 \), \( f \) is \( \varepsilon_E \)-concentrated on \( E \) and \( \mathcal{F}_{\nu}(f) \) is \( \varepsilon_W \) concentrated on \( W \). Then

\[ \text{mes}_{\nu}(E) \text{mes}_{\nu}(W) \geq (1 - \varepsilon_E - \varepsilon_W)^2. \]

**Proof.** Since 
\[ ||f||_{L^2(d\nu)} = ||\mathcal{F}_{\nu}(f)||_{L^2(d\nu)} = 1 \]

and \( \varepsilon_E + \varepsilon_W < 1 \), the measures of \( E \) and \( W \) must both be non-zero. Indeed, if not, then the \( \varepsilon_E \)-concentration of \( f \) implies that

\[ ||f - P_E f||_{L^2(d\nu)} = ||f||_{L^2(d\nu)} = 1 \leq \varepsilon_E, \]

which contradicts with \( \varepsilon_E < 1 \), likewise for \( \mathcal{F}_{\nu}(f) \). If at least one of \( \text{mes}_{\nu}(E) \) and \( \text{mes}_{\nu}(W) \) is infinity, then the inequality is clear. Therefore, it is enough to consider the case where both \( E \) and \( W \) have finite positive measures. Since 
\[ ||Q_W||_2 = 1, \]

it follows that

\[ ||f - Q_W P_E f||_{L^2(d\nu)} \leq ||f - Q_W f||_{L^2(d\nu)} + ||Q_W f - Q_W P_E f||_{L^2(d\nu)} \]

\[ \leq \varepsilon_W + ||Q_W||_2 ||f - P_E f||_{L^2(d\nu)} \]

\[ \leq \varepsilon_E + \varepsilon_W \]

and thus,

\[ ||Q_W P_E f||_{L^2(d\nu)} \geq ||f||_{L^2(d\nu)} - ||f - Q_W P_E f||_{L^2(d\nu)} \geq 1 - \varepsilon_E - \varepsilon_W. \]

Hence 
\[ ||Q_W P_E||_2 \geq 1 - \varepsilon_E - \varepsilon_W. \] (6.38) and Lemma 6 yields the desired inequality. \( \square \)

Let \( B_{L^p(d\nu)}(T) \), \( 1 \leq p \leq 2 \), the subspace of all \( g \in L^p(d\nu) \) such that \( Q_T g = g \). We say that \( f \) is \( \varepsilon \)-bandlimited to \( T \) if there is a \( g \in B_{L^p(d\nu)}(T) \) with 
\[ ||f - g||_{L^p(d\nu)} < \varepsilon ||f||_{L^p(d\nu)}. \]

Here we denote by 
\[ ||P_E||_p \]

the operator norm of \( P_E \) on \( L^p(d\nu) \) and by 
\[ ||P_E||_{p,T} \]

the operator norm of \( P_E : B_{L^p(d\nu)}(T) \to L^p(d\nu) \). Corresponding to (6.38) and Lemma 6 in the \( L^2(d\nu) \) case, we can obtain the following.
Lemma 7. Let $E$ and $T$ be measurable sets of $\mathbb{R}^2_+$. For $p \in [1, 2]$, we have

$$\|P_E\|_{p,T} \leq \left( \text{mes}_{\nu_a}(E) \text{mes}_{\gamma_a}(T) \right)^{\frac{1}{p}}.$$ 

Proof. For $f \in B_{L^p(d\nu_a)}(T)$ we see that

$$f(t,y) = \int_T \varphi_{\mu,\lambda}(t,y) F_\alpha(f)(\mu,\lambda) d\gamma_a(\mu,\lambda).$$

By (2.1), Hölder’s inequality and Proposition 4

$$|f(r,x)| \leq \left( \text{mes}_{\gamma_a}(T) \right)^{\frac{1}{p}} \|F_\alpha(f)\|_{L^{p'}(d\gamma_a)} \leq \left( \text{mes}_{\gamma_a}(T) \right)^{\frac{1}{p}} \|f\|_{L^p(d\nu_a)}.$$ 

Therefore

$$\|P_E f\|_{L^p(d\nu_a)} = \left( \int_E |f(r,x)|^p d\nu_a(r,x) \right)^{\frac{1}{p}} \leq \left( \text{mes}_{\nu_a}(E) \text{mes}_{\gamma_a}(T) \right)^{\frac{1}{p}} \|f\|_{L^p(d\nu_a)}.$$ 

Then, it follows that for $f \in B_{L^p(d\nu_a)}(W)$,

$$\frac{\|P_E f\|_{L^p(d\nu_a)}}{\|f\|_{L^p(d\nu_a)}} \leq \left( \text{mes}_{\nu_a}(E) \text{mes}_{\gamma_a}(T) \right)^{\frac{1}{p}},$$

which implies the desired inequality.

Proposition 7. Let $f \in L^p(d\nu_a)$. If $f$ is $\varepsilon_E$-concentrated to $E$ and $\varepsilon_T$ bandlimited to $W$, then

$$\left( \text{mes}_{\nu_a}(E) \text{mes}_{\gamma_a}(T) \right)^{\frac{1}{p}} \geq \frac{1 - \varepsilon_E - \varepsilon_T}{1 + \varepsilon_T}.$$ 

Proof. Without loss of generality, we may suppose that $\|f\|_{L^p(d\nu_a)} = 1$. Since $f$ is $\varepsilon_E$-concentrated to $E$, it follows that $\|P_E f\|_{L^p(d\nu_a)} \geq \|f\|_{L^p(d\nu_a)} - \|f - P_E f\|_{L^p(d\nu_a)} \geq 1 - \varepsilon_E$. Moreover, since $f$ is $\varepsilon_T$-bandlimited, there is a $g \in B_{L^p(d\nu_a)}(W)$ with $\|g - f\|_{L^p(d\nu_a)} \leq \varepsilon_T$. Therefore, it follows that

$$\|P_E g\|_{L^p(d\nu_a)} \geq \|P_E f\|_{L^p(d\nu_a)} - \|P_E (g - f)\|_{L^p(d\nu_a)} \geq \|P_E f\|_{L^p(d\nu_a)} - \varepsilon_T \geq 1 - \varepsilon_E - \varepsilon_T$$

and $\|g\|_{L^p(d\nu_a)} \leq \|f\|_{L^p(d\nu_a)} + \varepsilon_T = 1 + \varepsilon_T$. Then, we see that

$$\frac{\|P_E g\|_{L^p(d\nu_a)}}{\|g\|_{L^p(d\nu_a)}} \geq \frac{1 - \varepsilon_E - \varepsilon_T}{1 + \varepsilon_T}.$$ 

Hence $\|P_E\|_{p,W} \geq \frac{1 - \varepsilon_E - \varepsilon_T}{1 + \varepsilon_T}$ and Lemma 7 yields the desired inequality.
Proposition 8. Let \( f \in L^1(d\nu_\alpha) \cap L^2(d\nu_\alpha) \) with \( \|f\|_{L^2(d\nu_\alpha)} = 1 \). If \( f \) is \( \varepsilon_E \)-concentrated to \( E \) in \( L^1(d\nu_\alpha) \)-norm and \( \mathcal{F}_\alpha(f) \) is \( \varepsilon_T \)-concentrated to \( T \) in \( L^2(d\gamma_\alpha) \)-norm, then

\[
\mes_{\nu_\alpha}(E) \geq (1 - \varepsilon_E)^2 \|f\|_{L^1(d\nu_\alpha)}^2 \quad \text{and} \quad \mes_{\gamma_\alpha}(T) \|f\|_{L^1(d\nu_\alpha)}^2 \geq (1 - \varepsilon_T^2).
\]

In particular,

\[
\mes_{\nu_\alpha}(E)\mes_{\gamma_\alpha}(T) \geq (1 - \varepsilon_E)^2(1 - \varepsilon_T^2).
\]

Proof. By the orthogonality of the projection operator \( P_T \), \( \|f\|_{L^2(d\nu_\alpha)} = \|\mathcal{F}_\alpha(f)\|_{L^2(d\gamma_\alpha)} = 1 \) and \( f \) is \( \varepsilon_T \)-concentrated to \( W \) in \( L^2_{\gamma_\alpha} \)-norm, it follows that

\[
\|P_T(\mathcal{F}_\alpha(f))\|_{L^2(d\gamma_\alpha)}^2 = \|\mathcal{F}_\alpha(f)\|_{L^2(d\gamma_\alpha)}^2 - \|\mathcal{F}_\alpha(f) - P_T(\mathcal{F}_\alpha(f))\|_{L^2(d\gamma_\alpha)}^2 \geq 1 - \varepsilon_T^2,
\]

and thus,

\[
1 - \varepsilon_T^2 \leq \int_T |\mathcal{F}_\alpha(f)(\xi)|^2 d\gamma_\alpha(\mu, \lambda)
\]

\[
\leq \mes_{\gamma_\alpha}(T)\|\mathcal{F}_\alpha(f)\|_{L^\infty(d\gamma_\alpha)}^2 \leq \mes_{\gamma_\alpha}(T)\|f\|_{L^1(d\nu_\alpha)}^2.
\]

Similarly, \( f \) is \( \varepsilon_E \)-concentrated to \( E \) in \( L^1(d\nu_\alpha) \)-norm,

\[
(1 - \varepsilon_E)\|f\|_{L^1(d\nu_\alpha)} \leq \int_E |f(x)| d\nu_\alpha(x) \leq \sqrt{\mes_{\nu_\alpha}(E)}
\]

Here we used the Cauchy-Schwarz inequality and the fact that \( \|f\|_{L^2(d\nu_\alpha)} = 1 \).

Proposition 9. Let \( E \) and \( T \) be measurable subsets of \( \mathbb{R}_+^2 \), and \( f \in L^p(d\nu_\alpha) \) for \( p \in (1, 2] \). If \( f \) is \( \varepsilon_E \)-concentrated to \( E \) in \( L^p(d\nu_\alpha) \)-norm and \( \mathcal{F}_\alpha(f) \) is \( \varepsilon_T \)-concentrated to \( T \) in \( L^{p'}(d\gamma_\alpha) \)-norm, then

\[
\left(\mes_{\nu_\alpha}(E)\mes_{\gamma_\alpha}(T)\right)^{\frac{1}{p'}} \geq \frac{(1 - \varepsilon_E)\|\mathcal{F}_\alpha(f)\|_{L^{p'}(d\gamma_\alpha)} - \varepsilon_T\|f\|_{L^p(d\nu_\alpha)}}{\|f\|_{L^p(d\nu_\alpha)}}.
\]

Proof. Let \( f \in L^p(d\nu_\alpha) \) for \( p \in (1, 2] \). As above

\[
\|\mathcal{F}_\alpha(f) - \mathcal{F}_\alpha(Q_TP_Ef)\|_{L^{p'}(d\nu_\alpha)} \leq \|\mathcal{F}_\alpha(f) - \mathcal{F}_\alpha(Q_Tf)\|_{L^{p'}(d\nu_\alpha)} + \|\mathcal{F}_\alpha(Q_Tf) - \mathcal{F}_\alpha(Q_TP_Ef)\|_{L^{p'}(d\nu_\alpha)}
\]

\[
\leq \varepsilon_T\|\mathcal{F}_\alpha(f)\|_{L^{p'}(d\nu_\alpha)} + \|f - P_Ef\|_{L^p(d\nu_\alpha)}
\]

\[
\leq \varepsilon_T\|\mathcal{F}_\alpha(f)\|_{L^{p'}(d\nu_\alpha)} + \varepsilon_E\|f\|_{L^p(d\nu_\alpha)}
\]
and thus,
\[
\|\mathcal{F}_\alpha(Q_T P_\nu f)\|_{L^{p'}(d\nu_T)} \geq \|\mathcal{F}_\alpha(f)\|_{L^{p'}(d\nu_T)} - \|\mathcal{F}_\alpha(Q_T P_\nu f)\|_{L^{p'}(d\nu_T)} \\
\geq (1 - \varepsilon_T)\|\mathcal{F}_\alpha(f)\|_{L^{p'}(d\nu_T)} - \varepsilon_T \|f\|_{L^p(d\nu_T)}.
\]

On the other hand, it is easy to obtain
\[
\frac{\|\mathcal{F}_\alpha(Q_T P_\nu f)\|_{L^{p'}(d\nu_T)}}{\|f\|_{L^p(d\nu_T)}} \leq \left(\text{mes}_{\nu_\alpha}(E) \text{mes}_{\nu_\alpha}(T)\right)^{\frac{1}{p'}}.
\]

Hence
\[
\left(\text{mes}_{\nu_\alpha}(E) \text{mes}_{\nu_\alpha}(T)\right)^{\frac{1}{p'}} \|f\|_{L^{p'}(d\nu_T)} \geq (1 - \varepsilon_E)(1 - \varepsilon_T)\|\mathcal{F}_\alpha(f)\|_{L^{p'}(d\nu_\alpha)}.
\]

which gives the desired result.

**Proposition 10.** Let \(f \in L^1(d\nu_\alpha) \cap L^p(d\nu_\alpha), p \in (1, 2]\). If \(f\) is \(\varepsilon_E\)-concentrated to \(E\) in \(L^1(d\nu_\alpha)\)-norm and \(\mathcal{F}_\alpha(f)\) is \(\varepsilon_T\)-concentrated to \(T\) in \(L^{p'}(d\nu_\alpha)\)-norm, then
\[
\left(\text{mes}_{\nu_\alpha}(E) \text{mes}_{\nu_\alpha}(T)\right)^{\frac{1}{p'}} \|f\|_{L^{p'}(d\nu_\alpha)} \geq (1 - \varepsilon_E)(1 - \varepsilon_T)\|\mathcal{F}_\alpha(f)\|_{L^{p'}(d\nu_\alpha)}.
\]

**Proof.** Let \(f \in L^1(d\nu_\alpha) \cap L^p(d\nu_\alpha), p \in (1, 2]\). As \(\mathcal{F}_\alpha(f)\) is \(\varepsilon_T\)-concentrated to \(T\) in \(L^{p'}(d\nu_\alpha)\)-norm, it follows that
\[
\|\mathcal{F}_\alpha(f)\|_{L^{p'}(d\nu_\alpha)} \leq \varepsilon_T\|\mathcal{F}_\alpha(f)\|_{L^{p'}(d\nu_\alpha)} + \left(\int_T |\mathcal{F}_\alpha(f)(\lambda, \mu)|^{p'} d\nu_\alpha(\lambda, \mu)\right)^{\frac{1}{p'}}
\leq \varepsilon_T\|\mathcal{F}_\alpha(f)\|_{L^{p'}(d\nu_\alpha)} + (\text{mes}_{\nu_\alpha}(T))^{\frac{1}{p'}}\|\mathcal{F}_\alpha(f)\|_{L^\infty(d\nu_\alpha)}.
\]

Thus from Proposition (2.9),
\[
(1 - \varepsilon_T)\|\mathcal{F}_\alpha(f)\|_{L^{p'}(d\nu_\alpha)} \leq (\text{mes}_{\nu_\alpha}(T))^{\frac{1}{p'}}\|f\|_{L^1(d\nu_\alpha)}.
\] (6.39)

Similarly, using \(f\) is \(\varepsilon_E\)-concentrated to \(E\) in \(L^1(d\nu_\alpha)\)-norm, and Hölder inequality, we obtain
\[
(1 - \varepsilon_E)\|f\|_{L^1(d\nu_\alpha)} \leq (\text{mes}_{\nu_\alpha}(E))^{\frac{1}{p'}}\|f\|_{L^p(d\nu_\alpha)}.
\] (6.40)

Combining (6.39) and (6.40), we obtain the result.

**Proposition 11.** Let \(s > 0\). Then there exists a constant \(C_1(\alpha, s)\) such that for all \(f \in L^1(d\nu_\alpha) \cap L^2(d\nu_\alpha)\)
\[
\|f\|_{L^2(d\nu_\alpha)}^{2 + \frac{4s}{2 + 4s}} \leq C_1(\alpha, s)\|f\|_{L^1(d\nu_\alpha)}^{\frac{4s}{2 + 4s}} \|\|\theta(\lambda, \mu)||^{s} \mathcal{F}(f)\|_{L^2(d\nu_\alpha)}.
\] (6.41)
Proof. Let $A > 0$. From Plancherel’s theorem we have
\[
\|f\|^2_{L^2(d\nu)} = \|\mathcal{F}_\alpha(f)\|^2_{L^2(d \gamma)}
= \|1_{\theta^{-1}(B_+(0,A))}\mathcal{F}_\alpha(f)\|^2_{L^2(d \gamma)} + \|(1 - 1_{\theta^{-1}(B_+(0,A))})\mathcal{F}_\alpha(f)\|^2_{L^2(d \gamma)}
\]
By (2.2) and (2.10)
\[
\|1_{\theta^{-1}(B_+(0,A))}\mathcal{F}_\alpha(f)\|^2_{L^2(d \gamma)} \leq \|f\|^2_{L^1(d\nu)} \int_{\mathbb{R}^d} 1_{B_+(0,A)}(r,x) d\nu(r,x).
\]
By a simple calculations we find
\[
\|1_{\theta^{-1}(B_+(0,A))}\mathcal{F}_\alpha(f)\|^2_{L^2(d \gamma)} \leq \frac{A^{2\alpha + 3}}{2^{\alpha + 3} \Gamma(\alpha + \frac{5}{2})} \|f\|^2_{L^1(d\nu)}.
\]
On the other hand
\[
\|(1 - 1_{\theta^{-1}(B_+(0,A))})\mathcal{F}_\alpha(f)\|^2_{L^2(d \gamma)} \leq A^{-2\alpha} \|\theta(\lambda, \mu)\|^2 \|\mathcal{F}_\alpha(f)\|^2_{L^2(d \gamma)}
\leq A^{-2\alpha} \||\theta(\lambda, \mu)|^s \mathcal{F}_\alpha(f)\|^2_{L^2(d \gamma)}.
\]
It follows then
\[
\|f\|^2_{L^2(d\nu)} \leq \frac{A^{2\alpha + 3}}{2^{\alpha + 3} \Gamma(\alpha + \frac{5}{2})} \|f\|^2_{L^1(d\nu)} + A^{-2\alpha} \||\theta(\lambda, \mu)|^s \mathcal{F}_\alpha(f)\|^2_{L^2(d \gamma)}.
\]
Minimizing the right hand side of that inequality over $A > 0$ gives
\[
\|f\|^2_{L^2(d\nu)} \leq C(\alpha, s) \|f\|^2_{L^1(d\nu)} \||\theta(\lambda, \mu)|^s \mathcal{F}(f)\|^2_{L^2(d \gamma)},
\]
(6.42)
The desired result follows immediately from (6.42).

Proposition 12. Let $s > 0$. Then there exists a constant $C_2(\alpha, s)$ such that for all
\[
f \in L^1(d\nu) \cap L^2(d\nu)
\]
\[
\|f\|_{L^1(d\nu)}^{1 + \frac{4s}{2\alpha + 3}} \leq C_2(\alpha, s) \|f\|_{L^2(d\nu)}^{\frac{4s}{2\alpha + 3}} \|(r,x)\|^{2s} f\|_{L^1(d\nu)}.
\]
(6.43)

Proof. Let $A > 0$. We have
\[
\|f\|_{L^1(d\nu)} \leq \|1_{B_+(0,A)}f\|_{L^1(d\nu)} + \|(1 - 1_{B_+(0,A)})f\|_{L^1(d\nu)}.
\]
By Cauchy-Schwarz inequality we obtain
\[
\|1_{B_+(0,A)}f\|_{L^1(d\nu)} \leq \left(\frac{A^{2\alpha + 3}}{2^{\alpha + \frac{3}{2} \Gamma(\alpha + \frac{5}{2})}}\right)^{\frac{1}{2}} f\|_{L^2(d\nu)}.
\]
On the other hand
\[ \| (1 - 1_{B_r(0, A)}) f \|_{L^1(\mu)} \leq A^{-2s} \| (r, x) \|^{2s} (1 - 1_{B_r(0, A)}) f \|_{L^1(\mu)} \].

It follows then
\[ \| f \|_{L^1(\mu)} \leq \left( \frac{A^{2s+3}}{2^{2s+3} \Gamma(\alpha + \frac{5}{2})} \right)^{\frac{1}{2}} \| f \|_{L^2(\mu)} + A^{-2s} \| (r, x) \|^{2s} f \|_{L^1(\mu)} \].

Minimizing the right hand side of that inequality over \( A > 0 \) gives
\[ \| f \|_{L^1(\mu)} \leq C(\alpha, s) \| f \|_{L^2(\mu)}^{\frac{4s}{2s+3}} \| (r, x) \|^{2s} f \|_{L^1(\mu)}^{\frac{2s+3}{4s+2s+3}} \]. (6.44)

The desired result follows immediately from (6.44).

From the previous results we deduce the following variation on Heisenberg’s uncertainty inequality for the generalized Fourier transform.

**Theorem 5.** Let \( s > 0 \). Then for all \( f \in L^1(\mu) \cap L^2(\mu) \)
\[ \| f \|_{L^2(\mu)}^2 \| f \|_{L^1(\mu)} \leq C_1(\alpha, s) C_2(\alpha, s) \| (r, x) \|^{2s} f \|_{L^1(\mu)} \| \theta(\lambda, \mu) \|^{s} F_{\alpha}(f) \|_{L^2(\mu)} \]. (6.45)

**Proof.** The result follows immediately by multiplying inequality (6.41) by (6.43)

**Proposition 13.** Let \( s > 0 \) and let \( W \) a measurable subset of \( \Gamma \) with \( 0 < \text{mes}_{\gamma}(W) < \infty \). Then there exists a constant \( C(\alpha, s) \) such that for all \( f \in L^1(\mu) \cap L^2(\mu) \)
\[ \| 1_W F_{\alpha}(f) \|_{L^2(\mu)} \leq C(\alpha, s) \sqrt{\text{mes}_{\gamma}(W)} \| f \|_{L^2(\mu)}^{\frac{4s}{4s+2s+3}} \| (r, x) \|^{2s} f \|_{L^1(\mu)}^{\frac{2s+3}{4s+2s+3}} \]. (6.46)

**Proof.** We have
\[ \| 1_W F_{\alpha}(f) \|_{L^2(\mu)} \leq \sqrt{\text{mes}_{\gamma}(W)} \| F_{\alpha}(f) \|_{L^m(\mu)} \leq \sqrt{\text{mes}_{\gamma}(W)} \| f \|_{L^1(\mu)} \].

The desired result follows from Carlson Inequality (6.44).

We adapt the method of Ghorbal-Jaming [13], we obtain.

**Theorem 6.** Let \( E, W \) be a pair of measurable subsets such that
\[ 0 < \text{mes}_{\gamma}(E), \text{mes}_{\gamma}(W) < \infty \).

Then the following uncertainty principles hold.
1) Local uncertainty principle of \( F_{\alpha} \)
(i) For \( 0 < s < \frac{2\alpha + 3}{2} \), there exists a constant \( C(\alpha, s) \) such that for all \( f \in L^2(d\nu_\alpha) \)

\[
\|1_w \mathcal{F}_\alpha(f)\|_{L^2(d\gamma_a)} \leq C(\alpha, s)(\text{mes}_\gamma(W))^{\frac{s}{2\alpha + 3}} \|\| (r, x) \|\| t f\|_{L^2(d\nu_\alpha)}.
\]

(6.47)

(ii) For \( s > \frac{2\alpha + 3}{2} \), there exists a constant \( C(\alpha, s) \) such that for all \( f \in L^2(d\nu_\alpha) \)

\[
\|1_w \mathcal{F}_\alpha(f)\|_{L^2(d\gamma_a)} \leq C(\alpha, s)\sqrt{\text{mes}_\gamma(W)} \|\| (r, x) \|\| t f\|_{L^2(d\nu_\alpha)}^{\frac{2\alpha + 3}{(2\alpha + 3)^2}} \|f\|_{L^2(d\nu_\alpha)}. 
\]

(6.48)

2) Global uncertainty principle of \( \mathcal{F}_\alpha \)

For \( s, t > 0 \), there exists a constant \( C(\alpha, s) \) such that for all \( f \in L^2(d\nu_\alpha) \)

\[
\|\| (r, x) \|\| t f\|_{L^2(d\nu_\alpha)}^{\frac{2\alpha + 3}{2(2\alpha + 3)^2}} \|\| \theta(\lambda, \mu)\|\| \mathcal{F}_\alpha(f)\|_{L^2(d\gamma_a)}^{\frac{2\alpha + 3}{2(2\alpha + 3)^2}} \geq C(\alpha, s)\|f\|_{L^2(d\nu_\alpha)}^2.
\]

(6.49)

We put

\[
h_t(\lambda, \mu) := e^{-t||\theta(\lambda, \mu)||^2}, \quad \text{for all } \lambda, \mu \in \mathbb{R}.
\]

**Lemma 8.** Let \( 1 \leq q < \infty \). We have

\[
\|h_t\|_{L^q(d\gamma_a)} \leq Ct^{-\frac{2\alpha + 3}{2q}}.
\]

**Proof.** Let \( 1 \leq q < \infty \). Using the relation (2.2), we obtain the result. \( \square \)

**Lemma 9.** Let \( 1 < p \leq 2 \) and \( 0 < a < \frac{2\alpha + 3}{p} \). Then for all \( f \in L^p(d\nu_\alpha) \) and \( t > 0 \),

\[
\|e^{-t||\theta(\lambda, \mu)||^2} \mathcal{F}_\alpha(f)\|_{L^p(d\gamma_a)} \leq Ct^{-\frac{2}{p}} \|\| (r, x) \|\| t f\|_{L^p(d\nu_\alpha)}. 
\]

(6.50)

**Proof.** Inequality (6.50) holds if \( \|\| (r, x) \|\| t f\|_{L^p(d\nu_\alpha)} = \infty \).

Assume that \( \|\| (r, x) \|\| t f\|_{L^p(d\nu_\alpha)} < \infty \). For \( s > 0 \) let \( f_s = f \mathcal{X}_{B(0, s)} \) and \( f^2 = f - f_s \). Then since, \( \|\mathcal{F}^2(r, x)\| \leq s^{-a} \|\| (r, x) \|\| t a f(r, x)\| \)

\[
\|e^{-t||\theta(\lambda, \mu)||^2} \mathcal{F}_\alpha(f \mathcal{X}_{B(0, s)})\|_{L^p(d\gamma_a)} \leq \|e^{-t||\theta(\lambda, \mu)||^2} \mathcal{F}_\alpha(f \mathcal{X}_{B(0, s)})\|_{L^p(d\gamma_a)} \leq \|f \mathcal{X}_{B(0, s)}\|_{L^p(d\nu_\alpha)} \leq s^{-a} \|\| (r, x) \|\| t a f\|_{L^p(d\nu_\alpha)}. 
\]

By Proposition 4 and Hölder’s inequality

\[
\|e^{-t||\theta(\lambda, \mu)||^2} \mathcal{F}_\alpha(f \mathcal{X}_{B(0, s)})\|_{L^p(d\gamma_a)} \leq \|e^{-t||\theta(\lambda, \mu)||^2} \mathcal{F}_\alpha(f \mathcal{X}_{B(0, s)})\|_{L^p(d\gamma_a)} \leq \|e^{-t||\theta(\lambda, \mu)||^2} \mathcal{F}_\alpha(f \mathcal{X}_{B(0, s)})\|_{L^p(d\gamma_a)}.
\]

On the other hand,

\[
\|f \mathcal{X}_{B(0, s)}\|_{L^1(d\nu_\alpha)} \leq \|\| (r, x) \|\| t a \mathcal{X}_{B(0, s)}\|_{L^p(d\nu_\alpha)} \|\| (r, x) \|\| t a f\|_{L^p(d\nu_\alpha)}.
\]
A simple calculation gives that
\[ ||(r, x)||^{-a} \mathcal{X}_B(0, x) ||_{L^{p'}(dv \alpha)} = C(\alpha) s^{\frac{2a+3}{p'}}. \]

So
\[ ||e^{-t||\theta(\lambda, \mu)||^2} \mathcal{F}_\alpha(f)||_{L^{p'}(d\gamma_a)} \leq (1 - e^{-t||\theta(\lambda, \mu)||^2}) \mathcal{F}_\alpha(f)||_{L^{p'}(d\gamma_a)} \]
\[ \leq C s^{-a}(1 + ||e^{-t||\theta(\lambda, \mu)||^2}||_{L^{p'}(d\gamma_a)}^{\frac{2a+3}{p'}})||f||_{L^{p}(dv \alpha)}. \]

Choosing \( s = \frac{1}{t^2} \), we obtain (6.50).

**Theorem 7.** Let \( 1 < p \leq 2 \) and \( 0 < a < \frac{2a+3}{p'} \) and \( b > 0 \). Then for all \( f \in L^{p}(d\nu \alpha) \)
\[ ||\mathcal{F}_\alpha(f)||_{L^{p'}(d\gamma_a)} \leq C ||(r, x)||^a f||_{L^{p}(dv \alpha)} ||\theta(\mu, \lambda)||^b \mathcal{F}_\alpha(f)||_{L^{p'}(d\gamma_a)}, \quad (6.51) \]

**Proof.** Let \( 1 < p \leq 2 \) and \( 0 < a < \frac{2a+3}{p'} \). Assume that \( b \leq 2 \). From the previous lemma, for all \( t > 0 \)
\[ ||\mathcal{F}_\alpha(f)||_{L^{p'}(d\gamma_a)} \leq (1 - e^{-t||\theta(\lambda, \mu)||^2}) \mathcal{F}_\alpha(f)||_{L^{p'}(d\gamma_a)} + ||f||_{L^{p}(dv \alpha)}(1 - e^{-t||\theta(\lambda, \mu)||^2}) \mathcal{F}_\alpha(f)||_{L^{p'}(d\gamma_a)}. \]
On the other hand,
\[ ||(1 - e^{-t||\theta(\lambda, \mu)||^2}) \mathcal{F}_\alpha(f)||_{L^{p'}(d\gamma_a)} = t \frac{b}{2} ||(t||\theta(\lambda, \mu)||^2)^{-\frac{b}{2}} (1 - e^{-t||\theta(\lambda, \mu)||^2})||\theta(\mu, \lambda)||^{b} \mathcal{F}_\alpha(f)||_{L^{p'}(d\gamma_a)}. \]
Since \( (1 - e^{-t})t^{-\frac{b}{2}} \) is bounded for \( t \geq 0 \) if \( b \leq 2 \). Then, we obtain
\[ ||\mathcal{F}_\alpha(f)||_{L^{p'}(d\gamma_a)} \leq C \left( t^{\frac{b}{2}} ||(r, x)||^a f||_{L^{p}(dv \alpha)} + t^{\frac{b}{2}} ||\theta(\lambda, \mu)||^b \mathcal{F}_\alpha(f)||_{L^{p'}(d\gamma_a)} \right). \]
from which, optimizing in \( t \), we obtain (6.51) for \( 0 < a < \frac{2a+3}{p'} \) and \( b \leq 2 \).

If \( b > 2 \), let \( b' \leq 2 \). For \( u \geq 0 \) and \( b' < b \), we have \( u^{b'} \leq 1 + u^b \), which for \( u = \frac{||\theta(\lambda, \mu)||}{\epsilon} \) gives the inequality \( \left( \frac{||\theta(\lambda, \mu)||}{\epsilon} \right)^{b'} < 1 + \left( \frac{||\theta(\lambda, \mu)||}{\epsilon} \right)^b \) for all \( \epsilon > 0 \).

It follows that
\[ ||\theta(\lambda, \mu)||^{b'} \mathcal{F}_\alpha(f)||_{L^{p'}(d\gamma_a)} \leq \epsilon^{b'} + \epsilon^{b'} - b||\theta(\lambda, \mu)||^b \mathcal{F}_\alpha(f)||_{L^{p'}(d\gamma_a)}. \]
Optimizing in \( \epsilon \), we get the result for \( b > 2 \).
\[ ||\theta(\lambda, \mu)||^{b'} \mathcal{F}_\alpha(f)||_{L^{p'}(d\gamma_a)} \leq ||\mathcal{F}_\alpha(f)||_{L^{p'}(d\gamma_a)^{\frac{b-a}{p}}} ||\theta(\lambda, \mu)||^b \mathcal{F}_\alpha(f)||_{L^{p'}(d\gamma_a)}. \]
Together with (6.51) for \( b > 2 \).
Corollary 4. Let \(a, b > 0\). For all \(f \in L^2(d\nu_\lambda)\), we have

\[
\|f\|_{L^2(d\nu_\lambda)} \leq C \|\|\langle r, x\rangle\|^a f\|_{L^2(d\nu_\lambda)}^{\frac{b}{a+b}} \|\theta(\mu, \lambda)\|^b F_\alpha(f)\|_{L^2(d\nu_\lambda)}^{\frac{a}{a+b}}.
\] (6.52)

Proof. Using the previous theorem for \(p = 2\), and applying Plancherel formula, we obtain the result when \(0 < a < \frac{2\alpha + 3}{2}\). If \(a \geq \frac{2\alpha + 3}{2}\), let \(a' < \frac{2\alpha + 3}{2}\). For \(u \geq 0\), \(u^{a'} \leq 1 + u^a\) which for \(u = \frac{\|\langle r, x\rangle\|}{\varepsilon}\) gives the inequality

\[
\left(\frac{\|\langle r, x\rangle\|}{\varepsilon}\right)^{a'} \leq 1 + \left(\frac{\|\langle r, x\rangle\|}{\varepsilon}\right)^a, \quad \text{for all } \varepsilon > 0.
\]

It follows that

\[
\|\|\langle r, x\rangle\|^a f\|_{L^2(d\nu_\lambda)} \leq \varepsilon^{a'} \|f\|_{L^2(d\nu_\lambda)} + \varepsilon^{a'-a} \|\|\langle r, x\rangle\|^a f\|_{L^2(d\nu_\lambda)}.
\]

Optimizing in \(\varepsilon\), we obtain

\[
\|\|\langle r, x\rangle\|^a f\|_{L^2(d\nu_\lambda)} \leq C\|f\|_{\frac{a'-a}{L^2(d\nu_\lambda)}} \|\|\langle r, x\rangle\|^a f\|_{\frac{a}{L^2(d\nu_\lambda)}}.
\] (6.53)

Then, by (6.52) for \((a', b)\), and (6.53), we deduce that

\[
\|f\|_{L^2(d\nu_\lambda)} \leq C\|\|\langle r, x\rangle\|^a f\|_{\frac{a'-a}{L^2(d\nu_\lambda)}} \|\lambda\|^b F_\alpha(f)\|_{\frac{a'}{L^2(\mathbb{R})}} \leq C\|f\|_{\frac{a'-a}{L^2(d\nu_\lambda)}} \|\|\langle r, x\rangle\|^a f\|_{\frac{a}{L^2(d\nu_\lambda)}} \|\theta(\mu, \lambda)\|^b F_\alpha(f)\|_{\frac{a'}{L^2(d\nu_\lambda)}}.
\]

Thus

\[
\|f\|_{L^2(d\nu_\lambda)} \leq C\|\|\langle r, x\rangle\|^a f\|_{\frac{a'-a}{L^2(d\nu_\lambda)}} \|\theta(\mu, \lambda)\|^b F_\alpha(f)\|_{\frac{a'}{L^2(d\nu_\lambda)}},
\]

which gives the result for \(a \geq \frac{2\alpha + 3}{2}\). \(\square\)

Remark 3. The previous corollary generalize the result proved in [26].

Let \(T\) be a measurable subset of \(\mathbb{R}_+^2\). Let \(b > 0\) and let \(f \in L^p(d\nu_\lambda), p \in [1, 2]\). We say that \(\|\theta(\mu, \lambda)\|^b F_\alpha(f)\) is \(\varepsilon_T\)-concentrated to \(T\) in \(L^p(d\nu_\lambda)\)-norm, if there is a function \(h\) vanishing outside \(T\) such that

\[
\|\|\theta(\mu, \lambda)\|^b F_\alpha(f) - h\|_{L^p(d\nu_\lambda)} \leq \varepsilon_T \|\|\theta(\mu, \lambda)\|^b F_\alpha(f)\|_{L^p(d\nu_\lambda)}.
\]

From (6.37), it follows that \(\|\theta(\mu, \lambda)\|^b F_\lambda(f)\) is \(\varepsilon_T\)-concentrated to \(T\) in \(L^p(d\nu_\lambda)\)-norm, if and only if

\[
\|\|\theta(\mu, \lambda)\|^b F_\alpha(f) - ||\theta(\mu, \lambda)\|^b F_\alpha(QT f)\|_{L^p(d\nu_\lambda)} \leq \varepsilon_T \|\|\theta(\mu, \lambda)\|^b F_\alpha(f)\|_{L^p(d\nu_\lambda)}. \tag{6.54}
\]
Corollary 5. Let $T$ be a measurable subset of $\mathbb{R}^2_+$, and let $1 < p \leq 2$, $f \in L^p(d\nu_{\alpha})$ and $b > 0$. If $\|\theta(\mu, \lambda)\|^b \mathcal{F}_{\alpha}(f)$ is $\varepsilon_T$-concentrated to $T$ in $L^{p'}(d\gamma_{\alpha})$-norm, then for $0 < a < \frac{2\alpha+3}{p'}$

$$
\|\mathcal{F}_{\alpha}(f)\|_{L^{p'}(d\gamma_{\alpha})} \leq \frac{C}{(1 - \varepsilon_T)^{\frac{a}{a+b}}} \|\varepsilon_T\| L^{p'}(d\nu_{\alpha}) \|\theta(\mu, \lambda)\|^b \mathcal{F}_{\alpha}(Q_T f)\|^a_{L^{p'}(d\gamma_{\alpha})}.
$$

(6.55)

Proof. Let $f \in L^p(d\nu_{\alpha})$, $1 < p \leq 2$. Since $\|\theta(\mu, \lambda)\|^b \mathcal{F}_{\alpha}(f)$ is $\varepsilon_T$-concentrated to $T$ in $L^{p'}(d\gamma_{\alpha})$-norm, then we have

$$
\|\|\theta(\mu, \lambda)\|^b \mathcal{F}_{\alpha}(f)\|_{L^{p'}(d\gamma_{\alpha})}
$$

$$
\leq \varepsilon_T \|\|\theta(\mu, \lambda)\|^b \mathcal{F}_{\alpha}(f)\|_{L^{p'}(d\gamma_{\alpha})} + \|\|\theta(\mu, \lambda)\|^b \mathcal{F}_{\alpha}(Q_T f)\|_{L^{p'}(d\gamma_{\alpha})}.
$$

Thus

$$
\|\|\theta(\mu, \lambda)\|^b \mathcal{F}_{\alpha}(f)\|_{L^{p'}(d\gamma_{\alpha})} \leq \frac{1}{(1 - \varepsilon_T)^{\frac{a}{a+b}}} \|\|\theta(\mu, \lambda)\|^b \mathcal{F}_{\alpha}(Q_T f)\|_{L^{p'}(d\gamma_{\alpha})}^{\frac{a}{a+b}}.
$$

Multiply this inequality by $C\|\varepsilon_T\| L^{p'}(d\nu_{\alpha}) \|\theta(\mu, \lambda)\|^b \mathcal{F}_{\alpha}(Q_T f)\|^a_{L^{p'}(d\gamma_{\alpha})}$ and applying theorem 7 we deduce the desired result.  

We proceed as the previous corollary and using Corollary 4 we obtain the following.

Corollary 6. Let $T$ be a measurable subset of $\mathbb{R}^2_+$, and let $f \in L^2(d\nu_{\alpha})$ and $a, b > 0$.

If $\|\theta(\mu, \lambda)\|^b \mathcal{F}_{\alpha}(f)$ is $\varepsilon_T$-concentrated to $T$ in $L^2(d\gamma_{\alpha})$-norm, then

$$
\|f\|_{L^2(d\nu_{\alpha})} \leq \frac{C}{(1 - \varepsilon_T)^{\frac{a}{a+b}}} \|\varepsilon_T\| L^2(d\nu_{\alpha}) \|\theta(\mu, \lambda)\|^b \mathcal{F}_{\alpha}(Q_T f)\|^a_{L^2(d\gamma_{\alpha})}.
$$

(6.56)
REFERENCES


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