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QUALITATIVE AND QUANTITATIVE UNCERTAINTY PRINCIPLES FOR THE GENERALIZED FOURIER TRANSFORM ASSOCIATED WITH THE RIEMANN-LIOUVILLE OPERATOR

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The aim of this paper is to establish an extension of qualitative and quantitative uncertainty principles for the Fourier transform connected with the Riemann-Liouville operator.

1. Introduction

Classical uncertainty principles give us information about a function and its Fourier transform. If we try to limit the behavior of one we lose control of the other. Uncertainty principles have implications in two main areas: quantum physics and signal analysis. In quantum physics they tell us that a particles speed and position cannot both be measured with infinite precision. In signal analysis they tell us that if we observe a signal only for a finite period of time, we will lose information about the frequencies the signal consists of. The mathematical equivalent is that a function and its Fourier transform cannot both be

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arbitrarily localized. There is two categories of uncertainty principles: Quantitative uncertainty principles and Qualitative uncertainty principles.

Quantitative uncertainty principles is just another name for some special inequalities. These inequalities give us information about how a function and its Fourier transform relate. They are called uncertainty principles since they are similar to the classical Heisenberg Uncertainty Principle, which has had a big part to play in the development and understanding of quantum physics. For example: Benedicks [3], Slepian and Pollak [29], Landau and Pollak [18], and Donoho and Stark [10] paid attention to the supports of functions and gave qualitative uncertainty principles for the Fourier transforms.

Qualitative uncertainty principles are not inequalities, but are theorems that tell us how a function (and its Fourier transform) behave under certain circumstances. For example: Hardy [14], Morgan [23], Cowling and Price [8], Beurling [4], Miyachi [22] theorems enter within the framework of the quantitative uncertainty principles.

The quantitative and qualitative uncertainty principles has been studied by many authors for various Fourier transforms, for examples (cf. [6, 7, 12, 13, 19, 20, 30]).

In [2], the authors considered the singular partial differential operators defined by

$$\begin{aligned} \Delta_1 &= \frac{\partial}{\partial x}, \\ \Delta_2 &= \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r}\frac{\partial}{\partial r} - \frac{\partial^2}{\partial x^2}, \quad (r,x) \in (0,\infty)\times\mathbb{R}, \quad \alpha \geq 0 \end{aligned}$$

and they associated to Δ_1 and Δ_2 the following integral transform, called the Riemann-Liouville operator, defined on $C_*(\mathbb{R}^2)$ by

$$\mathcal{R}_\alpha(f)(r,x) = \begin{cases} \frac{\alpha}{\pi}\int_{-1}^{1}\int_{-1}^{1} f(rs\sqrt{1-t^2}, x+rt)(1-t^2)^{\alpha-\frac{1}{2}}(1-s^2)^{\alpha-1}dtds, & if\alpha > 0 \\ \frac{1}{\pi}\int_{-1}^{1} f(r\sqrt{1-t^2}, x+rt)(1-t^2)^{-\frac{1}{2}}dt, & if\alpha = 0 \end{cases}$$

In addition, a convolution product and a Fourier transform $\mathcal{F}_\alpha$ connected with the mapping $\mathcal{R}_\alpha$ have been studied and many harmonic analysis results have been established for the Fourier transform $\mathcal{F}_\alpha$ (Inversion formula, Plancherel formula, Paley-Winer and Plancherel theorems, ...). Our purpose in this work is to study the uncertainty principles for the Fourier transform $\mathcal{F}_\alpha$ connected with $\mathcal{R}_\alpha$.

Our aim here is to consider quantitative and qualitative uncertainty principles when the transform under consideration is the Fourier transform connected with the Riemann-Liouville operator .

The remaining part of the paper is organized as follows. In §2, we recall the main results about the Riemann-Liouville operator. §3 is devoted to generalize Cowling-Price's theorem for the generalized Fourier transform $\mathcal{F}_\alpha$. In

§4 we generalize Miyachi's theorem and in §5 Beurling's theorem for $\mathcal{F}_\alpha$. §6 is devoted to Donoho-Stark's uncertainty principle and variants of Heisenberg's inequalities for $\mathcal{F}_\alpha$.

2. Riemann-Liouville operator

In this section, we define and recall some properties of the Riemann-Liouville operator. For more details see ([2, 21]). We denote by

- $C_*(\mathbb{R}^2)$ the space of continuous functions on $\mathbb{R}^2$, even with respect to the first variable.
- $C_{*,c}(\mathbb{R}^2)$ the subspace of $C_*(\mathbb{R}^2)$ formed by functions with compact support.
- $\mathcal{E}_*(\mathbb{R}^2)$ the space of infinitely differentiable functions on $\mathbb{R}^2$, even with respect to the first variable.
- $\mathcal{S}_*(\mathbb{R}^2)$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}^2$, even with respect to the first variable.
- S^1 the unit sphere in $\mathbb{R}^2$,

$$S^1 = \Big\{(\eta,\xi) \in \mathbb{R}^2 : \eta^2 + \xi^2 = 1\Big\}.$$

- $\mathbb{R}^2_+ = \Big\{(r,x) \in \mathbb{R}^2 :\ r > 0\Big\}$.

It is well known [2] that for all $(\mu,\lambda) \in \mathbb{C}^2$, the system

$$\begin{cases} \Delta_1 u(r,x) &= -i\lambda u(r,x), \\ \Delta_2 u(r,x) &= -\mu^2 u(r,x) \\ u(0,0) &= 1, \quad \frac{\partial u}{\partial r}(0,x) = 0,\ \forall x \in \mathbb{R}, \end{cases}$$

admits a unique solution $\varphi_{\mu,\lambda}$, given by

$$\varphi_{\mu,\lambda}(r,x) = j_\alpha(r\sqrt{\mu^2+\lambda^2})e^{-i\lambda x},$$

where j_α is the normalized Bessel function defined by

$$\forall z \in \mathbb{C}, \quad j_\alpha(z) = \Gamma(\alpha+1)\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+1+\alpha)}(z/2)^{2k}.$$

Definition 1. *The Riemann-Liouville operator is defined on $C_*(\mathbb{R}^2)$ by:* $\forall (r,x) \in \mathbb{R}^2_+$

$$\mathcal{R}_\alpha f(r,x) = \begin{cases} \frac{\alpha}{\pi}\int_{-1}^{1}\int_{-1}^{1} f(rs\sqrt{1-t^2}, x+rt)(1-t^2)^{\alpha-\frac{1}{2}}(1-s^2)^{\alpha-1}dtds & if \alpha > 0 \\ \frac{1}{\pi}\int_{-1}^{1} f(r\sqrt{1-t^2}, x+rt)(1-t^2)^{-\frac{1}{2}}dt & if \alpha = 0. \end{cases}$$

Remark 1. *(i) The function $\varphi_{\mu,\lambda}$, $(\mu,\lambda) \in \mathbb{C}^2$, can be written as*

$$\forall (r,x) \in \mathbb{R}^2_+,\ \varphi_{\mu,\lambda}(r,x) = \mathcal{R}_\alpha(\cos(\mu.)e^{-i\lambda.})(r,x).$$

(ii) For all $\nu \in \mathbb{N}^2, (r,x) \in \mathbb{R}^2$ and $z = (\mu,\lambda) \in \mathbb{C}^2$,

$$|D_z^\nu \varphi_{\mu,\lambda}(r,x)| \leq ||(r,x)||^{|\nu|} \exp(2||(r,x)||\,||\mathrm{Im} z||), \tag{2.1}$$

where

$$D_z^\nu = \frac{\partial^{|\nu|}}{\partial z_1^{\nu_1}\partial z_2^{\nu_2}} \quad \text{and} \quad |\nu| = \nu_1 + \nu_2.$$

Now let Γ be the set

$$\Gamma = \mathbb{R}^2 \cup \Big\{(it,x); (t,x) \in \mathbb{R}^2, |t| \leq |x|\Big\}.$$

Γ_+ the subset of Γ, given by

$$\Gamma_+ = \mathbb{R}^2 \cup \Big\{(it,x); (t,x) \in \mathbb{R}^2, 0 \leq t \leq |x|\Big\}.$$

We have for all $(\mu,\lambda) \in \Gamma$,

$$\sup_{(r,x)\in\mathbb{R}^2} |\varphi_{\mu,\lambda}(r,x)| = 1.$$

In the following, we denote by

- $d\nu_\alpha(r,x)$ the measure defined on $\mathbb{R}^2_+$ by

$$d\nu_\alpha(r,x) = k_\alpha r^{2\alpha+1} dr \otimes dx,$$

with

$$k_\alpha = \frac{1}{2^\alpha \Gamma(\alpha+1)(2\pi)^{1/2}}.$$

- $L^p(d\nu_\alpha), 1 \leq p \leq \infty$, the space of measurable functions on $\mathbb{R}^2_+$, satisfying

$$\|f\|_{L^p(d\nu_\alpha)} = \left(\int_{\mathbb{R}^2_+} |f(r,x)|^p d\nu_\alpha(r,x)\right)^{1/p} < \infty,\ 1 \leq p < \infty,$$

$$\|f\|_{L^\infty(d\nu_\alpha)} = \operatorname*{ess\,sup}_{(r,x)\in\mathbb{R}^2_+} |f(r,x)| < \infty,\ p = \infty.$$

- $\mathcal{B}_{\Gamma_+}$ the σ-algebra defined on Γ_+ by

$$\mathcal{B}_{\Gamma_+} = \Big\{\theta^{-1}(B) : \quad B \in \mathcal{B}_{Bor}(\mathbb{R}_+^2)\Big\},$$

where θ defined on the set Γ_+ by $\theta(\lambda,\mu) = (\sqrt{\mu^2+\lambda^2},\lambda)$.

- $d\gamma_\alpha$ the measure defined on $\mathcal{B}_{\Gamma_+}$ by

$$\forall A \subset \mathcal{B}_{\Gamma_+}, \quad \gamma_\alpha(A) = \nu_\alpha(\theta(A)).$$

- $L^p(d\gamma_\alpha), 1 \le p \le \infty$, the space of measurable functions on Γ_+, satisfying

$$\begin{aligned} \|f\|_{L^p(d\gamma_\alpha)} &= \Big(\int_{\Gamma_+} |f(\mu,\lambda)|^p d\gamma_\alpha(\mu,\lambda)\Big)^{1/p} < \infty, \ 1 \le p < \infty, \\ \|f\|_{L^\infty(d\gamma_\alpha)} &= \underset{(\mu,\lambda)\in\Gamma_+}{ess\ sup}\ |f(\mu,\lambda)| < \infty, \ p = \infty. \end{aligned}$$

We have the following properties.

Proposition 1. i) For every nonnegative measurable function g on Γ_+, we have

$$\int_{\Gamma_+} f(\mu,\lambda)d\gamma_\alpha(\mu,\lambda) = k_\alpha\Big[\int_{\mathbb{R}_+^2} f(\mu,\lambda)(\mu^2+\lambda^2)^\alpha \mu d\mu d\lambda$$
$$+ \int_{\mathbb{R}^2}\int_0^{|\lambda|} f(i\mu,\lambda)(\lambda^2-\mu^2)^\alpha \mu d\mu d\lambda\Big].$$

ii) For every nonnegative measurable function f on $\mathbb{R}_+^2$ (resp. integrable on $\mathbb{R}_+^2$ with respect to the measure $d\nu$), $f\circ\theta$ is a measurable nonnegative function on Γ_+, (resp. integrable on Γ_+ with respect to the measure $d\gamma_\alpha$) and we have

$$\int_{\Gamma_+} f\circ\theta(\mu,\lambda)d\gamma_\alpha(\mu,\lambda) = \int_{\mathbb{R}_+^2} f(r,x)d\nu_\alpha(r,x). \tag{2.2}$$

In the following we recall some results on the dual of the Riemann-Liouville operator $\mathcal{R}_\alpha$.

Definition 2. *The dual ${}^t\mathcal{R}_\alpha$ of the Riemann-Liouville operator $\mathcal{R}_\alpha$ is defined by : $\forall (s,y) \in \mathbb{R}^2$,*

$$ {}^t\mathcal{R}_\alpha(f)(s,y) = \begin{cases} \frac{\alpha}{\pi}\int_r^\infty \int_{-\sqrt{u^2-r^2}}^{\sqrt{u^2-r^2}} f(u,x+v)(u^2-v^2-r^2)^{\alpha-1}(1-s^2)^{\alpha-1} u du dv & if\, \alpha > 0 \\ \frac{1}{\pi}\int_{\mathbb{R}} f(r^2+(x-y)^2),y)dy, & if\, \alpha = 0 \end{cases} \tag{2.3}$$

Example 1. *Let $p \in [1,\infty)$. For all $a > 0$, $\beta > 0$ we have*

$$\forall\,(s,y) \in \mathbb{R}^2, \quad {}^t\mathcal{R}_\alpha(E^p_{a,\beta})(s,y) = C(a,\beta,p)E^p_{\frac{a\beta}{1+\beta},1+\beta}(s,y), \tag{2.4}$$

with $E_{a,\beta}$ is the Gauss kernel associated with the Riemann-Liouville operator $\mathcal{R}_\alpha$ defined by

$$\forall\,(r,x) \in \mathbb{R}^2,\ E_{a,\beta}(r,x) = k(a,\beta)e^{-a(\beta r^2+x^2)}, \tag{2.5}$$

where

$$k(a,\beta) = \frac{2\sqrt{\pi}a^{2\alpha+\frac{3}{2}}}{\Gamma(\alpha+1)}(\frac{\beta}{\pi})^{\alpha+1}, \quad \text{and} \quad C(a,\beta,p) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}}\Big[\frac{(1+\beta)^{p-1}}{a\beta^p p}\Big]^{\frac{2\alpha+1}{2}}$$

Proposition 2. The function ${}^t\mathcal{R}_\alpha(f)$ defined almost everywhere on $\mathbb{R}^2_+$, by relation (2.3), is Lebesgue integrable on $\mathbb{R}^2_+$. Moreover for all bounded function $g \in C_*(\mathbb{R}^2)$, we have the formula

$$\int_{\mathbb{R}^2_+} {}^t\mathcal{R}_\alpha(f)(s,y)g(s,y)dsdy = \int_{\mathbb{R}^2_+} \mathcal{R}_\alpha(g)(r,x)f(r,x)r^{2\alpha+1}drdx. \tag{2.6}$$

Remark 2. *Let f be in $L^1(d\nu_\alpha)$. By taking $g \equiv 1$ in the relation (2.6) we deduce that*

$$\int_{\mathbb{R}^2_+} {}^t\mathcal{R}_\alpha(f)(s,y)dsdy = \int_{\mathbb{R}^2_+} f(r,x)r^{2\alpha+1}drdx. \tag{2.7}$$

We consider the generalized Fourier transform $\mathcal{F}_\alpha$ associated with the Riemann Liouville operator $\mathcal{R}_\alpha$ and we recall its main properties.

Definition 3. *The Fourier transform associated with the Riemann Liouville mean operator is defined on $L^1(d\nu_\alpha)$ by*

$$\forall\,(\mu,\lambda) \in \Gamma, \mathcal{F}_\alpha(f)(\mu,\lambda) = \int_{\mathbb{R}^2_+} f(r,x)\varphi_{\mu,\lambda}(r,x)d\nu_\alpha(r,x). \tag{2.8}$$

Example 2. *Let $a,\beta > 0$. The Fourier transform of Gauss kernel associated with Riemann-Liouville operator is given by*

$$\forall\ (\mu,\lambda) \in \Gamma,\ \mathcal{F}_\alpha(E_{a,\beta})(\mu,\lambda) = C(a,\beta,\alpha)E_{\frac{1+\beta}{4a\beta},\frac{1}{1+\beta}}(\mu,\lambda),$$

where

$$C(a,\beta,\alpha) = 2^{4\alpha+2}\Gamma(\alpha+1)(a\beta)^{2\alpha+\frac{3}{2}}(\frac{\pi}{1+\beta})^{\frac{2\alpha+1}{2}}.$$

Proposition 3. For all f in $L^1(d\nu_\alpha)$, we have the relation

$$\forall\,(\mu,\lambda)\in\Gamma, \mathcal{F}_\alpha(f)(\mu,\lambda)=\mathcal{F}_0\circ {}^t\mathcal{R}_\alpha(f)(\mu,\lambda), \tag{2.9}$$

where $\mathcal{F}_0$ is the Fourier-cosine transform on $\mathbb{R}^2$ defined for f in $\mathcal{S}_*(\mathbb{R}^2)$ by

$$\forall\,(\mu,\lambda)\in\mathbb{R}^2,\ \mathcal{F}_0(f)(\mu,\lambda)=\int_{\mathbb{R}^2_+} f(r,x)e^{-i\lambda x}\cos(r\mu)drdx.$$

In the follow we recall some properties on the Fourier transform $\mathcal{F}_\alpha$.

For all $f\in L^1(d\nu_\alpha)$,

$$||\mathcal{F}_\alpha(f)||_{L^\infty(d\gamma_\alpha)}\leq ||f||_{L^1(d\nu_\alpha)}. \tag{2.10}$$

For $f\in L^1(d\nu_\alpha)$ such that $\mathcal{F}_\alpha(f)\in L^1(d\gamma_\alpha)$, we have the inversion formula for $\mathcal{F}_\alpha$: for almost every $(r,x)\in\mathbb{R}^2_+$,

$$f(r,x)=\int_{\Gamma_+}\mathcal{F}_\alpha(f)(\mu,\lambda)\overline{\varphi_{\mu,\lambda}(r,x)}d\gamma_\alpha(\mu,\lambda). \tag{2.11}$$

Theorem 1. (Plancherel formula). For every f in $\mathcal{S}_*(\mathbb{R}^2)$, we have

$$\int_\Gamma|\mathcal{F}_\alpha(f)(\lambda,\mu)|^2d\gamma_\alpha(\lambda,\mu)=\int_{\mathbb{R}^2_+}|f(r,x)|^2d\nu_\alpha(r,x). \tag{2.12}$$

In particular, the Fourier transform $\mathcal{F}$ can be extended to an isometric isomorphism from $L^2(d\nu_\alpha)$ onto $L^2(d\gamma_\alpha)$.

Proposition 4. Let f be in $L^p(d\nu_\alpha)$, $p\in[1,2]$. Then $\mathcal{F}_\alpha(f)$ belongs to $L^{p'}(d\gamma_\alpha)$ with $\frac{1}{p}+\frac{1}{p'}=1$, and we have

$$\|\mathcal{F}_\alpha(f)\|_{L^{p'}(d\gamma_\alpha)}\leqslant\|f\|_{L^p(d\nu_\alpha)}.$$

For $(r,x)\in\mathbb{R}^2, s>0$, we note $N_s(r,x)$, by

$$N_s(r,x):=e^{-s(r^2+x^2)}. \tag{2.13}$$

We have

$$\mathcal{F}_\alpha\left(N_s(r,x)\right)(t,y)=C(s)e^{-\frac{(t^2+2y^2)}{4s}}.$$

We define the following functions W_l^s, $\widetilde{W}_l^s$, $l\in\mathbb{N}^2$, $s>0$ by

$$\forall\,(r,x)\in\mathbb{R}^2,\quad W_l^s(r,x)=r^{2k}x^me^{-s(r^2+x^2)},\quad l=(k,m), \tag{2.14}$$

and

$$\forall\,(r,x)\in\mathbb{R}^2,\quad \widetilde{W}_l^s(r,x)=\mathcal{F}_\alpha^{-1}(\lambda^{2k}\mu^me^{-s(\lambda^2+\mu^2)})(r,x),\quad l=(k,m), \tag{2.15}$$

Notation. We denote by $\mathcal{P}_m(\mathbb{R}^2)$ the set of homogeneous polynomials of degree m.

Proposition 5. ([26]). Let $l \in \mathbb{N}^2$. For all $s > 0$, there exists a homogeneous $Q \in \mathcal{P}_l(\mathbb{R}^2)$ such that

$$\forall (r,x) \in \mathbb{R}^2, \quad \mathcal{F}_\alpha(W_l^s)(r,x) = Q(r,x)e^{-\frac{1}{4s}(r^2+2x^2)}. \tag{2.16}$$

3. Generalized Cowling-Price theorem for the Generalized Fourier transform

Theorem 2. Let f be a measurable function on $\mathbb{R}^2_+$ such that

$$\int_{\mathbb{R}^2_+} \frac{e^{ap||(r,x)||^2}|f(r,x)|^p}{(1+||(r,x)||)^n} d\nu_\alpha(r,x) < \infty \tag{3.17}$$

and

$$\int_{\mathbb{R}^2_+} \frac{e^{4bq||\theta(\mu,\xi)||^2}|\mathcal{F}_\alpha(f)(\mu,\xi)|^q}{(1+||(\mu,\xi)||)^m} d\mu d\xi < \infty, \tag{3.18}$$

for some constants $a > 0$, $b > 0$, $1 \le p,q < \infty$, and for any $n \in (2\alpha+3, 2\alpha+3+p]$ and $m \in (2, 2+q]$. Then

i) If $ab > \frac{1}{4}$, we have $f = 0$ almost everywhere.

ii) If $ab = \frac{1}{4}$, we have $f = CN_b$.

iii) If $ab < \frac{1}{4}$, for all $\delta \in]b, \frac{1}{4a}[$, the functions of the form $f(r,x) = N_\delta(r,x)$, where $P \in \mathcal{P}$, satisfy (3.17) and (3.18).

Proof. We shall show that $\mathcal{F}_\alpha(f)(z)$ exists and is an entire function in $z \in \mathbb{C}^2$ and

$$|\mathcal{F}_\alpha(f)(z)| \le Ce^{\frac{1}{a}||\theta(Imz)||^2}(1+||Imz||)^s, \quad \text{for all} \, z \in \mathbb{C}^2, \quad \text{for some} \quad s > 0. \tag{3.19}$$

The first assertion follows from the hypothesis on the function f and Hölder's inequality using (3.17) and the derivation theorem under the integral sign. We want to prove (3.19). Actually, it follows from (2.8) and (2.1) that for all $z = (z_1, z_2) = (\mu + i\lambda, \xi + i\eta) \in \mathbb{C}^2$,

$$|\mathcal{F}_\alpha(f)(\mu+i\lambda,\xi+i\eta)| \le \int_{\mathbb{R}^2_+} |f(r,x)||\varphi_{(\mu+i\lambda,\xi+i\eta)}(r,x)| d\nu_\alpha(r,x)$$

$$\le e^{\frac{||(\lambda,\eta)||^2}{a}} \int_{\mathbb{R}^2_+} \frac{e^{a||(r,x)||^2}|f(r,x)|}{(1+||(r,x)||)^{\frac{n}{p}}}(1+||(r,x)||)^{\frac{n}{p}} e^{-a(||(r,x)||-||\frac{(\lambda,\eta)}{a}||)^2} d\nu_\alpha(r,x)$$

Then by using the Hölder inequality, (3.17) we can obtain that

$$\begin{aligned}|\mathcal{F}_\alpha(f)(\mu+i\lambda,\xi+i\eta)| &\le Ce^{\frac{\lambda^2+\eta^2}{a}}\Big(\int_{\mathbb{R}^2_+}(1+||(r,x)||)^{\frac{np'}{p}} e^{-ap'(||(r,x)||-||\frac{(\lambda,\eta)}{a}||)^2} d\nu_\alpha(r,x)\Big)^{\frac{1}{p'}}\\
&\le Ce^{\frac{\lambda^2+\eta^2}{a}}\Big(\int_0^\infty (1+t)^{\frac{np'}{p}+2\alpha+2} e^{-ap'(t-||\frac{(\lambda,\eta)}{a}||)^2} dt\Big)^{\frac{1}{p'}}\\
&\le Ce^{\frac{||\theta(\lambda,\eta)||^2}{a}}(1+||(\lambda,\eta)||)^{\frac{n}{p}+\frac{2\alpha+2}{p'}}\\
&= Ce^{\frac{1}{a}||\theta(Imz)||^2}(1+||Imz||)^{\frac{n}{p}+\frac{2\alpha+2}{p'}}.\end{aligned}$$

Thus (3.19) is proved.

• If $ab = \frac{1}{4}$, then

$$|\mathcal{F}_\alpha(f)(z)| \leq Ce^{4b||\theta(Imz)||^2}(1+||Imz||)^{\frac{n}{p}+\frac{2\alpha+2}{p'}}.$$

Therefore, if we let $g(z) = e^{4b(z_1^2+2z_2^2)}\mathcal{F}_\alpha(f)(z)$, then

$$|g(z)| \leq Ce^{4b||\theta(\mathrm{Re}z)||^2}(1+||\mathrm{Im}z||)^{\frac{n}{p}+\frac{2\alpha+2}{p'}}.$$

Hence it follows from (3.18) that

$$\int_{\mathbb{R}^2_+} \frac{|g(\mu,\xi)|^q}{(1+||(\mu,\xi)||)^m} d\mu d\xi < \infty.$$

Here we use the following lemma.

Lemma 1. ([28]) Let h be an entire function on $\mathbb{C}^2$ such that

$$|h(z)| \leq Ce^{a||\theta(\mathrm{Re}z)||^2}(1+||\mathrm{Im}z||)^m$$

for some $m > 0$, $a > 0$ and

$$\int_{\mathbb{R}^2} \frac{|h(x)|^q}{(1+|||(r,x)|||)^s}|Q(x)|dx < \infty$$

for some $q \geq 1$, $s > 1$ and $Q \in \mathcal{P}_M(\mathbb{R}^2)$.
Then h is a polynomial with $\deg h \leq \min\{m, \frac{s-M-2}{q}\}$ and, if $s \leq q+M+2$, then h is a constant.

Hence by this lemma g is a polynomial, we say P_b, with $\deg P_b := d \leq \min\{\frac{n}{p}+\frac{2\alpha+2}{p'}, \frac{m-2}{q}\}$. Then

$$\mathcal{F}_\alpha(f)(\lambda,\mu) = P_b(\lambda,\mu)e^{-4b(\lambda^2+2\mu^2)}.$$

Thus, by using (2.16), we can find constants c_l^s such that

$$f(r,x) = \sum_{|l|\leq d} c_l^s W_l^a(r,x) \quad \text{for all } (r,x) \in \mathbb{R}^2.$$

Therefore, nonzero f satisfies (3.17) provided that

$$n > 2\alpha+3+p\min\left\{\frac{n}{p}+\frac{2\alpha+2}{p'}, \frac{m-2}{q}\right\}.$$

Furthermore, if $m \leq q+2$, then g is a constant by the Lemma 1 and thus

$$\mathcal{F}_\alpha(f)(\lambda,\mu) = Ce^{-4b(\lambda^2+2\mu^2)} \quad \text{and} \quad f(r,x) = C_b e^{-a\|(r,x)\|^2}.$$

When $n > 2\alpha + 3$ and $m > 2$, these functions satisfy (3.18) and (3.17) respectively. This proves ii).

• If $ab > \frac{1}{4}$, then we can choose positive constants, a_1, b_1 such that $a > a_1 = \frac{1}{4b_1} > \frac{1}{4b}$. Then f and $\mathcal{F}_\alpha(f)$ also satisfy (3.17) and (3.18) with a and b replaced by a_1 and b_1 respectively. Therefore, it follows that $\mathcal{F}_\alpha(f)(\lambda,\mu) = P_{b_1}(\lambda,\mu)e^{-4b_1(\lambda^2+2\mu^2)}$. But then $\mathcal{F}_\alpha(f)$ cannot satisfy (3.18) unless $P_{b_1} \equiv 0$, which implies $f \equiv 0$. This proves i).

• If $ab < \frac{1}{4}$, then for all $\delta \in (b, \frac{1}{4a})$, the functions of the form $f(r,x) = W_l^\delta(r,x)$, where $P \in \mathcal{P}$, satisfy (3.17) and (3.18). This proves iii). □

The following is an immediate consequence of Theorem 2.

Corollary 1. Let f be a measurable function on $\mathbb{R}_+^2$ such that

$$|f(r,x)| \leq Me^{-a||(r,x)||^2}(1+||(r,x)||)^m \text{ a.e.} \tag{3.20}$$

and for all $(\mu,\xi) \in \mathbb{R}_+^2$,

$$|\mathcal{F}_\alpha(f)(\mu,\xi)| \leq Me^{-4b||\theta(\mu,\xi)||^2} \tag{3.21}$$

for some constants $a,b > 0$, $r \geq 0$ and $M > 0$.

i) If $ab > \frac{1}{4}$, then $f = 0$ almost everywhere.

ii) If $ab = \frac{1}{4}$, then f is of the form $f(r,x) = CN_b(r,x)$.

iii) If $ab < \frac{1}{4}$, then there are infinity many nonzero f satisfying (3.20) and (3.21).

4. Miyachi's theorem for the Generalized Fourier transform

Theorem 3. Let f be a measurable function on $\mathbb{R}_+^2$ even with respect to the first variable such that

$$E_{a,\beta}^{-1} f \in L^p(d\nu_\alpha) + L^q(d\nu_\alpha) \tag{4.22}$$

and

$$\int_{\mathbb{R}^2} \log^+ \frac{E^{-1}_{\frac{b(1+\beta)}{\beta},\frac{1}{1+\beta}}(\mu,\xi)|\mathcal{F}_\alpha(f)(\mu,\xi)|}{\lambda} d\mu d\xi < \infty, \tag{4.23}$$

for some constants $a > 0$, $b > 0$ $\lambda > 0$, $1 \leq p,q \leq \infty$. Then

If $ab > \frac{1}{4}$, we have $f = 0$ almost everywhere.

If $ab = \frac{1}{4}$, we have $f = CE_{b,\beta}$ with $|C| \leq \lambda$.

If $ab < \frac{1}{4}$, for all $\delta \in (b, \frac{1}{4a})$, the functions of the form $f(x) = CE_{\delta,\beta}$, satisfy (4.22) and (4.23).

To prove this result we need the following lemmas.

Lemma 2. ([20]). Let h be an entire on $\mathbb{C}^2$ function such that

$$|h(z)| \leq Ae^{B||Rez||^2} \text{ and } \int_{\mathbb{R}^2} \log^+ |h(y)|dy < \infty, \quad (4.24)$$

for some positive constants A, B. Then h is a constant on $\mathbb{C}^2$.

Lemma 3. Let r be in $[1,\infty]$. We consider a function g in $L^r(d\nu_\alpha)$. Then there exists a positive constant C such that:

$$||E^{-1}_{\frac{a\beta}{1+\beta},1+\beta}\,{}^t\mathcal{R}_\alpha(E_{a,\beta}g)||_{L^r(\mathbb{R}^2_+)} \leq C||g||_{L^r(d\nu_\alpha)},$$

where $||\cdot||_{L^r(\mathbb{R}^2_+)}$ is the norm of the usual Lebesgue space $L^r(\mathbb{R}^2_+)$ and $a > 0$.

Proof. From the hypothesis it follows that $E^{-1}_{a,\beta}g$ belongs to $L^1(d\nu_\alpha)$. Then by Proposition 2, the function ${}^t\mathcal{R}_\alpha(E^{-1}_{a,\beta}g)$ is defined almost everywhere on $\mathbb{R}^2$. Now we consider two cases.

i) If $r \in [1,\infty)$, we have

$$||E^{-1}_{\frac{a\beta}{1+\beta},1+\beta}\,{}^t\mathcal{R}_\alpha(E_{a,\beta}g)||^r_{L^r(\mathbb{R}^2_+)} = \int_{\mathbb{R}^2_+} E^{-r}_{\frac{a\beta}{1+\beta},1+\beta}(s,y)|{}^t\mathcal{R}_\alpha(E_{a,\beta}g)(s,y|^r dsdy.$$

By applying Hölder's inequality we obtain

$$||E^{-1}_{\frac{a\beta}{1+\beta},1+\beta}\,{}^t\mathcal{R}_\alpha(E_{a,\beta}g)||^r_{L^r(\mathbb{R}^2_+)} \leq \int_{\mathbb{R}^2_+} E^{-r}_{\frac{a\beta}{1+\beta},1+\beta}(s,y)\Big(|{}^t\mathcal{R}_\alpha(|g|^r)(s,y)| \times \Big(|{}^t\mathcal{R}_\alpha(E^{r'}_{a,\beta})(s,y)|\Big)^{r/r'} dyds,$$

where r' is the conjugate exponent of r. But from (2.4) we deduce that

$$||E^{-1}_{\frac{a\beta}{1+\beta},1+\beta}\,{}^t\mathcal{R}_\alpha(E_{a,\beta}g)||^r_{L^r(\mathbb{R}^2_+)} \leq C\int_{\mathbb{R}^2_+} {}^t\mathcal{R}_\alpha(|g|^r)(s,y)dsdy.$$

Thus using the relation (2.7) we obtain

$$||E^{-1}_{\frac{a\beta}{1+\beta},1+\beta}\,{}^t\mathcal{R}_\alpha(E_{a,\beta}g)||^r_{L^r(\mathbb{R}^2_+)} \leq C\int_{\mathbb{R}^2_+} |g(s,y)|^r d\nu_\alpha(s,y) < \infty.$$

ii) If $r = \infty$, we have

$$|E^{-1}_{\frac{a\beta}{1+\beta},1+\beta}(s,y)\,{}^t\mathcal{R}_\alpha(E_{a,\beta}g)(s,y)| \leq E^{-1}_{\frac{a\beta}{1+\beta},1+\beta}(s,y)\,{}^t\mathcal{R}_\alpha(E_{a,\beta})(s,y)||g||_{L^\infty(d\nu_\alpha)},$$

and from (2.4) we deduce that

$$|E^{-1}_{\frac{a\beta}{1+\beta},1+\beta}(s,y)\,{}^t\mathcal{R}_\alpha(E_{a,\beta}g)(s,y)| \leq C||g||_{L^\infty(d\nu_\alpha)} < \infty.$$

This completes the proof. □

Lemma 4. Let p,q in $[1,\infty]$ and f a measurable function on $\mathbb{R}^2_+$ such that

$$E^{-1}_{a,\beta} f \in L^p(d\nu_\alpha) + L^q(d\nu_\alpha), \tag{4.25}$$

for some $a > 0$, $\beta > 0$. Then the function defined on $\mathbb{C}^2$ by

$$\mathcal{F}_\alpha(f)(\mu,\lambda) = \int_{\mathbb{R}^2_+} f(r,x)\varphi_{(\mu,\lambda)}(r,x)d\nu_\alpha(r,x), \tag{4.26}$$

is well defined and entire on $\mathbb{C}^2$. Moreover there exists a positive constant C such that for all $\xi,\eta,\ \mu,\theta \in \mathbb{R}$ we have

$$|\mathcal{F}_\alpha(f)(\mu+i\theta,\xi+i\eta)| \leq Ce^{\frac{(1+\beta)\eta^2+\theta^2}{4a\beta}}. \tag{4.27}$$

Proof. The first assertion follows from the hypothesis on the function f and Hölder's inequality using (4.25) and the derivation theorem under the integral sign. We want to prove (4.27).
The condition (4.25) implies that the function f belongs to $L^1(d\nu_\alpha)$. Hence we deduce from (2.9) that for all $\xi,\eta,\ \alpha,\theta \in \mathbb{R}$, we have

$$|\mathcal{F}_\alpha(f)(\mu+i\theta,\xi+i\eta)| = |\int_{\mathbb{R}^2_+} {}^t\mathcal{R}_\alpha(f)(s,y)e^{-iy(\xi+i\eta)}\cos(s(\mu+i\theta))dsdy|.$$

$$\leq \int_{\mathbb{R}^2_+} \left|{}^t\mathcal{R}_\alpha(f)(s,y)\right| e^{\langle y,\eta\rangle} e^{|\theta|s} dsdy.$$

The integral of the second member can also be written in the form

$$c_0 E^{-1}_{\frac{1+\beta}{4a\beta},\frac{1}{1+\beta}}(\theta,\eta)\int_{\mathbb{R}^2_+} E^{-1}_{\frac{a\beta}{1+\beta},1+\beta}(s,y)\,{}^t\mathcal{R}_\alpha(|f|)(s,y)E_{\frac{a\beta}{1+\beta},1+\beta}(s-\frac{|\theta|}{2a\beta},y-\frac{1+\beta}{2a\beta}\eta)dsdy$$

where c_0 is a positive constant. On the follow we will to estimate

$$\int_{\mathbb{R}^2_+} E^{-1}_{\frac{a\beta}{1+\beta},1+\beta}(s,y)\,{}^t\mathcal{R}_\alpha(|f|)(s,y)E_{\frac{a\beta}{1+\beta},1+\beta}(s-\frac{|\theta|}{2a\beta},y-\frac{1+\beta}{2a\beta}\eta)dsdy.$$

Indeed from (4.25) there exists u in $L^p(d\nu_\alpha)$ and v in $L^q(d\nu_\alpha)$ such that

$$f = E_{a,\beta}(u+v).$$

Thus using the Lemma 3 and Hölder inequality we obtain

$$\int_{\mathbb{R}^2_+} E^{-1}_{\frac{a\beta}{1+\beta},1+\beta}(s,y)\,{}^t\mathcal{R}_\alpha(|f|)(s,y)E_{\frac{a\beta}{1+\beta},1+\beta}(s-\frac{|\theta|}{2a\beta},y-\frac{1+\beta}{2a\beta}\eta)dsdy$$
$$\leq C(||u||_{L^p(d\nu_\alpha)} + ||v||_{L^q(d\nu_\alpha)}) < \infty.$$

Hence there exists a positive constant C such that

$$|\mathcal{F}(f)(\mu+i\theta,\xi+i\eta)| \leq Ce^{\frac{(1+\beta)\eta^2+\theta^2}{4a\beta}}.$$

□

Proof. of Theorem 3.

We will divide the proof in several cases.

1 st case $ab > \frac{1}{4}$.

Consider the function h defined on $\mathbb{C}^2$ by

$$h(\gamma,\zeta) = E^{-1}_{\frac{1+\beta}{4a\beta},\frac{1}{1+\beta}}(\gamma,\zeta)\mathcal{F}_\alpha(f)(\gamma,\zeta), \tag{4.28}$$

with $\gamma = \mu + i\theta \in \mathbb{C}$ and $\zeta = \xi + i\eta \in \mathbb{C}$. This function is entire on $\mathbb{C}^2$ and using (4.27) we obtain:

$$|h(\gamma,\zeta)| \leq CE^{-1}_{\frac{1+\beta}{4a\beta},\frac{1}{1+\beta}}(\mu,\xi), \tag{4.29}$$

for all $\zeta,\gamma \in \mathbb{C}$. On the other hand we have

$$\begin{aligned}
\int_{\mathbb{R}^2_+} \log^+ |h(\mu,\xi)|d\mu d\xi &= \int_{\mathbb{R}^2_+} \log^+ |E^{-1}_{\frac{1+\beta}{4a\beta},\frac{1}{1+\beta}}(\mu,\xi)\mathcal{F}_\alpha(f)(\mu,\xi)|d\mu d\xi, \\
&= \int_{\mathbb{R}^2_+} \log^+[\frac{E^{-1}_{\frac{b(1+\beta)}{\beta},\frac{1}{1+\beta}}(\mu,\xi)|\mathcal{F}_\alpha(f)(\mu,\xi)|}{\lambda}]\lambda E_{\frac{(1+\beta)(4ab-1)}{4a\beta},\frac{1}{1+\beta}}(\mu,\xi)d\mu d\xi \\
&\leq \int_{\mathbb{R}^2_+} \log^+[\frac{E^{-1}_{\frac{b(1+\beta)}{\beta},\frac{1}{1+\beta}}(\mu,\xi)|\mathcal{F}_\alpha(f)(\mu,\xi)|}{\lambda}]d\mu d\xi + \\
&\quad \int_{\mathbb{R}^2_+} \lambda E_{\frac{(1+\beta)(4ab-1)}{4a\beta},\frac{1}{1+\beta}}(\mu,\xi)d\mu d\xi,
\end{aligned}$$

because $\log^+(cd) \leq \log^+(c) + d$ for all $c,d > 0$. Since $ab > \frac{1}{4}$, (4.23) implies that

$$\int_{\mathbb{R}^2_+} \log^+ |h(\mu,\xi)|d\mu d\xi < \infty. \tag{4.30}$$

From the relations (4.29) and (4.30), it follows from Lemma 2 that there exists a constant C such that

$$h(\mu,\zeta) = C, \ (\mu,\zeta) \in \mathbb{C}^2.$$

Thus

$$\mathcal{F}_\alpha(f) = CE_{\frac{1+\beta}{4a\beta},\frac{1}{1+\beta}}.$$

Using now the condition (4.23) and that $ab > \frac{1}{4}$, we deduce that $C = 0$ and hence we obtain

$$\forall\ (\mu,\zeta) \in \Gamma,\ \mathcal{F}_\alpha(f)(\mu,\zeta) = 0.$$

Then the injectivity of $\mathcal{F}$ implies the result of the theorem.
Second case $ab = \frac{1}{4}$.

The same proof as for the the first step give that

$$\mathcal{F}_\alpha(f) = CE_{\frac{1+\beta}{4a\beta}, \frac{1}{1+\beta}},$$

with $|C| \leq \lambda$. Thus

$$f = CE_{\frac{b}{4a\beta}, \frac{1}{1+\beta}}.$$

Third case $ab < \frac{1}{4}$

In the sequel we will construct a family of nonzero functions which satisfy the conditions (4.22),(4.23). By considering the family of functions $cE_{\delta,\beta}$, we see that

$$\mathcal{F}_\alpha(f) = cE_{\frac{1+\beta}{4\delta\beta}, \frac{1}{1+\beta}}.$$

These functions clearly satisfy the conditions (4.22),(4.23) for all $\delta \in (b, \frac{1}{4a})$. The proof of the Theorem is complete. □

The following is an immediate corollary of Theorem 3.

Corollary 2. Let f be a measurable function on $\mathbb{R}^2_+$ such that

$$E^{-1}_{a,\beta} f \in L^p(d\nu_\alpha) + L^q(d\nu_\alpha) \tag{4.31}$$

and

$$\int_{\mathbb{R}^2_+} E^{-r}_{\frac{b(1+\beta)}{\beta}, \frac{1}{1+\beta}}(\mu,\xi) |\mathcal{F}_\alpha(f)(\mu,\xi)| d\mu d\xi < \infty, \tag{4.32}$$

for some constants $a > 0$, $b > 0$, $1 \leq p, q \leq \infty$, $0 < r \leq \infty$. Then

If $ab \geq \frac{1}{4}$, we have $f = 0$ almost everywhere.

If $ab < \frac{1}{4}$, for all $\delta \in (b, \frac{1}{4a})$, the functions of the form $CE_{\delta,\beta}$ satisfy (4.31) and (4.32).

5. Beurling's theorem for the Generalized Fourier transform

Beurling's theorem and Bonami, Demange, and Jaming's extension are generalized for the generalized Fourier transform as follows.

Theorem 4. Let $N \in \mathbb{N}$, $\delta > 0$ and $f \in L^2(d\nu_\alpha)$ satisfy

$$\int_{\mathbb{R}^2_+} \int_{\mathbb{R}^2_+} \frac{|f(r,x)||\mathcal{F}_\alpha(f)(t,y)||R(t,y)|^\delta}{(1 + ||(r,x)|| + ||(t,y)||)^N} e^{||(r,x)||\,||(t,y)||} d\nu_\alpha(r,x) dt dy < \infty, \tag{5.33}$$

where R is a polynomial of degree m. If $N \geq m\delta + 4$, then

$$f(r,x) = \sum_{|l| < \frac{N-m\delta-2}{2}} a_l^s \widetilde{W}_l^s(r,x)\ a.e., \tag{5.34}$$

where $s > 0$, $a_l^s \in \mathbb{C}$ and $\widetilde{W}_l^s$ is given by (2.15). Otherwise, $f(r,x) = 0$ almost everywhere.

Proof. We start the following lemma.

Lemma 5. We suppose that $f \in L^2(d\nu_\alpha)$ satisfies (5.33). Then $f \in L^1(d\nu_\alpha)$.

Proof. We may suppose that f is not negligible. (5.33) and the Fubini theorem imply that for almost every $(t,y) \in \mathbb{R}_+^2$,

$$\frac{|\mathcal{F}_\alpha(f)(y)||R(t,y)|^\delta}{(1+||(t,y)||)^N} \int_{\mathbb{R}_+^2} \frac{|f(r,x)|}{(1+||(r,x)||)^N} e^{||(r,x)||\,||(t,y)||} d\nu_\alpha(r,x) < \infty.$$

Since f and thus, $\mathcal{F}_\alpha(f)$ are not negligible, there exist $(t_0,y_0) \in \mathbb{R}_+^2$, $(t_0,y_0) \neq (0,0)$, such that

$$\mathcal{F}_\alpha(f)(t_0,y_0)R(t_0,y_0) \neq 0.$$

Therefore,

$$\int_{\mathbb{R}_+^2} \frac{|f(r,x)|}{(1+||(r,x)||)^N} e^{||(r,x)||\,||(t_0,y_0)||} d\nu_\alpha(r,x) < \infty.$$

Since $\dfrac{e^{||(r,x)||\,||(t_0,y_0)||}}{(1+||(r,x)||)^N} \geq 1$ for large $||(r,x)||$, it follows that $\int_{\mathbb{R}_+^2} |f(r,x)| d\nu_\alpha(r,x) < \infty$. □

This lemma and Proposition 2 imply that ${}^t\mathcal{R}_\alpha(f)$ is well-defined almost everywhere on $\mathbb{R}_+^2$. By the same techniques used in [7], we can deduce that

$$\int_{\mathbb{R}_+^2} \int_{\mathbb{R}_+^2} \frac{e^{||(r,x)||\,||(t,y)||} |{}^t\mathcal{R}_\alpha(f)(r,x)||\mathcal{F}_0({}^t\mathcal{R}_\alpha)(f)(t,y)||R(t,y)|^\delta}{(1+||(r,x)||+||(t,y)||)^N} d\nu_\alpha(r,x)dtdy < \infty.$$

According to Theorem 2.3 in [25], we conclude that for all $(r,x) \in \mathbb{R}_+^2$,

$${}^t\mathcal{R}_\alpha(f)(r,x) = P(r,x)e^{-\frac{||(r,x)||^2}{4s}},$$

where $s > 0$ and P a polynomial of degree strictly lower than $\frac{N-m\delta-2}{2}$. Then by (2.9),

$$\mathcal{F}_\alpha(f)(t,y) = \mathcal{F}_0 \circ {}^t\mathcal{R}_\alpha(f)(t,y) = \mathcal{F}_0\Big(P(r,x)e^{-\frac{||(r,x)||^2}{4s}}\Big)(t,y) = Q(t,y)e^{-s||(t,y)||^2},$$

where Q is a polynomial of degree $\deg P$. Then by using (2.15), we can find constants a_l^s such that

$$\mathcal{F}_\alpha(f)(t,y) = \mathcal{F}_\alpha\Big(\sum_{|l|<\frac{N-m\delta-2}{2}} a_l^s \widetilde{W}_l^s\Big)(t,y).$$

By the injectivity of $\mathcal{F}_\alpha$ the desired result follows. □

As an application of Theorem 4, by using the same techniques in [19], we can deduce the following Gelfand-Shilov type theorem for the generalized Fourier transform.

Corollary 3. Let $N,m \in \mathbb{N}$, $\delta > 0$, $a,b > 0$ with $ab \geq \frac{1}{4}$, and $1 < p,q < \infty$ with $\frac{1}{p}+\frac{1}{q} = 1$. Let $f \in L^2(d\nu_\alpha)$ satisfy

$$\int_{\mathbb{R}_+^2} \frac{|f(r,x)|e^{\frac{(2a)^p}{p}||(r,x)||^p}}{(1+\|(r,x)\|)^N} d\nu_\alpha(r,x) < \infty \tag{5.35}$$

and

$$\int_{\mathbb{R}_+^2} \frac{|\mathcal{F}_\alpha(f)(t,y)|e^{\frac{(2b)^q}{q}||(t,y)||^q}|R(t,y)|^\delta}{(1+||(t,y)||)^N} dtdy < \infty \tag{5.36}$$

for some $R \in \mathcal{P}_m$.

i) If $ab > \frac{1}{4}$ or $(p,q) \neq (2,2)$, then $f(r,x) = 0$ almost everywhere.

ii) If $ab = \frac{1}{4}$ and $(p,q) = (2,2)$, then f is of the form (5.34) whenever $N \geq \frac{m\delta}{2}+2$ and $r = 2b^2$. Otherwise, $f(x) = 0$ almost everywhere.

Proof. Since

$$4ab\|(r,x)\|\|(t,y)\| \leq \frac{(2a)^p}{p}\|(r,x)\|^p + \frac{(2b)^q}{q}\|(t,y)\|^q,$$

it follows from (5.35) and (5.36) that

$$\int_{\mathbb{R}_+^2}\int_{\mathbb{R}_+^2} \frac{|f(r,x)||\mathcal{F}_\alpha(f)(t,y)||R(t,y)|^\delta}{(1+||(r,x)||+||(t,y)||)^{2N}} e^{4ab||(r,x)||\,||(t,y)||} d\nu_\alpha(r,x)dtdy < \infty.$$

Then (5.33) is satisfied, because $4ab \geq 1$. Therefore, according to the proof of Theorem 4, we can deduce that

$$\int_{\mathbb{R}_+^2}\int_{\mathbb{R}_+^2} \frac{e^{4ab||(r,x)||\,||(t,y)||}|{}^t\mathcal{R}_\alpha(f)(r,x)||\mathcal{F}_0({}^t\mathcal{R}_\alpha)(f)(t,y)||R(t,y)|^\delta}{(1+||(r,x)||+||(t,y)||)^{2N}} d\nu_\alpha(r,x)dtdy < \infty,$$

and ${}^t\mathcal{R}_\alpha(f)$ and f are of the forms

$$ {}^t\mathcal{R}_\alpha(f)(r,x) = P(r,x)e^{-\frac{||(r,x)||^2}{4s}} \text{ and } \mathcal{F}_\alpha(f)(t,y) = Q(t,y)e^{-s||(t,y)||^2}, $$

where $s > 0$ and P,Q are polynomials of the same degree strictly lower than $\frac{2N-m\delta-2}{2}$. Therefore, substituting these from, we can deduce that

$$ \int_{\mathbb{R}^2_+}\int_{\mathbb{R}^2_+} \frac{e^{-(\sqrt{s}\|(t,y)\|-\frac{1}{2\sqrt{s}}\|(r,x)\|)^2} e^{(4ab-1)||(r,x)||\,||(t,y)||}|P(r,x)||Q(r,x)||R(t,y)|^\delta}{(1+||(r,x)||+||(t,y)||)^{2N}} d\nu_\alpha(r,x)dtdy < \infty. $$

When $4ab > 1$, this integral is not finite unless $f = 0$ almost everywhere. Moreover, it follows from (5.35) and (5.36) that

$$ \int_{\mathbb{R}^2_+} \frac{|P(r,x)|e^{-\frac{1}{4s}\|(r,x)\|^2} e^{\frac{(2a)^p}{p}||(r,x)||^p}}{(1+\|(r,x)\|)^N} d\nu_\alpha(r,x) < \infty $$

and

$$ \int_{\mathbb{R}^2_+} \frac{|Q(t,y)|e^{-s\|(t,y)\|^2} e^{\frac{(2b)^q}{q}||(t,y)||^q}|R(t,y)|^\delta}{(1+||(t,y)||)^N} dtdy < \infty. $$

Hence, one of these integrals is not finite unless $(p,q) = (2,2)$. When $4ab = 1$ and $(p,q) = (2,2)$, the finiteness of above integrals implies that $r = 2b^2$ and the rest follows from Theorem 4. □

6. Quantitative Uncertainty Principle For the generalized Fourier transform

We shall investigate the case where f and $\mathcal{F}_\alpha(f)$ are close to zero outside measurable sets. Here the notion of "close to zero" is formulated as follows. If $f \in L^p(d\nu_\alpha)$, $1 \le p \le 2$, is ε-concentrated on a measurable set $E \subset \mathbb{R}^2_+$ if there is a measurable function g vanishing outside E such that $||f-g||_{L^p(d\nu_\alpha)} \le \varepsilon\|f\|_{L^p(d\nu_\alpha)}$. Therefore, if we introduce a projection operator P_E as

$$ P_E f(r,x) = \begin{cases} f(r,x) & \text{if } (r,x) \in E \\ 0 & \text{if } (r,x) \notin E, \end{cases} $$

then f is ε-concentrated on E if and only if $||f - P_E f||_{L^p(d\nu_\alpha)} \le \varepsilon\|f\|_{L^p(d\nu_\alpha)}$.

We define a projection operator Q_W as

$$ Q_W f(r,x) = \mathcal{F}_\alpha^{-1}\Big(P_W(\mathcal{F}_\alpha(f))\Big)(r,x). $$

Similarly, we say that $\mathcal{F}_\alpha(f)$ is ε_W-concentrated to W in $L^{p'}(d\gamma_\alpha)$ if and only if

$$\|\mathcal{F}_\alpha(f)-\mathcal{F}_\alpha(Q_W f)\|_{L^{p'}(d\gamma_\alpha)} \leq \varepsilon_W \|\mathcal{F}_\alpha(f)\|_{L^{p'}(d\gamma_\alpha)}. \tag{6.37}$$

We note that, for measurable set $E \subset \mathbb{R}^2_+$ and $W \subset \Gamma$,

$$Q_W P_E f(r,x) = \int_{\mathbb{R}^2_+} q(t,y;r,x) f(t,y) d\nu_\alpha(t,y),$$

where

$$q(t,y;r,x) = \begin{cases} \int_W \varphi_{\mu,\lambda}(t,y)\overline{\varphi_{\mu,\lambda}(r,x)} d\gamma_\alpha(\mu,\lambda) & \text{if } (t,y) \in E \\ 0 & \text{if } (t,y) \notin E. \end{cases}$$

Indeed, by the Fubini's theorem we see that

$$\begin{aligned} Q_W P_E f(r,x) &= \int_W \mathcal{F}_\alpha(P_E f)(\mu,\lambda)\overline{\varphi_{\mu,\lambda}(r,x)} d\gamma_\alpha(\mu,\lambda) \\ &= \int_W \Big(\int_E f(t,y)\varphi_{\mu,\lambda}(t,y) d\nu_\alpha(t,y)\Big)\overline{\varphi_{\mu,\lambda}(r,x)} d\gamma_\alpha(\mu,\lambda) \\ &= \int_E f(t,y)\Big(\int_W \varphi_{\mu,\lambda}(t,y)\overline{\varphi_{\mu,\lambda}(r,x)} d\gamma_\alpha(\mu,\lambda)\Big) d\nu_\alpha(t,y). \end{aligned}$$

The Hilbert-Schmidt norm $\|Q_W P_E\|_{HS}$ is given by

$$||Q_W P_E||_{HS} = \Big(\int_{\mathbb{R}^2_+}\int_{\mathbb{R}^2_+} |q(t,y;r,x)|^2 d\nu_\alpha(t,y) d\nu_\alpha(r,x)\Big)^{\frac{1}{2}}.$$

We denote by $\|T\|_2$ the operator norm on $L^2(d\nu_\alpha)$. Since P_E and Q_W are projections, it is clear that $\|P_E\|_2 = \|Q_W\|_2 = 1$. Moreover, it follows that

$$||Q_W P_E||_{HS} \geq ||Q_W P_E||_2. \tag{6.38}$$

Lemma 6. If E and W are sets of finite measure, then

$$||Q_W P_E||_{HS} \leq \sqrt{mes_{\nu_\alpha}(E) mes_{\gamma_\alpha}(W)},$$

where

$$mes_{\nu_\alpha}(E) := \int_E d\nu_\alpha(r,x), \quad mes_{\gamma_\alpha}(W) := \int_W d\gamma_\alpha(\mu,\lambda).$$

Proof. For $(t,y) \in E$, let $g_{t,y}(r,x) = q(t,y;r,x)$. (2.11) implies that

$$\mathcal{F}_\alpha(g_{t,y})(\mu,\lambda) = P_W(\varphi_{\mu,\lambda}(t,y)).$$

Then by Parseval's identity (2.12) and (2.1) it follows that

$$\int_{\mathbb{R}_+^2} |q(t,y;r,x)|^2 d\nu_\alpha(r,x) = \int_{\mathbb{R}_+^2} |g_{t,y}(r,x)|^2 d\nu_\alpha(r,x)$$

$$= \int_\Gamma |\mathcal{F}_\alpha(g_{t,y})(\mu,\lambda)|^2 d\gamma_\alpha(\mu,\lambda) \le mes_{\gamma_\alpha}(W)$$

Hence, integrating over $(t,y) \in E$, we see that $||Q_W P_E||_{HS}^2 \le mes_{\nu_\alpha}(E) mes_{\gamma_\alpha}(W)$.

□

Proposition 6. Let E and W be measurable sets and suppose that

$$\|f\|_{L^2(d\nu_\alpha)} = \|\mathcal{F}_\alpha(f)\|_{L^2(d\gamma_\alpha)} = 1.$$

Assume that $\varepsilon_E + \varepsilon_W < 1$, f is ε_E-concentrated on E and $\mathcal{F}_\alpha(f)$ is ε_W concentrated on W. Then

$$mes_{\nu_\alpha}(E) mes_{\gamma_\alpha}(W) \ge (1 - \varepsilon_E - \varepsilon_W)^2.$$

Proof. Since $||f||_{L^2(d\nu_\alpha)} = \|\mathcal{F}_\alpha(f)\|_{L^2(d\nu_\alpha)} = 1$ and $\varepsilon_E + \varepsilon_W < 1$, the measures of E and W must both be non-zero. Indeed, if not, then the ε_E-concentration of f implies that

$$||f - P_E f||_{L^2(d\nu_\alpha)} = ||f||_{L^2(d\nu_\alpha)} = 1 \le \varepsilon_E,$$

which contradicts with $\varepsilon_E < 1$, likewise for $\mathcal{F}_\alpha(f)$. If at least one of $mes_{\nu_\alpha}(E)$ and $mes_{\gamma_\alpha}(W)$ is infinity, then the inequality is clear. Therefore, it is enough to consider the case where both E and W have finite positive measures. Since $||Q_W||_2 = 1$, it follows that

$$\begin{aligned} ||f - Q_W P_E f||_{L^2(d\nu_\alpha)} &\le ||f - Q_W f||_{L^2(d\nu_\alpha)} + ||Q_W f - Q_W P_E f||_{L^2(d\nu_\alpha)} \\ &\le \varepsilon_W + ||Q_W||_2 ||f - P_E f||_{L^2(d\nu_\alpha)} \\ &\le \varepsilon_E + \varepsilon_W \end{aligned}$$

and thus,

$$||Q_W P_E f||_{L^2(d\nu_\alpha)} \ge ||f||_{L^2(d\nu_\alpha)} - ||f - Q_W P_E f||_{L^2(d\nu_\alpha)} \ge 1 - \varepsilon_E - \varepsilon_W.$$

Hence $||Q_W P_E||_2 \ge 1 - \varepsilon_E - \varepsilon_W$. (6.38) and Lemma 6 yields the desired inequality.

□

Let $B_{L^p(d\nu_\alpha)}(T)$, $1 \le p \le 2$, the subspace of all $g \in L^p(d\nu_\alpha)$ such that $Q_T g = g$. We say that f is ε-bandlimited to T if there is a $g \in B_{L^p(d\nu_\alpha)}(T)$ with $||f - g||_{L^p(d\nu_\alpha)} < \varepsilon \|f||_{L^p(d\nu_\alpha)}$. Here we denote by $\|P_E\|_p$ the operator norm of P_E on $L^p(d\nu_\alpha)$ and by $\|P_E\|_{p,T}$ the operator norm of $P_E : B_{L^p(d\nu_\alpha)}(T) \to L^p(d\nu_\alpha)$. Corresponding to (6.38) and Lemma 6 in the $L^2(d\nu_\alpha)$ case, we can obtain the following.

Lemma 7. Let E and T be measurable sets of $\mathbb{R}^2_+$. For $p \in [1,2]$, we have

$$\|P_E\|_{p,T} \leq \Big(mes_{\nu_\alpha}(E) mes_{\gamma_\alpha}(T)\Big)^{\frac{1}{p}}.$$

Proof. For $f \in B_{L^p(d\nu_\alpha)}(T)$ we see that

$$f(t,y) = \int_T \overline{\varphi_{\mu,\lambda}(t,y)} \mathcal{F}_\alpha(f)(\mu,\lambda) d\gamma_\alpha(\mu,\lambda).$$

By (2.1), Hölder's inequality and Proposition 4

$$\begin{aligned} |f(r,x)| &\leq \Big(mes_{\gamma_\alpha}(T)\Big)^{\frac{1}{p}} \|\mathcal{F}_\alpha(f)\|_{L^{p'}(d\gamma_\alpha)} \\ &\leq \Big(mes_{\gamma_\alpha}(T)\Big)^{\frac{1}{p}} \|f\|_{L^p(d\nu_\alpha)}. \end{aligned}$$

Therefore

$$||P_E f||_{L^p(d\nu_\alpha)} = \Big(\int_E |f(r,x)|^p d\nu_\alpha(r,x)\Big)^{\frac{1}{p}} \leq \Big(mes_{\nu_\alpha}(E) mes_{\gamma_\alpha}(T)\Big)^{\frac{1}{p}} ||f||_{L^p(d\nu_\alpha)}.$$

Then, it follows that for $f \in B_{L^p(d\nu_\alpha)}(W)$,

$$\frac{||P_E f||_{L^p(d\nu_\alpha)}}{||f||_{L^p(d\nu_\alpha)}} \leq \Big(mes_{\nu_\alpha}(E) mes_{\gamma_\alpha}(T)\Big)^{\frac{1}{p}},$$

which implies the desired inequality. □

Proposition 7. Let $f \in L^p(d\nu_\alpha)$. If f is ε_E-concentrated to E and ε_T bandlimited to W, then

$$\Big(mes_{\nu_\alpha}(E) mes_{\gamma_\alpha}(T)\Big)^{\frac{1}{p}} \geq \frac{1-\varepsilon_E-\varepsilon_T}{1+\varepsilon_T}.$$

Proof. Without loss of generality, we may suppose that $\|f\|_{L^p(d\nu_\alpha)} = 1$. Since f is ε_E-concentrated to E, it follows that $\|P_E f\|_{L^p(d\nu_\alpha)} \geq \|f\|_{L^p(d\nu_\alpha)} - \|f - P_E f\|_{L^p(d\nu_\alpha)} \geq 1 - \varepsilon_E$. Moreover, since f is ε_T-bandlimited, there is a $g \in B_{L^p(d\nu_\alpha)}(W)$ with $||g-f||_{L^p(d\nu_\alpha)} \leq \varepsilon_T$. Therefore, it follows that

$$||P_E g||_{L^p(d\nu_\alpha)} \geq ||P_E f||_{L^p(d\nu_\alpha)} - ||P_E(g-f)||_{L^p(d\nu_\alpha)} \geq ||P_E f||_{L^p(d\nu_\alpha)} - \varepsilon_T \geq 1-\varepsilon_E-\varepsilon_T$$

and $||g||_{L^p(d\nu_\alpha)} \leq ||f||_{L^p(d\nu_\alpha)} + \varepsilon_T = 1+\varepsilon_T$. Then, we see that

$$\frac{||P_E g||_{L^p(d\nu_\alpha)}}{||g||_{L^p(d\nu_\alpha)}} \geq \frac{1-\varepsilon_E-\varepsilon_T}{1+\varepsilon_T}.$$

Hence $\|P_E\|_{p,W} \geq \frac{1-\varepsilon_E-\varepsilon_T}{1+\varepsilon_T}$ and Lemma 7 yields the desired inequality. □

Proposition 8. Let $f \in L^1(d\nu_\alpha) \cap L^2(d\nu_\alpha)$ with $\|f\|_{L^2(d\nu_\alpha)} = 1$. If f is ε_E-concentrated to E in $L^1(d\nu_\alpha)$-norm and $\mathcal{F}_\alpha(f)$ is ε_T-concentrated to T in $L^2(d\gamma_\alpha)$-norm, then

$$mes_{\nu_\alpha}(E) \geq (1-\varepsilon_E)^2 \|f\|^2_{L^1(d\nu_\alpha)} \quad \text{and} \quad mes_{\gamma_\alpha}(T)\|f\|^2_{L^1(d\nu_\alpha)} \geq (1-\varepsilon_T^2).$$

In particular,

$$mes_{\nu_\alpha}(E) mes_{\gamma_\alpha}(T) \geq (1-\varepsilon_E)^2(1-\varepsilon_T^2).$$

Proof. By the orthogonality of the projection operator P_T, $\|f\|_{L^2(d\nu_\alpha)} = \|\mathcal{F}_\alpha(f)\|_{L^2(d\gamma_\alpha)} = 1$ and f is ε_T-concentrated to W in $L^2_{\gamma_\alpha}$-norm, it follows that

$$\|P_T(\mathcal{F}_\alpha(f))\|^2_{L^2(d\gamma_\alpha)} = \|\mathcal{F}_\alpha(f)\|^2_{L^2(d\gamma_\alpha)} - \|\mathcal{F}_\alpha(f) - P_T(\mathcal{F}_\alpha(f))\|^2_{L^2(d\gamma_\alpha)} \geq 1-\varepsilon_T^2,$$

and thus,

$$\begin{aligned} 1-\varepsilon_T^2 &\leq \int_T |\mathcal{F}_\alpha(f)(\xi)|^2 d\gamma_\alpha(\mu,\lambda) \\ &\leq mes_{\gamma_\alpha}(T)||\mathcal{F}_\alpha(f)||^2_{L^\infty(d\gamma_\alpha)} \leq mes_{\gamma_\alpha}(T)||f||^2_{L^1(d\nu_\alpha)}. \end{aligned}$$

Similarly, f is ε_E-concentrated to E in $L^1(d\nu_\alpha)$-norm,

$$(1-\varepsilon_E)\|f\|_{L^1(d\nu_\alpha)} \leq \int_E |f(x)| d\nu_\alpha)(x) \leq \sqrt{mes_{\nu_\alpha}(E)}$$

Here we used the Cauchy-Schwarz inequality and the fact that $\|f\|_{L^2(d\nu_\alpha)} = 1$. □

Proposition 9. Let E and T be measurable subsets of $\mathbb{R}^2_+$, and $f \in L^p(d\nu_\alpha)$ for $p \in (1,2]$. If f is ε_E-concentrated to E in $L^p(d\nu_\alpha)$-norm and $\mathcal{F}_\alpha(f)$ is ε_T-concentrated to T in $L^{p'}(d\gamma_\alpha)$-norm, then

$$(mes_{\nu_\alpha}(E) mes_{\gamma_\alpha}(T))^{\frac{1}{p'}} \geq \frac{(1-\varepsilon_E)\|\mathcal{F}_\alpha(f)\|_{L^{p'}(d\gamma_\alpha)} - \varepsilon_T ||f||_{L^p(d\nu_\alpha)}}{||f||_{L^p(d\nu_\alpha)}}.$$

Proof. Let $f \in L^p(d\nu_\alpha)$ for $p \in (1,2]$. As above

$$\begin{aligned} ||\mathcal{F}_\alpha(f) - \mathcal{F}_\alpha(Q_T P_E f)||_{L^{p'}(d\nu_\alpha)} &\leq ||\mathcal{F}_\alpha(f) - \mathcal{F}_\alpha(Q_T f)||_{L^{p'}(d\nu_\alpha)} \\ &\quad + ||\mathcal{F}_\alpha(Q_T f) - \mathcal{F}_\alpha(Q_T P_E f)||_{L^{p'}(d\nu_\alpha)} \\ &\leq \varepsilon_T ||\mathcal{F}_\alpha(f)||_{L^{p'}(d\nu_\alpha)} + ||f - P_E f||_{L^p(d\nu_\alpha)} \\ &\leq \varepsilon_T ||\mathcal{F}_\alpha(f)||_{L^{p'}(d\nu_\alpha)} + \varepsilon_E ||f||_{L^p(d\nu_\alpha)} \end{aligned}$$

and thus,

$$\begin{aligned} ||\mathcal{F}_\alpha(Q_T P_E f)||_{L^{p'}(d\nu_\alpha)} &\geq ||\mathcal{F}_\alpha(f)||_{L^{p'}(d\nu_\alpha)} - ||\mathcal{F}_\alpha(f) - \mathcal{F}_\alpha(Q_T P_E f)||_{L^{p'}(d\nu_\alpha)} \\ &\geq (1-\varepsilon_T)||\mathcal{F}_\alpha(f)||_{L^{p'}(d\nu_\alpha)} - \varepsilon_E ||f||_{L^p(d\nu_\alpha)}. \end{aligned}$$

On the other hand, it is easy to obtain

$$\frac{||\mathcal{F}_\alpha(Q_T P_E f)||_{L^{p'}(d\nu_\alpha)}}{||f||_{L^p(d\nu_\alpha)}} \leq \Big(mes_{\nu_\alpha}(E) mes_{\gamma_\alpha}(T)\Big)^{\frac{1}{p'}}.$$

Hence

$$(mes_{\nu_\alpha}(E) mes_{\gamma_\alpha}(T))^{\frac{1}{p'}} ||f||_{L^p(d\nu_\alpha)} \geq (1-\varepsilon_E)\|\mathcal{F}_\alpha(f)\|_{L^{p'}(d\gamma_\alpha)} - \varepsilon_T ||f||_{L^p(d\nu_\alpha)},$$

which gives the desired result. □

Proposition 10. Let $f \in L^1(d\nu_\alpha) \cap L^p(d\nu_\alpha)$, $p \in (1,2]$. If f is ε_E-concentrated to E in $L^1(d\nu_\alpha)$-norm and $\mathcal{F}_\alpha(f)$ is ε_T-concentrated to T in $L^{p'}(d\gamma_\alpha)$-norm, then

$$(mes_{\nu_\alpha}(E) mes_{\gamma_\alpha}(T))^{\frac{1}{p'}} \geq (1-\varepsilon_E)(1-\varepsilon_T) \frac{\|\mathcal{F}_\alpha(f)\|_{L^{p'}(d\gamma_\alpha)}}{\|f\|_{L^p(d\nu_\alpha)}}.$$

Proof. Let $f \in L^1(d\nu_\alpha) \cap L^p(d\nu_\alpha)$, $p \in (1,2]$. As $\mathcal{F}_\alpha(f)$ is ε_T-concentrated to T in $L^{p'}_{\gamma_\alpha}$-norm, it follows that

$$\begin{aligned} \|\mathcal{F}_\alpha(f)\|_{L^{p'}(d\gamma_\alpha)} &\leq \varepsilon_T \|\mathcal{F}_\alpha(f)\|_{L^{p'}(d\gamma_\alpha)} + \Big(\int_T |\mathcal{F}_\alpha(f)(\lambda,\mu)|^{p'} d\gamma_\alpha)(\lambda,\mu)\Big)^{\frac{1}{p'}} \\ &\leq \varepsilon_T \|\mathcal{F}_\alpha(f)\|_{L^{p'}(d\gamma_\alpha)} + (mes_{\gamma_\alpha}(T))^{\frac{1}{p'}} \|\mathcal{F}_\alpha(f)\|_{L^\infty(d\gamma_\alpha)}. \end{aligned}$$

Thus from Proposition (2.9),

$$(1-\varepsilon_T)\|\mathcal{F}_\alpha(f)\|_{L^{p'}(d\gamma_\alpha)} \leq (mes_{\gamma_\alpha}(T))^{\frac{1}{p'}} \|f\|_{L^1(d\nu_\alpha)}. \tag{6.39}$$

Similarly, using f is ε_E-concentrated to E in $L^1(d\nu_\alpha)$-norm, and Hölder inequality, we obtain

$$(1-\varepsilon_E)\|f\|_{L^1(d\nu_\alpha)} \leq (mes_{\gamma_\alpha}(E))^{\frac{1}{p'}} \|f\|_{L^p(d\nu_\alpha)}. \tag{6.40}$$

Combining (6.39) and (6.40), we obtain the result. □

Proposition 11. Let $s > 0$. Then there exists a constant $C_1(\alpha,s)$ such that for all
$f \in L^1(d\nu_\alpha) \bigcap L^2(d\nu_\alpha)$

$$||f||^{2+\frac{4s}{2\alpha+3}}_{L^2(d\nu_\alpha)} \leq C_1(\alpha,s)||f||^{\frac{4s}{2\alpha+3}}_{L^1(d\nu_\alpha)} || \, ||\theta(\lambda,\mu)||^s \mathcal{F}(f)||^2_{L^2(d\gamma_\alpha)}. \tag{6.41}$$

Proof. Let $A>0$. From Plancherel's theorem we have

$$\begin{aligned}\|f\|^2_{L^2(d\nu_\alpha)} &= \|\mathcal{F}_\alpha(f)\|^2_{L^2(d\gamma_\alpha)}\\ &= ||1_{\theta^{-1}(B_+(0,A))}\mathcal{F}_\alpha(f)||^2_{L^2(d\gamma_\alpha)} + ||(1-1_{\theta^{-1}(B_+(0,A))})\mathcal{F}_\alpha(f)||^2_{L^2(d\gamma_\alpha)}\end{aligned}$$

By (2.2) and (2.10)

$$||1_{\theta^{-1}(B_+(0,A))}\mathcal{F}_\alpha(f)||^2_{L^2(d\gamma_\alpha)} \leq \|f\|^2_{L^1(d\nu_\alpha)}\int_{\mathbb{R}^2_+} 1_{B_+(0,A)}(r,x)d\nu_\alpha(r,x).$$

By a simple calculations we find

$$||1_{\theta^{-1}(B_+(0,A))}\mathcal{F}_\alpha(f)||^2_{L^2(d\gamma_\alpha)} \leq \frac{A^{2\alpha+3}}{2^{\alpha+\frac{3}{2}}\Gamma(\alpha+\frac{5}{2})}\|f\|^2_{L^1(d\nu_\alpha)}.$$

On the other hand

$$\begin{aligned}||(1-1_{\theta^{-1}(B_+(0,A))})\mathcal{F}_\alpha(f)||^2_{L^2(d\gamma_\alpha)} &\leq A^{-2s}||(1-1_{\theta^{-1}(B_+(0,A))})\,||\theta(\lambda,\mu)||^s\mathcal{F}_\alpha(f)||^2_{L^2(d\gamma_\alpha)}\\ &\leq A^{-2s}||\,||\theta(\lambda,\mu)||^s\mathcal{F}_\alpha(f)||^2_{L^2(d\gamma_\alpha)}.\end{aligned}$$

It follows then

$$\|f\|^2_{L^2(d\nu_\alpha)} \leq \frac{A^{2\alpha+3}}{2^{\alpha+\frac{3}{2}}\Gamma(\alpha+\frac{5}{2})}\|f\|^2_{L^1(d\nu_\alpha)} + A^{-2s}||\,||\theta(\lambda,\mu)||^s\mathcal{F}_\alpha(f)||^2_{L^2(d\gamma_\alpha)}.$$

Minimizing the right hand side of that inequality over $A>0$ gives

$$||f||^2_{L^2(d\nu_\alpha)} \leq C(\alpha,s)||f||^{\frac{4s}{2\alpha+3+2s}}_{L^1(d\nu_\alpha)}||\,||\theta(\lambda,\mu)||^s\mathcal{F}(f)||^{\frac{2(2\alpha+3)}{2s+2\alpha+3}}_{L^2(d\gamma_\alpha)}. \tag{6.42}$$

The desired result follows immediately from (6.42). □

Proposition 12. Let $s>0$. Then there exists a constant $C_2(\alpha,s)$ such that for all
$f\in L^1(d\nu_\alpha)\bigcap L^2(d\nu_\alpha)$

$$||f||^{1+\frac{4s}{2\alpha+3}}_{L^1(d\nu_\alpha)} \leq C_2(\alpha,s)||f||^{\frac{4s}{2\alpha+3}}_{L^2(d\nu_\alpha)}||\,||(r,x)||^{2s}f||_{L^1(d\nu_\alpha)}. \tag{6.43}$$

Proof. Let $A>0$. We have

$$\|f\|_{L^1(d\nu_\alpha)} \leq ||1_{B_+(0,A)}f||_{L^1(d\nu_\alpha)} + ||(1-1_{B_+(0,A)})f||_{L^1(d\nu_\alpha)}.$$

By Cauchy-Schwarz inequality we obtain

$$||1_{B_+(0,A)}f||_{L^1(d\nu_\alpha)} \leq \Big(\frac{A^{2\alpha+3}}{2^{\alpha+\frac{3}{2}}\Gamma(\alpha+\frac{5}{2})}\Big)^{\frac{1}{2}}\|f\|_{L^2(d\nu_\alpha)}.$$

On the other hand

$$||(1-1_{B_+(0,A)})f||_{L^1(d\nu_\alpha)} \leq A^{-2s}||\,||(r,x)||^{2s}(1-1_{B_+(0,A)})f||_{L^1(d\nu_\alpha)}.$$

It follows then

$$\|f\|_{L^1(d\nu_\alpha)} \leq \Big(\frac{A^{2\alpha+3}}{2^{\alpha+\frac{5}{2}}\Gamma(\alpha+\frac{5}{2})}\Big)^{\frac{1}{2}}\|f\|_{L^2(d\nu_\alpha)} + A^{-2s}||\,||(r,x)||^{2s}f||_{L^1(d\nu_\alpha)}.$$

Minimizing the right hand side of that inequality over $A>0$ gives

$$||f||_{L^1(d\nu_\alpha)} \leq C(\alpha,s)||f||_{L^2(d\nu_\alpha)}^{\frac{4s}{2\alpha+3+4s}}||\,||(r,x)||^{2s}f||_{L^1(d\nu_\alpha)}^{\frac{2\alpha+3}{4s+2\alpha+3}}. \tag{6.44}$$

The desired result follows immediately from (6.44). □

From the previous results we deduce the following variation on Heisenberg's uncertainty inequality for the generalized Fourier transform.

Theorem 5. Let $s>0$. Then for all $f\in L^1(d\nu_\alpha)\bigcap L^2(d\nu_\alpha)$

$$||f||^2_{L^2(d\nu_\alpha)}||f||_{L^1(d\nu_\alpha)} \leq C_1(\alpha,s)C_2(\alpha,s)||\,||(r,x)||^{2s}f||_{L^1(d\nu_\alpha)}||\,||\theta(\lambda,\mu)||^{s}\mathcal{F}_\alpha(f)||^2_{L^2(d\gamma_\alpha)} \tag{6.45}$$

Proof. The result follows immediately by multiplying inequality (6.41) by (6.43) □

Proposition 13. Let $s>0$ and let W a measurable subset of Γ with $0<mes_{\gamma_\alpha}(W)<\infty$. Then there exists a constant $C(\alpha,s)$ such that for all $f\in L^1(d\nu_\alpha)\bigcap L^2(d\nu_\alpha)$

$$||1_W\mathcal{F}_\alpha(f)||_{L^2(d\gamma_\alpha)} \leq C(\alpha,s)\sqrt{mes_{\gamma_\alpha}(W)}||f||_{L^2(d\nu_\alpha)}^{\frac{4s}{4s+2\alpha+3}}||\,||(r,x)||^{2s}f||_{L^1(d\nu_\alpha)}^{\frac{2\alpha+3}{4s+2\alpha+3}}. \tag{6.46}$$

Proof. We have

$$||1_W\mathcal{F}_\alpha(f)||_{L^2(d\gamma_\alpha)} \leq \sqrt{mes_{\gamma_\alpha}(W)}||\mathcal{F}_\alpha(f)||_{L^\infty(d\gamma_\alpha)} \leq \sqrt{mes_{\gamma_\alpha}(W)}||f||_{L^1(d\nu_\alpha)}.$$

The desired result follows from Carlson Inequality (6.44). □

We adapt the method of Ghorbal-Jaming [13], we obtain.

Theorem 6. Let E,W be a pair of measurable subsets such that

$$0<mes_{\nu_\alpha}(E), mes_{\gamma_\alpha}(W)<\infty.$$

Then the following uncertainty principles hold.

1) Local uncertainty principle of $\mathcal{F}_\alpha$

(i) For $0 < s < \frac{2\alpha+3}{2}$, there exists a constant $C(\alpha,s)$ such that for all $f \in L^2(d\nu_\alpha)$

$$||1_W \mathcal{F}_\alpha(f)||_{L^2(d\gamma_\alpha)} \leq C(\alpha,s)(mes_{\gamma_\alpha}(W))^{\frac{s}{2\alpha+3}} || \, ||(r,x)||^s f||_{L^2(d\nu_\alpha)}. \quad (6.47)$$

(ii) For $s > \frac{2\alpha+3}{2}$, there exists a constant $C(\alpha,s)$ such that for all $f \in L^2(d\nu_\alpha)$

$$||1_W \mathcal{F}_\alpha(f)||_{L^2(d\gamma_\alpha)} \leq C(\alpha,s)\sqrt{mes_{\gamma_\alpha}(W)} || \, ||(r,x)||^s f||_{L^2(d\nu_\alpha)}^{\frac{2\alpha+3}{2s}} ||f||_{L^2(d\nu_\alpha)}^{1-\frac{2\alpha+3}{2s}}. \quad (6.48)$$

2) Global uncertainty principle of $\mathcal{F}_\alpha$

For $s,t > 0$, there exists a constant $C(\alpha,s)$ such that for all $f \in L^2(d\nu_\alpha)$

$$|| \, ||(r,x)||^s f||_{L^2(d\nu_\alpha)}^{\frac{2t}{s+t}} || \, ||\theta(\lambda,\mu)||^t \mathcal{F}_\alpha(f)||_{L^2(d\gamma_\alpha)}^{\frac{2s}{s+t}} \geq C(\alpha,s)||f||_{L^2(d\nu_\alpha)}^2. \quad (6.49)$$

We put

$$h_t(\lambda,\mu) := e^{-t||\theta(\lambda,\mu)||^2}, \quad \text{for all } \lambda,\mu \in \mathbb{R}.$$

Lemma 8. Let $1 \leq q < \infty$. We have

$$||h_t||_{L^q(d\gamma_\alpha)} \leq Ct^{-\frac{2\alpha+3}{2q}}.$$

Proof. Let $1 \leq q < \infty$. Using the relation (2.2), we obtain the result. □

Lemma 9. Let $1 < p \leq 2$ and $0 < a < \frac{2\alpha+3}{p'}$. Then for all $f \in L^p(d\nu_\alpha)$ and $t > 0$,

$$||e^{-t||\theta(\lambda,\mu)||^2} \mathcal{F}_\alpha(f)||_{L^{p'}(d\gamma_\alpha)} \leq Ct^{-\frac{a}{2}} || \, ||(r,x)||^a f||_{L^p(d\nu_\alpha)}. \quad (6.50)$$

Proof. Inequality (6.50) holds if $|| \, ||(r,x)||^a f||_{L^p(d\nu_\alpha)} = \infty$.

Assume that $|| \, ||(r,x)||^a f||_{L^p(d\nu_\alpha)} < \infty$. For $s > 0$ let $f_s = f\chi_{B(0,s)}$ and $f^s = f - f_s$. Then since, $|f^s(r,x)| \leq s^{-a}| \, ||(r,x)||^a f(r,x)|$,

$$\begin{aligned} ||e^{-t||\theta(\lambda,\mu)||^2} \mathcal{F}_\alpha(f\chi_{B(0,s)^c})||_{L^{p'}(d\gamma_\alpha)} &\leq ||e^{-t||\theta(\lambda,\mu)||^2}||_{L^\infty(d\gamma_\alpha)} ||\mathcal{F}_\alpha(f\chi_{B(0,s)^c})||_{L^{p'}(d\gamma_\alpha)} \\ &\leq ||f\chi_{B(0,s)^c}||_{L^p(d\nu_\alpha)} \\ &\leq s^{-a}|| \, ||(r,x)||^a f||_{L^p(d\nu_\alpha)}. \end{aligned}$$

By Proposition 4 and Hölder's inequality

$$\begin{aligned} ||e^{-t||\theta(\lambda,\mu)||^2} \mathcal{F}_\alpha(f\chi_{B(0,s)})||_{L^{p'}(d\gamma_\alpha)} &\leq ||e^{-t||\theta(\lambda,\mu)||^2}||_{L^{p'}(d\gamma_\alpha)} ||\mathcal{F}_\alpha(f\chi_{B(0,s)})||_{L^\infty(d\gamma_\alpha)} \\ &\leq ||e^{-t||\theta(\lambda,\mu)||^2}||_{L^{p'}(d\gamma_\alpha)} ||f\chi_{B(0,s)}||_{L^1(d\nu_\alpha)}. \end{aligned}$$

On the other hand,

$$||f\chi_{B(0,s)}||_{L^1(d\nu_\alpha)} \leq || \, ||(r,x)||^{-a}\chi_{B(0,s)}||_{L^{p'}(d\nu_\alpha)} || \, ||(r,x)||^a f||_{L^p(d\nu_\alpha)}.$$

A simple calculation give that

$$|| \, ||(r,x)||^{-a}\chi_{B(0,s)}||_{L^{p'}(d\nu_\alpha)} = C(\alpha)s^{\frac{2\alpha+3}{p'}-a}.$$

So

$$||e^{-t||\theta(\lambda,\mu)||^2}\mathcal{F}_\alpha(f)||_{L^{p'}(d\gamma_\alpha)} \leq ||e^{-t||\theta(\lambda,\mu)||^2}\mathcal{F}_\alpha(f_s)||_{L^{p'}(d\gamma_\alpha)} + ||e^{-t||\theta(\lambda,\mu)||^2}\mathcal{F}_\alpha(f^s)||_{L^{p'}(d\gamma_\alpha)}$$

$$\leq Cs^{-a}(1+||e^{-t||\theta(\lambda,\mu)||^2}||_{L^{p'}(d\gamma_\alpha)}s^{\frac{2\alpha+3}{p'}})||(r,x)||^a f||_{L^p(d\nu_\alpha)}.$$

Choosing $s=t^{\frac{1}{2}}$, we obtain (6.50). □

Theorem 7. Let $1<p\leq 2$ and $0<a<\frac{2\alpha+3}{p'}$ and $b>0$. Then for all $f\in L^p(d\nu_\alpha)$

$$||\mathcal{F}_\alpha(f)||_{L^{p'}(d\gamma_\alpha)} \leq C|| \, ||(r,x)||^a f||^{\frac{b}{a+b}}_{L^p(d\nu_\alpha)} || \, ||\theta(\mu,\lambda)||^b\mathcal{F}_\alpha(f)||^{\frac{a}{a+b}}_{L^{p'}(d\gamma_\alpha)}. \quad (6.51)$$

Proof. Let $1<p\leq 2$ and $0<a<\frac{2\alpha+3}{p'}$. Assume that $b\leq 2$. From the previous lemma, for all $t>0$

$$||\mathcal{F}_\alpha(f)||_{L^{p'}(d\gamma_\alpha)} \leq ||e^{-t||\theta(\lambda,\mu)||^2}\mathcal{F}_\alpha(f)||_{L^{p'}(d\gamma_\alpha)} + ||(1-e^{-t||\theta(\lambda,\mu)||^2})\mathcal{F}_\alpha(f)||_{L^{p'}(d\gamma_\alpha)}$$

$$\leq Ct^{-\frac{a}{2}}||(r,x)||^a f||_{L^p(d\nu_\alpha)} + ||(1-e^{-t||\theta(\lambda,\mu)||^2})\mathcal{F}_\alpha(f)||_{L^{p'}(d\gamma_\alpha)}.$$

On the other hand, $||(1-e^{-t||\theta(\lambda,\mu)||^2})\mathcal{F}_\alpha(f)||_{L^{p'}(d\gamma_\alpha)} = t^{\frac{b}{2}}||(t||\theta(\lambda,\mu)||^2)^{-\frac{b}{2}}(1 -e^{-t||\theta(\lambda,\mu)||^2})||\theta(\mu,\lambda)||^b\mathcal{F}_\alpha(f)||_{L^{p'}(d\gamma_\alpha)}$. Since $(1-e^{-t})t^{-\frac{b}{2}}$ is bounded for $t\geq 0$ if $b\leq 2$. Then, we obtain

$$||\mathcal{F}_\alpha(f)||_{L^{p'}(d\gamma_\alpha)} \leq C\Big(t^{\frac{a}{2}}||(r,x)||^a f||_{L^p(d\nu_\alpha)} + t^{\frac{b}{2}}|| \, ||\theta(\lambda,\mu)||^b\mathcal{F}_\alpha(f)||_{L^{p'}(d\gamma_\alpha)}\Big).$$

from which, optimizing in t, we obtain (6.51) for $0<a<\frac{2\alpha+2}{p'}$ and $b\leq 2$.

If $b>2$, let $b'\leq 2$. For $u\geq 0$ and $b'<b$, we have $u^{b'}\leq 1+u^b$, which for $u=\frac{||\theta(\lambda,\mu)||}{\varepsilon}$ gives the inequality $(\frac{||\theta(\lambda,\mu)||}{\varepsilon})^{b'}<1+(\frac{||\theta(\lambda,\mu)||}{\varepsilon})^b$ for all $\varepsilon>0$.

It follows that

$$|| \, ||\theta(\lambda,\mu)||^{b'}\mathcal{F}_\alpha(f)||_{L^{p'}(d\gamma_\alpha)}) \leq \varepsilon^{b'} + \varepsilon^{b'-b}|| \, ||\theta(\lambda,\mu)||^b\mathcal{F}_\alpha(f)||_{L^{p'}(d\gamma_\alpha)}).$$

Optimizing in ε, we get the result for $b>2$.

$$|| \, ||\theta(\lambda,\mu)||^{b'}\mathcal{F}_\alpha(f)||_{L^{p'}(d\gamma_\alpha)} \leq ||\mathcal{F}_\alpha(f)||^{\frac{b-b'}{b}}_{L^{p'}(d\gamma_\alpha)} || \, ||\theta(\lambda,\mu)||^b\mathcal{F}_\alpha(f)||^{\frac{b'}{b}}_{L^{p'}(d\gamma_\alpha)}.$$

Together with (6.51) for $b>2$. □

Corollary 4. Let $a,b>0$. For all $f\in L^2(d\nu_\alpha)$, we have

$$||f||_{L^2(d\nu_\alpha)}\leq C|| \, ||(r,x)||^a f||_{L^2(d\nu_\alpha)}^{\frac{b}{a+b}} ||\theta(\mu,\lambda)||^b\mathcal{F}_\alpha(f)||_{L^2(d\gamma_\alpha)}^{\frac{a}{a+b}}. \tag{6.52}$$

Proof. Using the previous theorem for $p=2$, and applying Plancherel formula, we obtain the result when $0<a<\frac{2\alpha+3}{2}$. If $a\geq\frac{2\alpha+3}{2}$, let $a'<\frac{2\alpha+3}{2}$. For $u\geq 0$, $u^{a'}\leq 1+u^a$ which for $u=\frac{||(r,x)||}{\varepsilon}$ gives the inequality

$$\left(\frac{||(r,x)||}{\varepsilon}\right)^{a'}\leq 1+\left(\frac{||(r,x)||}{\varepsilon}\right)^{a},\quad \text{for all } \varepsilon>0.$$

It follows that

$$|| \, ||(r,x)||^{a'}f||_{L^2(d\nu_\alpha)}\leq \varepsilon^{a'}||f||_{L^2(d\nu_\alpha)}+\varepsilon^{a'-a}|| \, ||(r,x)||^a f||_{L^2(d\nu_\alpha)}.$$

Optimizing in ε, we obtain

$$|| \, ||(r,x)||^{a'}f||_{L^2(d\nu_\alpha)}\leq C||f||_{L^2(d\nu_\alpha)}^{\frac{a-a'}{a}}|| \, ||(r,x)||^a f||_{L^2(d\nu_\alpha)}^{\frac{a'}{a}}. \tag{6.53}$$

Then, by (6.52) for (a' and b), and (6.53), we deduce that

$$\begin{aligned} ||f||_{L^2(d\nu_\alpha)} &\leq C|| \, ||(r,x)||^{a'}f||_{L^2(d\nu_\alpha)}^{\frac{b}{a'+b}}|| \, |\lambda|^b\mathcal{F}_\alpha(f)||_{L^2_\nu(\mathbb{R})}^{\frac{a'}{a'+b}} \\ &\leq C||f||_{L^2(d\nu_\alpha)}^{\frac{b(a-a')}{a(a'+b)}}|| \, ||(r,x)||^a f||_{L^2(d\nu_\alpha)}^{\frac{a'b}{a(a'+b)}}|| \, ||\theta(\mu,\lambda)||^b\mathcal{F}_\alpha(f)||_{L^2(d\gamma_\alpha)}^{\frac{a'}{a'+b}}. \end{aligned}$$

Thus

$$||f||_{L^2(d\nu_\alpha)}^{\frac{a'(a+b)}{a(a'+b)}}\leq C|| \, ||(r,x)||^a f||_{L^2(d\nu_\alpha)}^{\frac{a'b}{a(a'+b)}}|| \, ||\theta(\mu,\lambda)||^b\mathcal{F}_\alpha(f)||_{L^2(d\gamma_\alpha)}^{\frac{a'}{a'+b}},$$

which gives the result for $a\geq\frac{2\alpha+3}{2}$. □

Remark 3. *The previous corollary generalize the result proved in [26].*

Let T be a measurable subset of $\mathbb{R}^2_+$. Let $b>0$ and let $f\in L^p(d\nu_\alpha)$, $p\in[1,2]$. We say that $||\theta(\mu,\lambda)||^b\mathcal{F}_\alpha(f)$ is ε_T-concentrated to T in $L^{p'}(d\gamma_\alpha)$-norm, if there is a function h vanishing outside T such that

$$|| \, ||\theta(\mu,\lambda)||^b\mathcal{F}_\alpha(f)-h||_{L^{p'}(d\gamma_\alpha)}\leq\varepsilon_T|| \, ||\theta(\mu,\lambda)||^b\mathcal{F}_\alpha(f)||_{L^{p'}(d\gamma_\alpha)}.$$

From (6.37), it follows that $||\theta(\mu,\lambda)||^b\mathcal{F}_\Lambda(f)$ is ε_T-concentrated to T in $L^{p'}(d\gamma_\alpha)$-norm, if and only if

$$|| \, ||\theta(\mu,\lambda)||^b\mathcal{F}_\alpha(f)-||\theta(\mu,\lambda)||^b\mathcal{F}_\alpha(Q_Tf)||_{L^{p'}(d\gamma_\alpha)}\leq\varepsilon_T|| \, ||\theta(\mu,\lambda)||^b\mathcal{F}_\alpha(f)||_{L^{p'}(d\gamma_\alpha)}. \tag{6.54}$$

Corollary 5. Let T be a measurable subset of $\mathbb{R}_+^2$, and let $1 < p \leq 2$, $f \in L^p(d\nu_\alpha)$ and $b > 0$. If $||\theta(\mu,\lambda)||^b \mathcal{F}_\alpha(f)$ is ε_T-concentrated to T in $L^{p'}(d\gamma_\alpha)$-norm, then for $0 < a < \frac{2\alpha+3}{p'}$

$$||\mathcal{F}_\alpha(f)||_{L^{p'}(d\gamma_\alpha)} \leq \frac{C}{(1-\varepsilon_T)^{\frac{a}{a+b}}} || \, ||(r,x)||^a f||_{L^p(d\nu_\alpha)}^{\frac{b}{a+b}} || \, ||\theta(\mu,\lambda)||^b \mathcal{F}_\alpha(Q_T f)||_{L^{p'}(d\gamma_\alpha)}^{\frac{a}{a+b}}. \tag{6.55}$$

Proof. Let $f \in L^p(d\nu_\alpha)$, $1 < p \leq 2$. Since $||\theta(\mu,\lambda)||^b \mathcal{F}_\alpha(f)$ is ε_T-concentrated to T in $L^{p'}(d\gamma_\alpha)$-norm, then we have

$$|| \, ||\theta(\mu,\lambda)||^b \mathcal{F}_\alpha(f)||_{L^{p'}(d\gamma_\alpha)}$$

$$\leq \varepsilon_T || \, ||\theta(\mu,\lambda)||^b \mathcal{F}_\alpha(f)||_{L^{p'}(d\gamma_\alpha)} + || \, ||\theta(\mu,\lambda)||^b \mathcal{F}_\alpha(Q_T f)||_{L^{p'}(d\gamma_\alpha)}.$$

Thus

$$|| \, ||\theta(\mu,\lambda)||^b \mathcal{F}_\alpha(f)||_{L^{p'}(d\gamma_\alpha)}^{\frac{a}{a+b}} \leq \frac{1}{(1-\varepsilon_T)^{\frac{a}{a+b}}} || \, ||\theta(\mu,\lambda)||^b \mathcal{F}_\Lambda(Q_T f)||_{L^{p'}(d\gamma_\alpha)}^{\frac{a}{a+b}}.$$

Multiply this inequality by $C|| \, ||(r,x)||^a f||_{L^p(d\nu_\alpha)}^{\frac{b}{a+b}}$ and applying theorem 7 we deduce the desired result. □

We proceed as the previous corollary and using Corollary 4 we obtain the following.

Corollary 6. Let T be a measurable subset of $\mathbb{R}_+^2$, and let $f \in L^2(d\nu_\alpha)$ and $a, b > 0$.

If $||\theta(\mu,\lambda)||^b \mathcal{F}_\alpha(f)$ is ε_T-concentrated to T in $L^2(d\gamma_\alpha)$-norm, then

$$||f||_{L^2(d\nu_\alpha)} \leq \frac{C}{(1-\varepsilon_T)^{\frac{a}{a+b}}} || \, ||(r,x)||^a f||_{L^2(d\nu_\alpha)}^{\frac{b}{a+b}} || \, ||\theta(\mu,\lambda)||^b \mathcal{F}_\alpha(Q_T f)||_{L^2(d\gamma_\alpha)}^{\frac{a}{a+b}}. \tag{6.56}$$

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