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DUNKL-SEMICLASSICAL ORTHOGONAL POLYNOMIALS. THE SYMMETRIC CASE.

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In this paper, we introduce a new class of symmetric orthogonal polynomials that generalizes the class of Dunkl-classical ones. As applications, we give some new characterizations of the symmetric semiclassical orthogonal polynomials.

1. Introduction and basic notations

Classical orthogonal polynomials (Hermite, Laguerre, Jacobi and Bessel) can be defined as the only sequences of polynomials which are orthogonal with respect to a form (linear functional) u satisfying the Pearson differential equation

$$D(Au) + Bu = 0,$$

where A and B are fixed polynomials with $\deg A \leq 2$, and $\deg B = 1$. In 1939, Shohat extended these ideas introducing a new class of orthogonal polynomials. In fact, he studied orthogonal polynomials associated with forms satisfying the last equation, with no restrictions in the degrees of the polynomials A, and B. Obviously, orthogonal polynomials defined as above, generalize in a natural way the classical ones. They were called semiclassical orthogonal polynomials [15]. Among others, an approach to such polynomials taking into account

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the quasi-orthogonality of the derivatives of the polynomials sequence is given. This theory has been developed and extensively studied by Maroni where an algebraic theory of semiclassical orthogonal polynomials is presented [10], [11]. See also [9] for the discrete case.

Recently, Dunkl-classical orthogonal polynomials as T_μ-classical polynomials have been introduced in [4] (see also [12]), where T_μ is the Dunkl operator. The classification of the T_μ-classical symmetric orthogonal polynomials was started in a pioneering paper by Y. Ben Cheikh and M. Gaid [4]. Short time ago, several authors (see [3, 5, 8, 12, 13], among others), made some contributions in this direction. Our point of view in [12] provides a wider perspective in the subject. In the symmetric case, we consider as T_μ-classical every regular form u satisfying the Pearson differential equation

$$T_\mu\left(\Phi u\right)+\Psi u=0,$$

where Φ and Ψ are fixed polynomials with $\deg\Phi\leq 2$, and $\deg\Psi=1$.
In this work, we introduce the concept of the Dunkl-semiclassical forms as a generalization of the Dunkl-classical ones and we give some characterizations of these forms in the symmetric case.

The structure of the paper is as follows. The first section contains materials of preliminary and the definition of a Dunkl-semiclassical orthogonal polynomials. In the second section, we establish some characterizations of Dunkl-semiclassical symmetric forms. Essentially, we characterize these forms by a distributional equation of Pearson type. The third section deals with so-called class of Dunkl-semiclassical symmetric form. A criterion for determining it is given. In the fourth section, we characterize a symmetric Dunkl-semiclassical form through the fact that its Stieltjes function satisfies a first order linear difference equation with polynomial coefficients. Lastly, in the fifth section, we carry out the complete description of the symmetric Dunkl-semiclassical forms of class $s=1$ and we give an example of class $s=2$.

We begin by reviewing some preliminary results needed for the sequel. The vector space of polynomials with coefficients in $\mathbb{C}$ (the field of complex numbers) is denoted by $\mathcal{P}$ and by $\mathcal{P}'$ its dual space, whose elements are called forms. The set of all nonnegative integers will be denoted by $\mathbb{N}$. The action of $u\in\mathcal{P}'$ on $f\in\mathcal{P}$ is denoted by $\langle u,f\rangle$. In particular, we denote by $(u)_n:=\langle u,x^n\rangle, n\in\mathbb{N}$, the moments of u. For any form u, any $a\in\mathbb{C}-\{0\}$ and any polynomial g let $Du=u'$, hu, $h_a u$, δ_c and $(x-c)^{-1}u$ be the forms defined by: $\langle u',f\rangle:=-\langle u,f'\rangle$, $\langle gu,f\rangle:=\langle u,gf\rangle$, $\langle h_a u,f\rangle=:\langle u,h_a f\rangle=\langle u,f(ax)\rangle$, $\langle\delta_c,f\rangle:=f(c)$, and $\left\langle (x-c)^{-1}u,f\right\rangle:=\langle u,\theta_c f\rangle$ where $\left(\theta_c f\right)(x)=\dfrac{f(x)-f(c)}{x-c}$, $f\in\mathcal{P}, c\in\mathbb{C}$.

Then, it is straightforward to prove that for $f \in \mathcal{P}$ and $u \in \mathcal{P}'$, we have

$$(x-c)^{-1}((x-c)u) = u - (u)_0\delta_c \ , \tag{1}$$

$$(x-c)\left((x-c)^{-1}u\right) = u \ , \tag{2}$$

$$(fu)' = f'u + fu' \ . \tag{3}$$

For $f \in \mathcal{P}$ and $u \in \mathcal{P}'$, the product uf is the polynomial

$$\left\langle u, \frac{xf(x)-\zeta f(\zeta)}{x-\zeta} \right\rangle = \sum_{k=0}^{n} \left(\sum_{\nu=k}^{n} a_\nu (u)_{\nu-k} \right) x^k, \tag{4}$$

where $f(x) = \sum_{k=0}^{n} a_k x^k$. The Stieltjes function of $u \in \mathcal{P}'$ is defined by

$$S(u)(z) := -\sum_{n\geq 0} \frac{(u)_n}{z^{n+1}}. \tag{5}$$

We have the following formula

$$S(fu)(z) = f(z)S(u)(z) + (u\theta_0 f)(z). \tag{6}$$

Denoting by Δ the linear space generated by $\left\{\delta^{(n)}\right\}_{n\geq 0}$, where $\delta^{(n)}$ means the nth derivative of the Dirac delta in the origin, i.e., $\left\langle \delta^{(n)}, f \right\rangle = (-1)^n f^{(n)}(0)$, $f \in \mathcal{P}$ and by F the isomorphism : $\Delta \longrightarrow \mathcal{P}$ defined as follows [10]: for $u = \sum_{k=0}^{n} (u)_k \frac{(-1)^n}{n!} \delta^{(n)}$,

$$F(u) = \sum_{k=0}^{n} (u)_k z^k. \tag{7}$$

We will only consider sequences of polynomials $\{P_n\}_{n\geq 0}$ such that $\deg P_n \leq n, n \in \mathbb{N}$. If the set $\{P_n\}_{n\geq 0}$ spans $\mathcal{P}$, which occurs when $\deg P_n = n, n \in \mathbb{N}$, then it will be called a polynomial sequence (PS). Along the text, we will only deal with PS whose elements are monic, that is, monic polynomial sequences (MPS). The MPS $\{P_n\}_{n\geq 0}$ is orthogonal with respect to $u \in \mathcal{P}'$ when the following conditions hold: $\langle u, P_n P_m \rangle = r_n \delta_{n,m}$, $n,m \geq 0$, $r_n \neq 0$, $n \geq 0$ [6]. In this case, we say that $\{P_n\}_{n\geq 0}$ is a monic orthogonal polynomial sequence (MOPS) and the form u is said to be regular. Furthermore, the MOPS $\{P_n\}_{n\geq 0}$ fulfils the second order recurrence relation

$$\begin{array}{l} P_0(x) = 1 \quad , \quad P_1(x) = x - \beta_0 \\ P_{n+2} = (x-\beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x) \quad , \gamma_{n+1} \neq 0, \quad n \geq 0. \end{array} \tag{8}$$

The form u is said to be symmetric if and only if $(u)_{2n+1} = 0, n \geq 0$, or, equivalently, in (8) $\beta_n = 0, n \geq 0$.

A form u is said to be normalized if $(u)_0 = 1$. In this paper, we suppose that any form will be normalized.

Let us introduce the Dunkl operator

$$T_\mu(f) = f' + 2\mu H_{-1}f, \quad (H_{-1}f)(x) = \frac{f(x) - f(-x)}{2x}, \quad f \in \mathcal{P}, \mu \in \mathbb{C}. \tag{9}$$

This operator was introduced and studied for the first time by Dunkl [7]. Note that T_0 is reduced to the derivative operator D. The transposed ${}^tT_\mu$ of $T\mu$ is ${}^tT_\mu = -D - H_{-1} = -T_\mu$, leaving out a light abuse of notation without consequence. Thus, we have

$$\langle T_\mu u, f\rangle = -\langle u, T_\mu f\rangle, \quad u \in \mathcal{P}', \quad f \in \mathcal{P}, \quad \mu \in \mathbb{C}. \tag{10}$$

In particular, this yields $\langle T_\mu u, x^n\rangle = -\mu_n (u)_{n-1}, n \geq 0$, where $(u)_{-1} = 0$ and

$$\mu_n = n + \mu(1 - (-1)^n), \quad n \geq 0. \tag{11}$$

It is easy to see that

$$T_\mu(fu) = fT_\mu u + f'u + 2\mu\,(H_{-1}f)\,(h_{-1}u), \quad f \in \mathcal{P}, \quad u \in \mathcal{P}', \tag{12}$$

$$h_a \circ T_\mu = aT_\mu \circ h_a \quad in\ \mathcal{P}', \quad a \in \mathbb{C} - \{0\}. \tag{13}$$

Remark 1.1. When u is a symmetric form, (12) becomes

$$T_\mu(fu) = fT_\mu u + \left(T_\mu f\right) u, \quad f \in \mathcal{P}, \quad u \in \mathcal{P}'. \tag{14}$$

Now, consider a MPS $\{P_n\}_{n\geq 0}$ and let

$$P_n^{[1]}(x,\mu) = \frac{1}{\mu_{n+1}}\left(T_\mu P_{n+1}\right)(x), \quad \mu \neq -n - \frac{1}{2}, \quad n \geq 0. \tag{15}$$

Definition 1.2. [4, 12] A MOPS $\{P_n\}_{n\geq 0}$ is called Dunkl-classical or T_μ-classical if $\{P_n^{[1]}(.,\mu)\}_{n\geq 0}$ is also a MOPS. In this case, the corresponding form u is called either Dunkl-classical or T_μ-classical form.

Lemma 1.3. *[12] If $\{P_n\}_{n\geq 0}$ is Dunkl-classical symmetric MOPS, then $\{P_n^{[1]}(.,\mu)\}_{n\geq 0}$ is orthogonal with respect to Φu where Φ is a non-zero polynomial and* $\deg \Phi \leq 2$.

We recall the notion of the quasi-orthogonality.

Definition 1.4. *Let $v \in \mathcal{P}'$ and s is a non-negative integer. A MPS $\{B_n\}_{n\geq 0}$ is said to be quasi-orthogonal of order s with respect to v, if*

$$\begin{aligned} &\langle v, x^m B_n \rangle = 0, \quad 0 \leq m \leq n-s-1, \quad n \geq s+1, \\ &\exists r \geq s, \quad \langle v, x^{r-s} B_r \rangle \neq 0. \end{aligned} \tag{16}$$

If $\langle v, x^{r-s} B_r \rangle \neq 0$ for any $r \geq s$, then $\{B_n\}_{n\geq 0}$ is said to be strictly quasi-orthogonal of order s with respect to v.

Remark 1.5. A strictly quasi-orthogonal of order zero is orthogonal.

The following is a natural extension of definition 1.2.

Definition 1.6. *Let $\{P_n\}_{n\geq 0}$ be a MOPS with respect to the regular form u. We say that $\{P_n\}_{n\geq 0}$ is Dunkl-semiclassical or T_μ-semiclassical if there exists a non-negative integer s and a non-zero polynomial Φ, $\deg \Phi \leq s+2$ such that $\{P_n^{[1]}(.,\mu)\}_{n\geq 0}$ is quasi-orthogonal of order s with respect to Φu. The form u is called Dunkl-semiclassical or T_μ-semiclassical.*

2. Some Characterizations of the Dunkl-semiclassical symmetric forms

From now on, we assume that u is a symmetric regular form and $\{P_n\}_{n\geq 0}$ its corresponding MOPS.

The main result of this section follows:

Theorem 2.1. *The following statements are equivalent*
(a) The sequence $\{P_n\}_{n\geq 0}$ is Dunkl-semiclassical.
(b) There exist a non-negative integer s and two polynomials Φ monic, $\deg \Phi \leq s+2$, Ψ, $\deg \Psi \geq 1$ such that

$$T_\mu (\Phi u) + \Psi u = 0. \tag{17}$$

Proof. $(a) \Longrightarrow (b)$. Let s and Φ as in Definition 1.6. We have

$$\left\langle T_\mu (\Phi u), P_{n+1} \right\rangle = -\mu_{n+1} \left\langle \Phi u, P_n^{[1]}(.,\mu) \right\rangle, \quad n \geq 0. \tag{18}$$

But, we have $\left\langle \Phi u, P_n^{[1]}(.,\mu) \right\rangle = 0, n \geq s+1$, according to (16), with m=0. As consequence, since $\Phi u \neq 0$, there exists an integer p, $1 \leq p \leq s+1$, such that $\left\langle \Phi u, P_{p-1}^{[1]}(.,\mu) \right\rangle \neq 0$, $\left\langle \Phi u, P_n^{[1]}(.,\mu) \right\rangle = 0, n \geq p$. Using (18), it follows that $\left\langle T_\mu (\Phi u), P_p \right\rangle \neq 0$ and $\left\langle T_\mu (\Phi u), P_n \right\rangle = 0, n \geq p+1$. So, from the orthogonality

of $\{P_n\}_{n\geq 0}$, there exists a polynomial
$\Psi(x) = \sum_{k=1}^{p} \left\langle \Phi u, P_{k-1}^{[1]}(.,\mu) \right\rangle \left\langle u, P_k^2(.,\mu) \right\rangle^{-1} P_k, \deg \Psi = p \geq 1,$
such that $T_\mu(\Phi u) + \Psi u = 0$. Hence, (b) holds.
$(b) \Longrightarrow (a)$. Let $s = \max(\deg \Phi - 2, \deg \Psi - 1)$.
We have $\mu_{n+1} \left\langle \Phi u, x^m P_n^{[1]}(.,\mu) \right\rangle = -\left\langle T_\mu(x^m \Phi u), P_{n+1} \right\rangle$, for all non-negative integers m and n. On account of (12), we get

$$T_\mu(x^m \Phi u) = x^m T_\mu(\Phi u) + m x^{m-1} \Phi(x) u + \mu\left(1 - (-1)^m\right) x^{m-1} \Phi(-x)(h_{-1} u).$$

But, we have $T_\mu(\Phi u) = -\Psi u$ from (17) and $(h_{-1}u) = u$ because u is symmetric.
Then, $T_\mu(x^m \Phi u) = A_{m+s+1} u$, with
$A_{m+s+1}(x) = -x^m \Psi(x) + m x^{m-1} \Phi(x) + \mu\left(1 - (-1)^m\right) x^{m-1} \Phi(-x),$
where $\deg A_{m+s+1} \leq m+s+1, m \geq 0$. Thus,

$$\mu_{n+1} \left\langle \Phi u, x^m P_n^{[1]}(.,\mu) \right\rangle = -\left\langle u, A_{m+s+1} P_{n+1} \right\rangle. \tag{19}$$

Using the orthogonality of $\{P_n\}_{n\geq 0}$, we obtain

$$\mu_{n+1} \left\langle \Phi u, x^m P_n^{[1]}(.,\mu) \right\rangle = 0, \quad 0 \leq m \leq n-s-1, \quad n \geq s+1. \tag{20}$$

For $m = n-s$, in (19) we get

$$\mu_{n+1} \left\langle \Phi u, x^{n-s} P_n^{[1]}(.,\mu) \right\rangle = -\lambda_n \left\langle u, P_{n+1}^2 \right\rangle, \quad n \geq s, \tag{21}$$

where $\lambda_n = -\dfrac{\Psi^{(s+1)}(0)}{(s+1)!} + (\mu_n - \mu_s) \dfrac{\Phi^{(s+2)}(0)}{(s+2)!}$.
Then, we analyse two situations.
When $t = s+2$, then $\qquad \lambda_n = -\dfrac{\Psi^{(s+1)}(0)}{(s+1)!} + \mu_n - \mu_s.$

The MPS $\{P_n^{[1]}(.,\mu)\}_{n\geq 0}$ is strictly quasi-orthogonal of order s if and only if $\frac{\Psi^{(s+1)}(0)}{(s+1)!} + \mu_s \neq \mu_n, \quad n \geq 0.$

When $t < s+2$, then $\qquad \lambda_n = -\dfrac{\Psi^{(s+1)}(0)}{(s+1)!} \neq 0, \quad$ because we have necessarily $\deg \Psi = s+1$.
Thus, the MPS $\{P_n^{[1]}(.,\mu)\}_{n\geq 0}$ is quasi-orthogonal of order s. □

Remark 2.2. If u is a symmetric Dunkl-semiclassical form satisfying (17) such that $s = \max(\deg \Phi - 2, \deg \Psi - 1)$ is even and the pair (Φ, Ψ) is admissible (i.e one of the following conditions is satisfied: $\deg \Phi \neq \deg \Psi - 1$ or $\deg \Phi = \deg \Psi - 1$ with $a_p + \mu_n c_t \neq 0$ where a_p and c_t are the leading coefficients of Ψ and Φ , respectively) then $\{P_n^{[1]}(.,\mu)\}_{n\geq 0}$ is strictly quasi-orthogonal of order s with respect to Φu.

The following result allows us to characterize the symmetric Dunkl-semi-classical MOPS by the first structure relation.

Theorem 2.3. *For any symmetric MOPS $\{P_n\}_{n\geq 0}$, the following statements are equivalent*

(a) *The sequence $\{P_n\}_{n\geq 0}$ is Dunkl-semiclassical.*

(b) *There exists $(r,s)\in\mathbb{N}^2$, with $0\leq s\leq r$ such that*

$$\Phi(x)P_n^{[1]}(x,\mu)=\sum_{k=n-s}^{n+t}\lambda_{n,k}P_k(x),\quad n\geq s, \tag{22}$$

$$\lambda_{r,r-s}\neq 0. \tag{23}$$

Proof. $(a)\Longrightarrow(b)$. First, we always have $\Phi(x)P_n^{[1]}(x,\mu)=\sum_{k=0}^{n+t}\lambda_{n,k}P_k(x)$, with $t=\deg\Phi\leq s+2$ and $\lambda_{n,k}=\left\langle \Phi u,P_kP_n^{[1]}(.,\mu)\right\rangle\left\langle u,P_k^2\right\rangle^{-1},0\leq k\leq n+t,n\geq 0$. But, $\left\{P_n^{[1]}(.,\mu)\right\}_{n\geq 0}$ is quasi-orthogonal of order s with respect to Φu, according to Defintion 1.6. Therefore, there exists an integer $r\geq s$ such that $\left\langle \Phi u,P_kP_n^{[1]}(.,\mu)\right\rangle=0,0\leq k\leq n-s-1,n\geq s+1$ and $\left\langle \Phi u,P_{r-s}P_r^{[1]}(.,\mu)\right\rangle\neq 0$. Thus, we have $\lambda_{n,k}=0,0\leq k\leq n-s-1,n\geq s+1$, and $\lambda_{r,r-s}\neq 0$. Hence, (22) and (23) hold, with $\lambda_{r,r-s}=\left\langle \Phi u,P_{r-s}P_r^{[1]}(.,\mu)\right\rangle\left\langle u,P_{r-s}^2\right\rangle^{-1}\neq 0$.

$(b)\Longrightarrow(a)$. From the assumption and the orthogonality of $\{P_n\}_{n\geq 0}$, we get $\left\langle \Phi u,P_mP_n^{[1]}(.,\mu)\right\rangle=\sum_{k=n-s}^{n+t}\lambda_{n,k}\left\langle u,P_kP_m\right\rangle=\sum_{k=n-s}^{n+t}\lambda_{n,k}\left\langle u,P_k^2\right\rangle\delta_{m,k},\quad n\geq s$.

Then, $\left\langle \Phi u,P_mP_n^{[1]}(.,\mu)\right\rangle=0,0\leq m\leq n-s-1,n\geq s+1$ and

$\left\langle \Phi u,P_{r-s}P_r^{[1]}(.,\mu)\right\rangle=\lambda_{r,r-s}\left\langle u,P_{r-s}^2\right\rangle\neq 0$.

Thus, $\{P_n^{[1]}(.,\mu)\}_{n\geq 0}$ is quasi-orthogonal of order s with respect to Φu.
Hence, $\{P_n\}_{n\geq 0}$ is Dunkl-semiclassical. □

Let us recall that a form u is called semi-classical if it satisfies the functional equation (17) where $\mu=0$. We have the following result:

Theorem 2.4. *For any symmetric regular form u, the following statements are equivalent*

(a) *The form u is Dunkl-semiclassical.*

(b) *The form u is semiclassical.*

Proof. $(a)\Longrightarrow(b)$. Using (12), the equation (17) becomes after multiplying by x

$$T_\mu\left(x\Phi u\right)-\Phi u-2\mu\Phi(-x)u+x\Psi u=0. \tag{24}$$

By writing $\Phi(x)=\Phi^e(x^2)+x\Phi^o(x^2)$ and taking into account (9), (12) and the fact that u is symmetric, the last equation becomes

$$D((x\Phi)u)+\big(x\Psi-(1+2\mu)\Phi^e(x^2)-x\Phi^o(x^2)\big)u=0, \tag{25}$$

because $(H_{-1}x\Phi^e u)=0$ and $\big(H_{-1}x^2\Phi^o u\big)=-x\Phi^o(x^2)u$.
Hence, the form u is semiclassical.
$(b)\Longrightarrow(a)$. Assume that u is semiclassical. Then, it satisfies the functional equation

$$D(Au)+Bu=0. \tag{26}$$

Multiplying by x and taking into account (3) and (9), we obtain

$$T_\mu(xAu)-2\mu(H_{-1}xAu)+(xB-A)u=0. \tag{27}$$

By writing $A(x)=A^e(x^2)+xA^o(x^2)$, we get

$$(H_{-1}xAu)=-xA^o(x^2)u,$$

because $(H_{-1}xA^e u)=0$ and $\big(H_{-1}x^2A^o u\big)=-xA^o(x^2)u_0$.
Therefore, we obtain

$$T_\mu(xAu)+(xB-A+2\mu xA^o)u=0. \tag{28}$$

Hence, the form u is Dunkl-semiclassical. □

Remark 2.5. The set of Dunkl-semiclassical symmetric forms and the set of semiclassical ones are identical but, in general, the class numbers are different when $\mu\neq 0$ (see Definition 3.1). For instance, Dunkl-classical symmetric forms (Dunkl-semiclassical of class $s=0$) are among the semiclassical ones of class $\tilde{s}\leq 1$ (see Theorem 5.1).

Lemma 2.6. *If u is a Dunkl-semiclassical symmetric form , then $\tilde{u}=h_{a^{-1}}u$ is also for every $a\neq 0$.*

Proof. Applying the operator h_a to the functional equation (17) and using (13), we obtain

$$T_\mu\big(\tilde{\Phi}\tilde{u}\big)+\tilde{\Psi}\tilde{u}=0, \tag{29}$$

where $\tilde{\Phi}(x)=a^{-t}\Phi(ax),\quad \tilde{\Psi}(x)=a^{1-t}\Psi(ax),\quad t=\deg\Phi.$ □

3. The class of a Dunkl-semiclassical symmetric form

A Dunkl-semiclassical symmetric form satisfies an infinity number of equations of type (17). Indeed, multiplying (17) by χ and using (12), we obtain

$$\begin{aligned}0 &= \chi T_\mu(\Phi u) + \chi\Psi u\\ &= T_\mu(\chi\Phi u) + (\chi\Psi - \chi'\Phi - 2\mu(H_{-1}\chi)(h_{-1}\Phi))u.\end{aligned}$$

Then, for any pair (Φ,Ψ) satisfying (17) we associate the positive integer $\max(\deg\Phi-2, \deg\Psi-1)$. Denoting

$$\Upsilon(u) := \left\{s = \max(\deg\Phi - 2, \deg\Psi - 1), \quad T_\mu(\Phi u) + \Psi u = 0\right\},$$

what leads us to the following definition

Definition 3.1. *The minimum element of $\Upsilon(u)$ will be called the class of u. When u is of class s, the sequence $\{P_n\}_{n\geq 0}$ orthogonal with respect to u is said to be of class s.*

Lemma 3.2. *Let u be a Dunkl-semiclassical symmetric form satisfying*

$$T_\mu(\Phi_1 u) + \Psi_1 u = 0 \tag{30}$$

and

$$T_\mu(\Phi_2 u) + \Psi_2 u = 0 \tag{31}$$

where $\Phi_1, \Psi_1, \Phi_2, \Psi_2$ are polynomials, Φ_1, Φ_2 monic, $\deg\Psi_1 \geq 1, \deg\Psi_2 \geq 1$. *Denoting $s_1 = \max(\deg\Phi_1 - 2, \deg\Psi_1 - 1), s_2 = \max(\deg\Phi_2 - 2, \deg\Psi_2 - 1)$. Let $\Phi = gcd(\Phi_1, \Phi_2)$. Then, there exists a polynomial Ψ,* $\deg\Psi \geq 1$ *such that*

$$T_\mu(\Phi u) + \Psi u = 0 \tag{32}$$

with

$$\max(\deg\Phi - 2, \deg\Psi - 1) = s_1 - \deg\Phi_1 + \deg\Phi = s_2 - \deg\Phi_2 + \deg\Phi. \tag{33}$$

Proof. With $\Phi = gcd(\Phi_1, \Phi_2)$, there exist two coprime polynomials $\hat{\Phi}_1, \hat{\Phi}_2$ such that

$$\Phi_1 = \Phi\hat{\Phi}_1, \quad \Phi_2 = \Phi\hat{\Phi}_2. \tag{34}$$

Taking into account (12), equations (30) and (31) become

$$\hat{\Phi}_i T_\mu(\Phi u) + \left(\Psi_i + \hat{\Phi}_i'\Phi + 2\mu\left(H_{-1}\hat{\Phi}_i\right)(h_{-1}\Phi)\right)u = 0, \quad i \in \{1,2\}. \tag*{$(35)_i$}$$

The operation $\hat{\Phi}_2 \times (35)_1 - \hat{\Phi}_1 \times (35)_2$ gives

$$\begin{aligned}\Big\{\hat{\Phi}_2\left(\Psi_1 + \hat{\Phi}_1'\Phi + 2\mu\left(H_{-1}\hat{\Phi}_1\right)(h_{-1}\Phi)\right) - \\ \hat{\Phi}_1\left(\Psi_2 + \hat{\Phi}_2'\Phi + 2\mu\left(H_{-1}\hat{\Phi}_2\right)(h_{-1}\Phi)\right)\Big\}u = 0.\end{aligned}$$

From the regularity of u we get

$$\begin{aligned}\hat{\Phi}_2\left(\Psi_1+\hat{\Phi}_1'\Phi+2\mu\left(H_{-1}\hat{\Phi}_1\right)(h_{-1}\Phi)\right) &= \\ \hat{\Phi}_1\left(\Psi_2+\hat{\Phi}_2'\Phi+2\mu\left(H_{-1}\hat{\Phi}_2\right)(h_{-1}\Phi)\right).\end{aligned} \tag{36}$$

Thus, there exists a polynomial Ψ such that

$$\begin{cases}\Psi_1+\hat{\Phi}_1'\Phi+2\mu\left(H_{-1}\hat{\Phi}_1\right)(h_{-1}\Phi)=\Psi\hat{\Phi}_1,\\ \Psi_2+\hat{\Phi}_2'\Phi+2\mu\left(H_{-1}\hat{\Phi}_2\right)(h_{-1}\Phi)=\Psi\hat{\Phi}_2.\end{cases} \tag{37}$$

Then, formulas $(35)_1-(35)_2$ become

$$\hat{\Phi}_i\left\{T_\mu(\Phi u)+\Psi u\right\}=0,\quad i\in\{1,2\}$$

writing $\hat{\Phi}_i=\prod_{k=1}^{l_i}(x-c_{i,k})^{\alpha_{i,k}}, i\in\{1,2\}$, which yields

$$T_\mu(\Phi u)+\Psi u=\sum_{k=1}^{l_1}\beta_{1,k}\delta_{c_{1,k}}^{(\alpha_{1,k})}=\sum_{k=1}^{l_2}\beta_{2,k}\delta_{c_{2,k}}^{(\alpha_{2,k})}.$$

Since $\hat{\Phi}_1$ and $\hat{\Phi}_2$ have no common zero, we obtain (32). With (34) and (37) it is easy to prove (33). □

Proposition 3.3. *For each Dunkl-semiclassical symmetric form u, the pair* (Φ,Ψ) *which realizes the minimum of* $\Upsilon(u)$ *is unique.*

Proof. Indeed, if $s_1=s_2$ in (30), (31) and if this common value is the minimum element of $\Upsilon(u)$, we necessarily have $s=s_1=s_2$, hence $\deg\Phi_1=\deg\Phi=\deg\Phi_2$, therefore $\Phi_1=\Phi=\Phi_2$ and $\Psi_1=\Psi=\Psi_2$. □

Given a Dunkl-semiclassical symmetric form u, it is necessary to know whether the integer s associated with (Φ,Ψ) is the minimum of $\Upsilon(u)$.

Proposition 3.4. *The Dunkl-semiclassical symmetric form u satisfying (17) is of class* $s=\max(\deg\Phi-2,\deg\Psi-1)$ *if and only if*

$$\prod_c\left(\left|\left(T_\mu\Phi\right)(c)+\Psi(c)\right|+\left|\left\langle u,\theta_c\Psi+\theta_c^2\Phi-2\mu\theta_c o\theta_{-c}\Phi\right\rangle\right|\right)>0\,, \tag{38}$$

where c belongs to the set of zeros of Φ.

Proof. Let c be a zero of Φ. Put $\Phi(x)=(x-c)\Phi_c(x)$ and carry out the Euclidean division of $\Psi(x)+\Phi_c(x)+2\mu\Phi_c(-x)$ by $x-c$

$$\Psi(x)+\Phi_c(x)+2\mu\Phi_c(-x)=(x-c)Q_c(x)+r_c.$$

Then (17) becomes

$$(x-c)\left\{T_{\mu}\left(\Phi_{c}u\right)+Q_{c}u\right\}+r_{c}u=0,$$

on account of (1)-(2), the last equation is equivalent to

$$T_{\mu}\left(\Phi_{c}u\right)+Q_{c}u=\left(T_{\mu}\left(\Phi_{c}u\right)+Q_{c}u\right)_{0}\delta_{c}-(x-c)^{-1}r_{c}u. \tag{39}$$

By writing $\Phi_{c}(x)=\Phi_{c}^{e}(x^{2})+x\Phi_{c}^{o}(x^{2})$ and using the fact that u is symmetric, we get

$$\langle u,\theta_{c}(\Phi(-x))\rangle=\langle u,\theta_{-c}\Phi\rangle.$$

Moreover, it is easy to see that

$$\Phi_{c}(-c)=\left(H_{-1}\Phi\right)(c)\quad,\quad\Phi_{c}(x)=\left(\theta_{c}\Phi\right)(x).$$

Finally

$$\left\{\begin{array}{l} r_{c}=\left(T_{\mu}\Phi\right)(c)+\Psi(c),\\ Q_{c}(x)=\left(\theta_{c}\Psi\right)(x)+\left(\theta_{c}^{2}\Phi\right)+2\mu\left(\theta_{c}\Phi_{c}(-\xi)\right)(x),\\ \left(T_{\mu}\left(\Phi_{c}u\right)+Q_{c}u\right)_{0}=\langle u,Q_{c}\rangle=\left\langle u,\theta_{c}\Psi+\theta_{c}^{2}\Phi-2\mu\,\theta_{c}o\theta_{-c}\Phi\right\rangle. \end{array}\right. \tag{40}$$

The condition (38) is necessary. Let us suppose that there exists c, $\Phi(c)=0$, satisfying

$$r_{c}=0\quad,\quad\langle u,Q_{c}\rangle=0.$$

Then by (39), u verifies

$$T_{\mu}\left(\Phi_{c}u\right)+Q_{c}u=0, \tag{41}$$

with $s_{c}=\max\left(\deg Q_{c}-1,\deg\Phi_{c}-2\right)<s$, what contradicts that $s=\min\Upsilon(u)$.
The condition (38) is sufficient. Let us suppose u to be of class $\tilde{s}\leq s$ with $T_{\mu}\left(\tilde{\Phi}u\right)+\tilde{\Psi}u=0$. On account of the Lemma 3.2, there exists a polynomial χ such that $\Phi=\tilde{\Phi}\chi,\quad\Psi=\tilde{\Psi}\chi-\chi'\tilde{\Phi}-2\mu\left(H_{-1}\chi\right)\left(h_{-1}\Phi\right)$.
Suppose $\tilde{s}<s$. Then $\deg\chi\geq1$ and let c be a zero of χ: $\chi(x)=(x-c)\chi_{c}(x)$. We have

$$\Psi(x)+\Phi_{c}(x)+2\mu\Phi_{c}(-x)=(x-c)\left\{\chi_{c}\tilde{\Psi}-\chi'\tilde{\Phi}-2\mu\left(H_{-1}\chi\right)(x)\tilde{\Phi}(-x)\right\}.$$

Therefore $r_{c}=0$. On the other hand, by writing $\chi_{c}(x)=\chi_{c}^{e}(x^{2})+x\chi_{c}^{o}(x^{2})$ and using the fact that u is symmetric, we get

$$\left\langle u,\left(H_{-1}\chi\right)(x)\tilde{\Phi}(-x)\right\rangle=\left\langle u,\left(H_{-1}\chi\right)(x)\tilde{\Phi}(x)\right\rangle.$$

Then,

$$\begin{aligned}\left\langle u,\theta_{c}\Psi+\theta_{c}^{2}\Phi-2\mu\,\theta_{c}o\theta_{-c}\Phi\right\rangle&=\left\langle u,\chi_{c}\Psi-\left(T_{\mu}\chi\right)\tilde{\Phi}\right\rangle\\&=\left\langle T_{\mu}\left(\tilde{\Phi}u\right)+\tilde{\Psi}u,\chi_{c}\right\rangle=0.\end{aligned}$$

This is contradictory with (38), consequently $\tilde{s}=s$, $\tilde{\Phi}=\Phi$ and $\tilde{\Psi}=\Psi$. □

Remark 3.5. When $s=0$, the form u is usually called Dunkl-classical [12].

Proposition 3.6. *Let u a Dunkl-semiclassical symmetric form of class s satisfying (17). The following statements hold.*
i) When s is odd then the polynomial Φ *is odd and* Ψ *is even.*
ii) When s is even then the polynomial Φ *is even and* Ψ *is odd.*

Proof. Let us split up the polynomials Φ and Ψ according to their even and odd parts
$$\Phi(x)=\Phi^e(x^2)+x\Phi^o(x^2), \Psi(x)=\Psi^e(x^2)+x\Psi^o(x^2).$$
Equation (17) becomes
$$w^e+w^o=0 \tag{42}$$
where $w^e=T_\mu\left(\Phi^e(x^2)u\right)+x\Psi^o(x^2)u$ and $w^o=T_\mu\left(x\Phi^o(x^2)u\right)+\Psi^e(x^2)u$. Obviously, we get
$$\left\langle w^e,x^{2n+1}\right\rangle=0,\quad \left\langle w^o,x^{2n}\right\rangle=0,\quad n\geq 0.$$
Hence, from (42)
$$\left\langle w^e,x^{2n}\right\rangle=-\left\langle w^o,x^{2n}\right\rangle=0,\quad n\geq 0,$$
$$\left\langle w^o,x^{2n+1}\right\rangle=-\left\langle w^e,x^{2n+1}\right\rangle=0,\quad n\geq 0.$$
Therefore $w^e=w^o=0$. Consequently, u satisfies two functional equations
$$T_\mu\left(\Phi^e(x^2)u\right)+x\Psi^o(x^2)u=0, \tag{43}$$
$$T_\mu\left(x\Phi^o(x^2)u\right)+\Psi^e(x^2)u=0. \tag{44}$$
We denote $t=\deg\Phi$ and $p=\deg\Psi$.
i) When $s=2k+1$, with $s=\max(t-2,p-1)$ we get $t\leq 2k+3, p\leq 2k+2$, then $\deg\left(x\Psi^o(x^2)\right)\leq 2k+1, \deg\left(\Phi^e(x^2)\right)\leq 2k+2$. So, in accordance with (43), we obtain the contradiction $s=2k+1\leq 2k$. Necessary $\Phi^e=\Psi^o=0$.
ii) When $s=2k$, with $s=\max(t-2,p-1)$ we get $t\leq 2k+2, p\leq 2k+1$, then $\deg\left(\Psi^e(x^2)\right)\leq 2k, \deg\left(x\Phi^o(x^2)\right)\leq 2k+1$. So, in accordance with (44), we obtain the contradiction $s=2k\leq 2k-1$. Necessary $\Phi^o=\Psi^e=0$. Hence the desired result. □

Remark 3.7. *When* $\mu=0$ *we recover again the same result for the semiclassical case [1].*

4. Characterization of the Dunkl-semiclassical symmetric form by means of the formal Stieltjes function

One of the most important characterizations of Dunkl-semiclassical symmetric forms is given in the terms of a non homogeneous first order linear difference equation which its formal Stieltjes series satisfies. See also [1, 10] for the semi-classical case.

Proposition 4.1. *The symmetric form u is $T_\mu-$semiclassical of class s, if and only if, it is regular and there exist three coprime polynomials A (monic), C, D such that*

$$A(z)T_{-\mu}\left(S(u)\right)(z) = C(z)S(u)(z) + D(z), \tag{45}$$

with

$$s = \max\left(\deg C - 1\ ,\ \deg D\right). \tag{46}$$

Proof. Necessity. From (17), we have

$$0 = T_\mu\left(\Phi u\right) + \Psi u = \Phi T_\mu\left(u\right) + \left\{\Psi + T_\mu \Phi\right\} u$$

with (14). The isomorphism F yields

$$F\left(\Phi T_\mu\left(u\right) + \left\{\Psi + T_\mu \Phi\right\} u\right)(z) = 0.$$

From the definition of $S(u)$, we obtain

$$S\left(\Phi\left(T_\mu u\right)\right)(z) + S\left(\Psi u\right)(z) + S\left(\left(T_\mu \Phi\right) u\right)(z) = 0. \tag{47}$$

On account of (6) and (9), (47) becomes

$$\Phi T_{-\mu}\left(S(u)\right)(z) + \left(u'\theta_0\Phi\right)(z) + 2\mu\left(\left(H_{-1}u\right)\theta_0\Phi\right)(z) + \left\{T_{-\mu}\Phi + \Psi\right\}S(u)(z) + \\ + \left(u\theta_0\Phi'\right)(z) + 2\mu\left(u\theta_0\left(H_{-1}\Phi\right)\right)(z) + \left(u\theta_0\Psi\right)(z) = 0$$

By split up the polynomials Φ according to their even and odd parts

$$\Phi(x) = \Phi^e(x^2) + x\Phi^o(x^2)$$

we prove that, the last equation becomes

$$\Phi T_{-\mu}\left(S(u)\right)(z) + \left\{T_\mu\Phi + \Psi\right\}S(u)(z) + T_\mu\left(u\theta_0\Phi\right)(z) + \left(u\theta_0\Psi\right)(z) = 0. \tag{48}$$

Hence (45) with

$$\begin{cases} A(z) = \Phi(z), \\ C(z) = -\left(T_\mu\Phi\right)(z) - \Psi(z), \\ D(z) = -T_\mu\left(u\theta_0\Phi\right)(z) - \left(u\theta_0\Psi\right)(z). \end{cases} \tag{49}$$

Let c be a zero of Φ. From the first relation in (49), we remark that c is a zero of A. As u is of class s, in accordance with (38) we get

$$\left(T_\mu\Phi\right)(c)+\Psi(c)\neq 0 \quad \text{or} \quad \left\langle u,\theta_c\Psi+\theta_c^2\Phi-2\mu\,\theta_c o\theta_{-c}\Phi\right\rangle\neq 0.$$

But with the definitions of T_μ, θ_ξ, uf and formula (49), we obtain

$$\begin{cases} C(c)=-\left(T_\mu\Phi\right)(c)-\Psi(c), \\ D(c)=-\left\langle u,\theta_c\Psi+\theta_c^2\Phi-2\mu\,\theta_c o\theta_{-c}\Phi\right\rangle. \end{cases}$$

Consequently, A,C and D have no common zero. Then A,C and D are coprime.

Sufficiency. Let u be a regular and symmetric form with Stieltjes series $S(u)$ satisfies (45). From (6) and (9), formula (45) becomes

$$S\left(A\left(T_\mu u\right)-Cu\right)=\left(\left(T_\mu u\right)\theta_0 A\right)(z)-\left(u\theta_0 C\right)(z)+D(z). \tag{50}$$

From (14), (50) could be written as

$$S\left(T_\mu(Au)-\left\{T_\mu(A)+C\right\}u\right)=\left(\left(T_\mu u\right)\theta_0 A\right)(z)-\left(u\theta_0 C\right)(z)+D(z),$$

which implies

$$\begin{cases} T_\mu(Au)-\left\{T_\mu(A)+C\right\}u=0 \\ D(z)=\left(u\theta_0 C\right)(z)-\left(\left(T_\mu u\right)\theta_0 A\right)(z). \end{cases}$$

Denoting $\Phi(x)=A(x)$ and $\Psi(x)=-\left\{T_\mu(A)+C\right\}$.
Now, it is easy to see that

$$T_\mu(\Phi u)+\Psi u=0 \quad \text{with} \quad s=\max\left(\deg\Phi-2,\deg\Psi-1\right). \qquad \square$$

5. Some symmetric Dunkl-semiclassical orthogonal polynomials

Let us recall some results to be used in the sequel. In 1996, J. Alaya and P. Maroni have established the linear system fulfilled by the coefficients of the three-term recurrence relation of a symmetric Laguerre-Hahn MOPS of class 1 [1] that is to say the MOPS associated with a regular form u satisfying the functional equation

$$(\Phi(x)u)'+\Psi(x)u+B(x^{-1}u^2)=0$$

where Φ, Ψ and B are three polynomials with Φ monic and

$$\max\left(\max\left(\deg\Phi-2,\deg B-2\right),\deg\Psi-1\right)\leq 1.$$

Consequently, the authors deduce the classification of symmetric semiclassical forms of class 1 as a particular case ($B=0$). There are three situations [1,2]:

• The generalized Hermite form $\mathcal{H}(\alpha)$ satisfying

$$(x\mathcal{H}(\alpha))' + \left(2x^2 - (2\alpha+1)\right)\mathcal{H}(\alpha) = 0. \quad (51)$$

In addition, its MOPS $\{H_n^\alpha(x)\}_{n\geq 0}$ verifies (7) with

$$\beta_n = 0,\ \gamma_{n+1} = \frac{1}{2}\left(n+1+\alpha\left(1+(-1)^n\right)\right), \quad 2\alpha \neq -2n-1, \quad n \geq 0. \quad (52)$$

• The generalized Gegenbauer $\mathcal{G}(\alpha,\beta)$ satisfying

$$\left(x(x^2-1)\mathcal{G}(\alpha,\beta)\right)' + \left(-2(\alpha+\beta+2)x^2 + 2(\beta+1)\right)\mathcal{G}(\alpha,\beta) = 0. \quad (53)$$

In addition, its MOPS $\{S_n^{(\alpha,\beta)}(x)\}_{n\geq 0}$ verifies (7) with (for $n \geq 0$)

$$\beta_n = 0,\ \gamma_{n+1} = \frac{(n+1+\delta_n)(n+1+2\alpha+\delta_n)}{4(n+\alpha+\beta+1)(n+\alpha+\beta+2)},\ \delta_n = (2\beta+1)\frac{1+(-1)^n}{2}, \quad (54)$$

where the regularity conditions are $\alpha \neq -n, \beta \neq -n, \alpha+\beta \neq -n, n \geq 1$.

• The symmetrized Bessel form $\mathcal{B}(\alpha)$ satisfying

$$\left(x^3\mathcal{B}(\alpha)\right)' - \left(2(\alpha+1)x^2 + \frac{1}{2}\right)\mathcal{B}(\alpha) = 0. \quad (55)$$

In addition, its MOPS $\{B_n^\alpha(x)\}_{n\geq 0}$ verifies (7) with

$$\beta_n = 0, \quad \gamma_{n+1} = -\frac{1-2\alpha-(-1)^n(2n+2\alpha+1)}{16(n+\alpha)(n+\alpha+1)},\ \alpha \neq -n-1,\ n \geq 0. \quad (56)$$

5.1. Symmetric Dunkl-semiclassical orthogonal polynomials of class $s \leq 1$

Now, we are going to give a new proof of the following result which was established by many authors from different ways (see [4, 5, 13]).

Theorem 5.1. *Up to a dilatation, the only Dunkl-classical symmetric MOPS are:*
(a) The generalized Hermite polynomials $\{H_n^\mu(x)\}_{n\geq 0}$ *for* $\mu \neq -n-\frac{1}{2}, n \geq 1$. *Moreover,*

$$T_\mu\left(\mathcal{H}(\mu)\right) + 2x\mathcal{H}(\mu) = 0. \quad (57)$$

(b) The generalized Gegenbauer polynomials $\{S_n^{(\alpha,\mu-\frac{1}{2})}(x)\}_{n\geq 0}$ *for* $\alpha \neq n, \alpha+\mu \neq -n+\frac{1}{2}, \mu \neq -n-\frac{1}{2}, n \geq 1$. *Moreover,*

$$T_\mu\left((x^2-1)\mathcal{G}\left(\alpha,\mu-\frac{1}{2}\right)\right) - 2(\alpha+1)x\mathcal{G}\left(\alpha,\mu-\frac{1}{2}\right) = 0. \quad (58)$$

Proof. Since a Dunkl-classical symmetric form u is T_μ-semiclassical form of class zero, then by virtue of Proposition 3.4 and Proposition 3.6, it follows that u satisfies (17) with

$$\Phi(x) = c_2x^2 + c_0, \quad \Psi(x) = ax, \ \ a \neq 0. \tag{59}$$

Up to a dilatation, we distinguish three canonical cases for Φ:

$$\Phi(x) = 1 \ , \quad \Phi(x) = x(x^2-1) \ , \quad \Phi(x) = x^2.$$

Any so-called canonical situation will be denoted by $\hat{u}$.

1. $\Phi(x) = 1$. It is possible to choose $a = 2$ by the dilatation $h_{\sqrt{\frac{2}{a}}}$, then (17) becomes

$$T_\mu\left(\hat{u}\right) + 2x\hat{u} = 0 \tag{60}$$

which is equivalent to

$$(x\hat{u})' + \left(2x^2 - (2\mu+1)\right)\hat{u} = 0. \tag{61}$$

In fact, multiplying (60) by x, we obtain (61) by taking into account (8) and the fact $H_{-1}(x\hat{u}) = 0$. Conversely, multiplying (61) by x^{-1} and using (1), we obtain (60) since $\langle T_\mu\left(\hat{u}\right) + 2x\hat{u}, 1\rangle = 0$ and $H_{-1}(x\hat{u}) = 0$. In other word, from (61), we have the moments $(\hat{u})_n, n \geq 0$ satisfy

$$2(\hat{u})_{n+2} = (n+2\mu+1)(\hat{u})_n, \quad n \geq 0,$$

and the set of solutions is a 1-dimensional linear space since $\hat{u}$ is symmetric. Hence, in this case $\hat{u} = \mathcal{H}(\mu)$ by virtue of (51).

2. $\Phi(x) = x^2 - 1$. In this case, putting $a = -2(\alpha+1), \alpha \neq 1$, we get

$$T_\mu\left((x^2-1)\hat{u}\right) - 2(\alpha+1)x\hat{u} = 0. \tag{62}$$

Since $H_{-1}(x(x^2-1)\hat{u}) = 0$, by applying the same process as we did in the first case, we prove that (62) is equivalent to

$$\left(x(x^2-1)\hat{u}\right)' + \left((-2\alpha-2\mu-3)x^2 + (2\mu+1)\right)\hat{u} = 0$$

And, we deduce that in this case $\hat{u} = \mathcal{G}\left(\alpha, \mu - \frac{1}{2}\right)$ by comparing the last equation with (53).

3. $\Phi(x) = x^2$. This case is impossible. Indeed, (17) becomes $T_\mu\left(x^2\hat{u}\right) + ax\hat{u} = 0$ which leads to

$$\left\langle T_\mu\left(x^2\hat{u}\right) + ax\hat{u}, x^{2n+1}\right\rangle = 0, \quad n \geq 0$$

this gives $\left(a - (2n+1+2\mu)\right)(\hat{u})_{2n+2} = 0$. Then we deduce that $(\hat{u})_2 = \frac{a}{(1+2\mu)}$ and $(\hat{u})_{2n+2} = 0, n \geq 1$ which is a contradiction with the regularity of $\hat{u}$. □

Now, we assume that u is a symmetric T_μ-semiclassical form of class one. By virtue of Proposition 3.4 and Proposition 3.5, it follows that u satisfies (17) with

$$\Phi(x) = c_3x^3 + c_1x, \quad \Psi(x) = a_2x^2 + a_0 \tag{63}$$

Up to a dilatation, we distinguish three canonical cases for Φ:

$$\Phi(x) = x\ , \quad \Phi(x) = x(x^2-1)\ , \quad \Phi(x) = x^3$$

Theorem 5.2. *On certain regularity conditions, the following canonical cases arise:*
a) First case: $\Phi(x) = x$. The generalized Hermite form.

$$T_\mu\left(x\mathcal{H}(\alpha)\right) + \left(2x^2 - (2\alpha+1)\right)\mathcal{H}(\alpha) = 0. \tag{64}$$

• $\alpha \neq \mu$: $\mathcal{H}(\alpha)$ *is T_μ-semiclassical form of class $s = 1$.*
• $\alpha = \mu$: $\mathcal{H}(\mu)$ *is T_μ-classical form .*
b) Second case: $\Phi(x) = x(x^2-1)$. The generalized Gegenbauer form

$$T_\mu\left(x(x^2-1)\mathcal{G}(\alpha,\beta)\right) + \left(-2(\alpha+\beta+2)x^2 + 2(\beta+1)\right)\mathcal{G}(\alpha,\beta) = 0. \tag{65}$$

• $\beta \neq \mu - \frac{1}{2}$: $\mathcal{G}(\alpha,\beta)$ *is T_μ-semiclassical form of class $s = 1$.*
• $\beta = \mu - \frac{1}{2}$: $\mathcal{G}(\alpha,\beta)$ *is T_μ-classical form .*
c) Third case: $\Phi(x) = x^3$. Symmetrized Bessel form

$$T_\mu\left(x^3\mathcal{B}(\alpha)\right) - \left(2(\alpha+1)x^2 + \frac{1}{2}\right)\mathcal{B}(\alpha) = 0. \tag{66}$$

For every α : $\mathcal{B}(\alpha)$ is T_μ-semiclassical form of class $s = 1$.

Proof. According to (17) and (63), a T_μ-semiclassical symmetric form u of class $s = 1$ satisfies

$$T_\mu\left(\left(c_3x^3 + c_1x\right)u\right) + \left(a_2x^2 + a_0\right)\right)u = 0. \tag{67}$$

But $H_{-1}\left(\left(c_3x^3 + c_1x\right)u\right) = 0$ then (67) is equivalent to

$$\left(\left(c_3x^3 + c_1x\right)u\right)' + \left(a_2x^2 + a_0\right)\right)u = 0. \tag{68}$$

So, the T_μ-semiclassical symmetric forms u of class $s = 1$ are among the semiclassical symmetric forms u of class $s \leq 1$. Now, we are able to prove the different canonical cases
a) First case: $\Phi(x) = x$. In this case, according to (51), we deduce that $\Psi(x) = 2x^2 - (2\alpha+1)$. Then, we have $\left(T_\mu\Phi\right)(0) + \Psi(0) = 2(\mu - \alpha)$ and $\left\langle v, \theta_0\Psi + \theta_0^2\Phi - 2\mu\theta_0 o\theta_0\Phi\right\rangle = 0$. Then, using the standard criterion (38), we obtain the two different cases:

• $\alpha \neq \mu$: $\mathcal{H}(\alpha)$ is T_μ-semiclassical form of class $s=1$.
• $\alpha = \mu$, then it is possible to simplify by x. We obtain

$$T_\mu\left(\mathcal{H}(\mu)\right)+2x\mathcal{H}(\mu)=0.$$

Therefore , $\mathcal{H}(\mu)$ is T_μ-semiclassical form of class $s=0$ (e.i T_μ-classical form). In the two other cases, we are going to proceed with the same stages and techniques.

b) Second case: $\Phi(x)=x(x^2-1)$. In this case, we have
$\Psi(x)=-2(\alpha+\beta+2)x^2+2(\beta+1)$. Then, we have

$$\begin{cases} \left(T_\mu\Phi\right)(0)+\Psi(0)=2(\beta-\mu)+1, \quad \left\langle v,\theta_0\Psi+\theta_0^2\Phi-2\mu\theta_0 o\theta_0\Phi\right\rangle=0 \\ \left(T_\mu\Phi\right)(1)+\Psi(1)=2, \\ \left(T_\mu\Phi\right)(-1)+\Psi(-1)=2. \end{cases}$$

Then, using the standard criterion (38), we obtain the two different cases:
• $\beta \neq \mu-\frac{1}{2}$: $\mathcal{G}(\alpha,\beta)$ is T_μ-semiclassical form of class $s=1$.
• $\beta = \mu-\frac{1}{2}$: $\mathcal{G}(\alpha,\beta)$ is T_μ-classical form .

c) Third case: $\Phi(x)=x^3$. In this case, we have $\Psi(x)=2(\alpha+1)x^2+\frac{1}{2}$. Then, we have $\left(T_\mu\Phi\right)(0)+\Psi(0)=\frac{1}{2}$. Then, according to standard criterion (38), $\mathcal{B}(\alpha)$ is T_μ-semiclassical form of class $s=1$ for every α. □

5.2. An example of symmetric Dunkl-semiclassical orthogonal polynomials of class $s=2$

Let us consider the symmetric form u defined by

$$u=-\frac{2\lambda}{2\alpha+1}\mathcal{H}(\alpha)+\left(1+\frac{2\lambda}{2\alpha+1}\right)\delta_0, \quad \lambda\neq 0. \tag{69}$$

This form is regular for all complex numbers λ; except in a discrete set (see [14]). Indeed, the MOPS corresponding to u, which we denote by $\{\tilde{P}_n\}_{n\geq 0}$, satisfies the recurrence (7) with

$$\tilde{\beta}_n=0, \quad \tilde{\gamma}_1=-\lambda\ , \ \tilde{\gamma}_{2n+2}=\frac{(n+\alpha+\frac{3}{2})x_{n+1}}{x_n}\ , \ \tilde{\gamma}_{2n+3}=\frac{(n+1)x_n}{x_{n+1}}\ , \quad n\geq 0\ ,$$

where

$$x_n=(\alpha+\lambda+\frac{1}{2})-\frac{\lambda\Gamma(\alpha+\frac{3}{2})\Gamma(n+1)}{\Gamma(n+\alpha+\frac{3}{2})}\ , \quad n\geq 0\ .$$

The form u is regular for every λ such that $x_n\neq 0$, $n\geq 0$.
It is clear to see that (69) is equivalent to

$$xu=-\frac{2\lambda}{2\alpha+1}x\mathcal{H}(\alpha). \tag{70}$$

Multiplying (64) by x and using (12) and the fact that $\mathcal{H}(\alpha)$ is symmetric, we get

$$T_\mu\left(x^2\mathcal{H}(\alpha)\right)+\left(2x^3-2(\alpha-\mu+1)x\right)\mathcal{H}(\alpha)=0.$$

Taking into account (70) and simplifying by $-\frac{2\lambda}{2\alpha+1}$, the last equation becomes

$$T_\mu\left(x^2u\right)+\left(2x^3-2(\alpha-\mu+1)x\right)u=0. \tag{71}$$

Then, u is a Dunkl-semiclassical form verifying (17) with $\Phi(x)=x^2$ and $\Psi(x)=2x^3-2(\alpha-\mu+1)x$.
Here, we have
$\left(T_\mu\Phi\right)(0)+\Psi(0)=0$ and $\left\langle v,\theta_0\Psi+\theta_0^2\Phi-2\mu\theta_0 o\theta_0\Phi\right\rangle=-2\lambda-2\alpha-1$.
Then, using the standard criterion (38), we obtain the two different cases:
- $\lambda\neq-\frac{2\alpha+1}{2}$: u is Dunkl-semiclassical form of class $s=2$.
- $\lambda=-\frac{2\alpha+1}{2}$: $u=\mathcal{H}(\alpha)$ is Dunkl-semiclassical form of class $s\leq 1$(see Theorem 5.2).

Remark 5.3. When we take $\mu=0$ in any results of this paper, we obtain well known ones in the semiclassical case.

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