LE MATEMATICHE
Vol. LXXII (2017) – Fasc. I, pp. 123–135
doi: 10.4418/2017.72.1.9

# 2-ABSORBING AND STRONGLY 2-ABSORBING SECONDARY SUBMODULES OF MODULES

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In this paper, we will introduce the concept of 2-absorbing (resp. strongly 2-absorbing) secondary submodules of modules over a commutative ring as a generalization of secondary modules and investigate some basic properties of these classes of modules.

## 1. Introduction

Throughout this paper, $R$ will denote a commutative ring with identity and $\mathbb{Z}$ will denote the ring of integers.

Let $M$ be an $R$-module. A proper submodule $P$ of $M$ is said to be *prime* if for any $r \in R$ and $m \in M$ with $rm \in P$, we have $m \in P$ or $r \in (P :_R M)$ [13]. Let $N$ be a proper submodule of $M$. Then the *$M$-radical* of $N$, denoted by $M\text{-}rad(N)$, is defined to be the intersection of all prime submodules of $M$ containing $N$. If $M$ has no prime submodule containing $N$, then the $M$-radical of $N$ is defined to be $M$ [16]. A non-zero submodule $S$ of $M$ is said to be *second* if for each $a \in R$, the homomorphism $S \xrightarrow{a} S$ is either surjective or zero [20]. In this case $Ann_R(S)$ is a prime ideal of $R$.

The notion of 2- absorbing ideals as a generalization of prime ideals was introduced and studied in [8]. A proper ideal $I$ of $R$ is a *2-absorbing ideal* of

Entrato in redazione: 0 0

*AMS 2010 Subject Classification:* 13C13, 13C99

*Keywords:* Secondary, 2-absorbing secondary, strongly 2-absorbing secondary, second radical

This research was in part supported by a grant from IPM (No. 94130048)

$R$ if whenever $a,b,c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. It has been proved that $I$ is a 2-absorbing ideal of $R$ if and only if whenever $I_1$, $I_2$, and $I_3$ are ideals of $R$ with $I_1I_2I_3 \subseteq I$, then $I_1I_2 \subseteq I$ or $I_1I_3 \subseteq I$ or $I_2I_3 \subseteq I$ [8]. In [9], the authors introduced the concept of 2-absorbing primary ideal which is a generalization of primary ideal. A proper ideal $I$ of $R$ is called a *2-absorbing primary ideal* of $R$ if whenever $a,b,c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.

The notion of 2-absorbing ideals was extended to 2-absorbing submodules in [12]. A proper submodule $N$ of $M$ is called a *2-absorbing submodule* of $M$ if whenever $abm \in N$ for some $a,b \in R$ and $m \in M$, then $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$.

In [5], the present authors introduced the dual notion of 2-absorbing submodules (that is, *2-absorbing (resp. strongly 2-absorbing) second submodules*) of $M$ and investigated some properties of these classes of modules. A non-zero submodule $N$ of $M$ is said to be a *2-absorbing second submodule of* $M$ if whenever $a,b \in R$, $L$ is a completely irreducible submodule of $M$, and $abN \subseteq L$, then $aN \subseteq L$ or $bN \subseteq L$ or $ab \in Ann_R(N)$. A non-zero submodule $N$ of $M$ is said to be a *strongly 2-absorbing second submodule of* $M$ if whenever $a,b \in R$, $K$ is a submodule of $M$, and $abN \subseteq K$, then $aN \subseteq K$ or $bN \subseteq K$ or $ab \in Ann_R(N)$.

In [18], the authors introduced the notion of 2-absorbing primary submodules as a generalization of 2-absorbing primary ideals of rings and studied some properties of this class of modules. A proper submodule $N$ of $M$ is said to be a *2-absorbing primary submodule* of $M$ if whenever $a,b \in R$, $m \in M$, and $abm \in N$, then $am \in M\text{-}rad(N)$ or $bm \in M\text{-}rad(N)$ or $ab \in (N :_R M)$.

The purpose of this paper is to introduce the concepts of 2-absorbing and strongly 2-absorbing secondary submodules of an $R$-module $M$ as dual notion of 2-absorbing primary submodules and obtain some related results.

## 2. Main results

Let $M$ be an $R$-module. For a submodule $N$ of $M$ the the *second radical* (or second socle) of $N$ is defined as the sum of all second submodules of $M$ contained in $N$ and it is denoted by $sec(N)$ (or $soc(N)$). In case $N$ does not contain any second submodule, the second radical of $N$ is defined to be $(0)$. $N \neq 0$ is said to be a *second radical submodule of* $M$ if $sec(N) = N$ (see [11] and [2]).

A proper submodule $N$ of $M$ is said to be *completely irreducible* if $N = \bigcap_{i\in I} N_i$, where $\{N_i\}_{i\in I}$ is a family of submodules of $M$, implies that $N = N_i$ for some $i \in I$. It is easy to see that every submodule of $M$ is an intersection of completely irreducible submodules of $M$ [14].

We frequently use the following basic fact without further comment.

**Remark 2.1.** Let $N$ and $K$ be two submodules of an $R$-module $M$. To prove $N \subseteq K$, it is enough to show that if $L$ is a completely irreducible submodule of $M$ such that $K \subseteq L$, then $N \subseteq L$.

**Definition 2.2.** We say that a non-zero submodule $N$ of an $R$-module $M$ is a *2-absorbing secondary submodule* of $M$ if whenever $a, b \in R$, $L$ is a completely irreducible submodule of $M$ and $abN \subseteq L$, then $a(sec(N)) \subseteq L$ or $b(sec(N)) \subseteq L$ or $ab \in Ann_R(N)$. By a *2-absorbing secondary module*, we mean a module which is a 2-absorbing secondary submodule of itself.

**Example 2.3.** Clearly, every submodule of the $\mathbb{Z}$-module $\mathbb{Z}$ is not secondary. But as $sec(\mathbb{Z}) = 0$, every submodule of the $\mathbb{Z}$-module $\mathbb{Z}$ is 2-absorbing secondary.

**Lemma 2.4.** *Let $I$ be an ideal of $R$ and $N$ be a 2-absorbing secondary submodule of $M$. If $a \in R$, $L$ is a completely irreducible submodule of $M$, and $IaN \subseteq L$, then $a(sec(N)) \subseteq L$ or $I(sec(N)) \subseteq L$ or $Ia \in Ann_R(N)$.*

*Proof.* Let $a(sec(N)) \not\subseteq L$ and $Ia \notin Ann_R(N)$. Then there exists $b \in I$ such that $abN \neq 0$. Now as $N$ is a 2-absorbing secondary submodule of $M$, $baN \subseteq L$ implies that $b(sec(N)) \subseteq L$. We show that $I(sec(N)) \subseteq L$. To see this, let $c$ be an arbitrary element of $I$. Then $(b+c)aN \subseteq L$. Hence, either $(b+c)(sec(N)) \subseteq L$ or $(b+c)a \in Ann_R(N)$. If $(b+c)(sec(N)) \subseteq L$, then since $b(sec(N)) \subseteq L$ we have $c(sec(N)) \subseteq L$. If $(b+c)a \in Ann_R(N)$, then $ca \notin Ann_R(N)$. Thus $caN \subseteq L$ implies that $c(sec(N)) \subseteq L$. Hence, we conclude that $I(sec(N) \subseteq L$. □

**Theorem 2.5.** *Let $I$ and $J$ be two ideals of $R$ and $N$ be a 2-absorbing secondary submodule of an $R$-module $M$. If $L$ is a completely irreducible submodule of $M$ and $IJN \subseteq L$, then $I(sec(N)) \subseteq L$ or $J(sec(N)) \subseteq L$ or $IJ \subseteq Ann_R(N)$.*

*Proof.* Let $I(sec(N)) \not\subseteq L$ and $J(sec(N)) \not\subseteq L$. We show that $IJ \subseteq Ann_R(N)$. Assume that $c \in I$ and $d \in J$. By assumption, there exists $a \in I$ such that $a(sec(N)) \not\subseteq L$ but $aJN \subseteq L$. Now Lemma 2.4 shows that $aJ \subseteq Ann_R(N)$ and so $(I \setminus (L :_R sec(N)))J \subseteq Ann_R(N)$. Similarly, there exists $b \in (J \setminus (L :_R sec(N)))$ such that $Ib \subseteq Ann_R(N)$ and also $I(J \setminus (L :_R sec(N))) \subseteq Ann_R(N)$. Thus we have $ab \in Ann_R(N)$, $ad \in Ann_R(N)$, and $cb \in Ann_R(N)$. As $a+c \in I$ and $b+d \in J$, we have $(a+c)(b+d)N \subseteq L$. Therefore, $(a+c)(sec(N)) \subseteq L$ or $(b+d)(sec(N)) \subseteq L$ or $(a+c)(b+d) \in Ann_R(N)$. If $(a+c)(sec(N)) \subseteq L$, then $c(sec(N)) \not\subseteq L$. Hence $c \in I \setminus (L :_R sec(N))$ which implies that $cd \in Ann_R(N)$. Similarly, if $(b+d)(sec(N)) \subseteq L$, we can deduce that $cd \in Ann_R(N)$. Finally if $(a+c)(b+d) \in Ann_R(N)$, then $ab+ad+cb+cd \in Ann_R(N)$ so that $cd \in Ann_R(N)$. Therefore, $IJ \subseteq Ann_R(N)$. □

**Theorem 2.6.** *Let $N$ be a non-zero submodule of an $R$-module $M$. The following statements are equivalent:*

*(a) If $abN \subseteq L_1 \cap L_2$ for some $a, b \in R$ and completely irreducible submodules $L_1, L_2$ of $M$, then $a(sec(N)) \subseteq L_1 \cap L_2$ or $b(sec(N)) \subseteq L_1 \cap L_2$ or $ab \in Ann_R(N)$;*

*(b) If $IJN \subseteq K$ for some ideals $I, J$ of $R$ and a submodule $K$ of $M$, then $I(sec(N)) \subseteq K$ or $J(sec(N)) \subseteq K$ or $IJ \in Ann_R(N)$;*

*(c) For each $a, b \in R$, we have $a(sec(N)) \subseteq abN$ or $b(sec(N)) \subseteq abN$ or $abN = 0$.*

*Proof.* $(a) \Rightarrow (b)$. Assume that $IJN \subseteq K$ for some ideals $I, J$ of $R$, a submodule $K$ of $M$, and $IJ \not\subseteq Ann_R(N)$. Then by Theorem 2.5, for all completely irreducible submodules $L$ of $M$ with $K \subseteq L$ either $I(sec(N)) \subseteq L$ or $J(sec(N)) \subseteq L$. If $I(sec(N)) \subseteq L$ (resp. $J(sec(N)) \subseteq L$) for all completely irreducible submodules $L$ of $M$ with $K \subseteq L$, we are done. Now suppose that $L_1$ and $L_2$ are two completely irreducible submodules of $M$ with $K \subseteq L_1$, $K \subseteq L_2$, $I(sec(N)) \not\subseteq L_1$, and $J(sec(N)) \not\subseteq L_2$. Then $I(sec(N)) \subseteq L_2$ and $J(sec(N)) \subseteq L_1$. Since $IJN \subseteq L_1 \cap L_2$, we have either $I(sec(N)) \subseteq L_1 \cap L_2$ or $J(sec(N)) \subseteq L_1 \cap L_2$. If $I(sec(N)) \subseteq L_1 \cap L_2$, then $I(sec(N)) \subseteq L_1$ which is a contradiction. Similarly from $J(sec(N)) \subseteq L_1 \cap L_2$ we get a contradiction.

$(b) \Rightarrow (a)$. This is clear

$(a) \Rightarrow (c)$. By part (a), $N \neq 0$. Let $a, b \in R$. Then $abN \subseteq abN$ implies that $a(sec(N)) \subseteq abN$ or $b(sec(N)) \subseteq abN$ or $abN = 0$.

$(c) \Rightarrow (a)$. This is clear. □

**Definition 2.7.** We say that a non-zero submodule $N$ of an $R$-module $M$ is a *strongly 2-absorbing secondary submodule* of $M$ if satisfies the equivalent conditions of Theorem 2.6. By a *strongly 2-absorbing secondary module*, we mean a module which is a strongly 2-absorbing secondary submodule of itself.

Let $N$ be a submodule of an $R$-module $M$. Then part (d) of Theorem 2.6 shows that $N$ is a strongly 2-absorbing secondary submodule of $M$ if and only if $N$ is a strongly 2-absorbing secondary module.

**Example 2.8.** Clearly every strongly 2-absorbing secondary submodule is a 2-absorbing secondary submodule. But the converse is not true in general. For example, consider $M = \mathbb{Z}_6 \oplus \mathbb{Q}$ as a $\mathbb{Z}$-module. Then $M$ is a 2-absorbing secondary module. But since $0 \neq 6M \subseteq 0 \oplus \mathbb{Q}$, $sec(M) = M$, $2M \not\subseteq 0 \oplus \mathbb{Q}$, and $3M \not\subseteq 0 \oplus \mathbb{Q}$, $M$ is not a strongly 2-absorbing secondary module.

**Proposition 2.9.** *Let $N$ be a 2-absorbing second submodule of an $R$-module $M$. Then $N$ is a strongly 2-absorbing secondary submodule of $M$.*

*Proof.* Let $a,b \in R$ and $K$ be a submodule of $M$ such that $abN \subseteq K$. Then $aN \subseteq K$ or $bN \subseteq K$ or $abN = 0$ by assumption. Thus $a(sec(N)) \subseteq aN \subseteq K$ or $b(sec(N)) \subseteq aN \subseteq K$ or $abN = 0$, as required. □

The following example shows that the converse of the Proposition 2.9 is not true in general.

**Example 2.10.** Let $M$ be the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$. Then as $p^2\langle 1/p^3+\mathbb{Z}\rangle \subseteq \langle 1/p+\mathbb{Z}\rangle$, $p\langle 1/p^3+\mathbb{Z}\rangle \not\subseteq \langle 1/p+\mathbb{Z}\rangle$, and $p^2\langle 1/p^3+\mathbb{Z}\rangle \neq 0$, we have the submodule $\langle 1/p^3+\mathbb{Z}\rangle$ of $\mathbb{Z}_{p^\infty}$ is not 2-absorbing second submodule. But $sec(\langle 1/p^3+\mathbb{Z}\rangle) = \langle 1/p+\mathbb{Z}\rangle$ implies that $\langle 1/p^3+\mathbb{Z}\rangle$ is a strongly 2-absorbing secondary submodule of $M$.

An $R$-module $M$ is said to be a *comultiplication module* if for every submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $N = (0 :_M I)$, equivalently, for each submodule $N$ of $M$, we have $N = (0 :_M Ann_R(N))$ [1].

**Theorem 2.11.** *Let $M$ be a finitely generated comultiplication $R$-module. If $N$ is a strongly 2-absorbing secondary submodule of $M$, then $Ann_R(N)$ is a 2-absorbing primary ideal of $R$.*

*Proof.* Let $a,b,c \in R$ be such that $abc \in Ann_R(N)$, $ac \notin \sqrt{Ann_R(N)}$, and $bc \notin \sqrt{Ann_R(N)}$. Since by [4, 2.12], $Ann_R(sec(N)) = \sqrt{Ann_R(N)}$, there exist completely irreducible submodules $L_1$ and $L_2$ of $M$ such that $ac(sec(N)) \not\subseteq L_1$ and $bc(sec(N)) \not\subseteq L_2$. But $abcN = 0 \subseteq L_1 \cap L_2$ implies that $abN \subseteq (L_1 \cap L_2 :_M c)$. Now as $N$ is a strongly 2-absorbing secondary submodule of $M$, $a(sec(N)) \subseteq (L_1 \cap L_2 :_M c)$ or $b(sec(N)) \subseteq (L_1 \cap L_2 :_M c)$ or $abN = 0$. If $a(sec(N)) \subseteq (L_1 \cap L_2 :_M c)$ (resp. $b(sec(N)) \subseteq (L_1 \cap L_2 :_M c)$), then we have $ac(sec(N)) \subseteq L_1$ (resp. $bc(sec(N)) \subseteq L_2$), a contradiction. Hence $abN = 0$, as needed. □

**Theorem 2.12.** *Let $N$ be a submodule of a comultiplication $R$-module $M$. If $Ann_R(N)$ is a 2-absorbing primary ideal of $R$, then $N$ is a strongly 2-absorbing secondary submodule of $M$.*

*Proof.* Let $abN \subseteq K$ for some $a,b \in R$ and some submodule $K$ of $M$. As $M$ is a comultiplication module, there exists an ideal $I$ of $R$ such that $K = (0 :_M I)$. Hence $Iab \subseteq Ann_R(N)$ which implies that either $Ia \subseteq \sqrt{Ann_R(N)}$ or $Ib \subseteq \sqrt{Ann_R(N)}$ or $ab \in Ann_R(N)$. If $ab \in Ann_R(N)$, we are done. If $Ia \subseteq \sqrt{Ann_R(N)}$, as $\sqrt{Ann_R(N)} \subseteq Ann_R(sec(N))$, we have $Ia(sec(N)) = 0$. This implies that $a(sec(N)) \subseteq K$ because $M$ is a comultiplication module. Similarly, if $Ib \subseteq \sqrt{Ann_R(N)}$, we get $b(sec(N)) \subseteq K$. This completes the proof. □

The following example shows that Theorem 2.12 is not satisfied in general.

**Example 2.13.** Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_p \oplus \mathbb{Z}_q \oplus \mathbb{Q}$, where $p \neq q$ are two prime numbers. Then $M$ is not a comultiplication $\mathbb{Z}$-module and $Ann_{\mathbb{Z}}(M) = 0$ is a 2-absorbing primary ideal of $R$. But since $0 \neq pqM \subseteq 0 \oplus 0 \oplus \mathbb{Q}$, $sec(M) = M$, $pM \not\subseteq 0 \oplus 0 \oplus \mathbb{Q}$, and $qM \not\subseteq 0 \oplus 0 \oplus \mathbb{Q}$, $M$ is not a strongly 2-absorbing secondary module.

In [18, 2.6], it is shown that, if $M$ is a finitely generated multiplication $R$-module and $N$ is a 2-absorbing primary submodule of $M$, then $M$-$rad(N)$ is a 2-absorbing submodule of $M$. In the following lemma, we see that some of this conditions are redundant.

**Lemma 2.14.** *Let $N$ be a 2-absorbing primary submodule of an $R$-module $M$. Then $M$-$rad(N)$ is a 2-absorbing submodule of $M$.*

*Proof.* This follows from the fact that $M$-$rad(M$-$rad(N)) = M$-$rad(N)$ by [15, Proposition 2]. □

**Proposition 2.15.** *Let $M$ be an $R$-module. Then we have the following.*

*(a) If $N$ is a 2-absorbing (resp. strongly 2-absorbing) secondary submodule of an $R$-module $M$, then $sec(N)$ is a 2-absorbing (resp. strongly 2-absorbing) second submodule of $M$.*

*(b) If $N$ is a second radical submodule of $M$, then $N$ is a 2-absorbing (resp. strongly 2-absorbing) second submodule if and only if $N$ is a 2-absorbing (resp. strongly 2-absorbing) secondary submodule.*

*Proof.* (a) This follows from the fact that $sec(sec(N)) = sec(N)$ by [4, 2.1].
(b) This follows from part (a) □

Let $N$ and $K$ be two submodules of an $R$-module $M$. The *coproduct* of $N$ and $K$ is defined by $(0 :_M Ann_R(N)Ann_R(K))$ and denoted by $C(NK)$ [6].

**Theorem 2.16.** *Let $N$ be a submodule of an $R$-module $M$ such that $sec(N)$ is a second submodule of $M$. Then we have the following.*

*(a) $N$ is a strongly 2-absorbing secondary submodule of $M$.*

*(b) If $M$ is a comultiplication $R$-module, then $C(N^t)$ is a strongly 2-absorbing secondary submodule of $M$ for every positive integer $t \geq 1$, where $C(N^t)$ means the coproduct of $N$, $t$ times.*

*Proof.* (a) Let $a,b \in R$, $K$ be a submodule of $M$ such that $abN \subseteq K$, and let $b(sec(N)) \not\subseteq K$. Then as $sec(N)$ is a second submodule and $a(sec(N)) \subseteq aN \subseteq (K :_M b)$, we have $a(sec(N)) = 0$ by [3, 2.10]. Thus $a(sec(N)) \subseteq K$, as needed.

(b) Let $M$ be a comultiplication $R$-module. Then there exists an ideal $I$ of $R$ such that $N = (0 :_M I)$. Thus by [4, 2.1],

$$sec(c(N^t)) = sec((0 :_M I^t)) = sec((0 :_M I)) = sec(N).$$

Now the results follows from to the proof of part (a). □

**Theorem 2.17.** *Let $M$ be a comultiplication $R$-module. Then we have the following.*

*(a) If $N_1, N_2, ..., N_n$ are strongly 2-absorbing secondary submodules of $M$ with the same second radical, then $N = \sum_{i=1}^n N_i$ is a strongly 2-absorbing secondary submodule of $M$.*

*(b) If $N_1, N_2, ..., N_n$ are 2-absorbing secondary submodules of $M$ with the same second radical, then $N = \sum_{i=1}^n N_i$ is a 2-absorbing secondary submodule of $M$.*

*(c) If $N_1$ and $N_2$ are two secondary submodules of $M$, then $N_1 + N_2$ is a strongly 2-absorbing secondary submodule of $M$.*

*(d) If $M$ is finitely generated, $N$ is a submodule of $M$ which possess a secondary representation, and $sec(N) = K_1 + K_2$, where $K_1$ and $K_2$ are two minimal submodules of $M$, then $N$ is a strongly 2-absorbing secondary submodule of $M$.*

*Proof.* (a) Let $a,b \in R$ and $K$ be a submodule of $M$ such that $abN \subseteq K$. Thus for each $i = 1,2,...,n$, $abN_i \subseteq K$. If there exists $1 \leq j \leq n$ such that $a(sec(N_j)) \subseteq K$ or $b(sec(N_j)) \subseteq K$, then $a(sec(N)) \subseteq K$ or $b(sec(N)) \subseteq K$ (note that $sec(N) = sec(\sum_{i=1}^n N_i) = \sum_{i=1}^n sec(N_i) = sec(N_i)$ by [11, 2.6]). Otherwise, $abN_i = 0$ for each $i = 1,2,...,n$. Hence $abN = 0$, as desired.

(b) The proof is similar to the part (a).

(c) As $N_1$ and $N_2$ are secondary submodules of $M$, $Ann_R(N_1)$ and $Ann_R(N_2)$ are primary ideals of $R$. Hence $Ann_R(N_1 + N_2) = Ann_R(N_1) \cap Ann_R(N_2)$ is a 2-absorbing primary ideal of $R$ by [9, 2.4]. Thus by Theorem 2.12, $N_1 + N_2$ is a strongly 2-absorbing secondary submodule of $M$.

(d) Let $N = \sum_{i=1}^n N_i$ be a secondary representation. By [4, 2.6], $sec(N) = \sum_{i=1}^n sec(N_i)$. Since $sec(N_i)$'s are second submodules of $M$ by [4, 2.13], we have

$$\{sec(N_1), sec(N_2), ..., sec(N_n)\} = \{K_1, K_2\}.$$

Without loss of generality, we may assume that for some $1 \leq t < n$,

$$\{sec(N_1), ..., sec(N_t)\} = \{K_1\}$$

and $\{sec(N_{t+1}), ..., sec(N_n)\} = \{K_2\}$. Set $H_1 := N_1 + ... + N_t$ and $H_2 := N_{t+1} + ... N_n$. By [4, 2.12], $H_1$ and $H_2$ are secondary submodules of $M$. Therefore, by part (c), $N = H_1 + H_2$ is a strongly 2-absorbing secondary submodule of $M$. □

The following example shows that the direct sum of two strongly 2-absorbing secondary $R$-modules is not a strongly 2-absorbing secondary $R$-module in general.

**Example 2.18.** Clearly, the $\mathbb{Z}$-modules $\mathbb{Z}_6$ and $\mathbb{Z}_{10}$ are strongly 2-absorbing secondary $\mathbb{Z}$-modules. Let $M = \mathbb{Z}_6 \oplus \mathbb{Z}_{10}$. Then $M$ is not a strongly 2-absorbing second $\mathbb{Z}$-module. By [3, 2.1], $sec(M) = M$. Thus $M$ is not a strongly 2-absorbing secondary $\mathbb{Z}$-module by Proposition 2.15.

**Lemma 2.19.** *Let $f : M \to \acute{M}$ be a monomorphism of $R$-modules. Then we have the following.*

*(a) If $N$ is a submodule of $M$, then $sec(f(N)) = f(sec(N))$.*

*(b) If $\acute{N}$ is a submodule of $\acute{M}$ such that $\acute{N} \subseteq f(M)$, then $sec(f^{-1}(\acute{N})) = f^{-1}(sec(\acute{N}))$.*

*Proof.* (a) Let $\acute{S}$ be a second submodule of $f(N)$. Then one can see that $f^{-1}(\acute{S})$ is a second submodule of $N$. Hence $f(f^{-1}(\acute{S})) \subseteq f(sec(N))$. Thus $sec(f(N)) \subseteq f(sec(N))$. The reverse inclusion is clear.

(b) Let $S$ be a second submodule of $f^{-1}(\acute{N})$. Then one can see that $f(S)$ is a second submodule of $\acute{N}$. Hence $f^{-1}(S) \subseteq f^{-1}(sec(\acute{N}))$. Thus $sec(f^{-1}(\acute{N})) \subseteq f^{-1}(sec(\acute{N}))$. To see the reverse inclusion, let $\acute{S}$ be a second submodule of $\acute{N}$. Then $f^{-1}(\acute{S})$ is a second submodule of $f^{-1}(\acute{N})$. It follows that $f^{-1}(sec(\acute{N})) \subseteq sec(f^{-1}(\acute{N}))$. □

**Theorem 2.20.** *Let $f : M \to \acute{M}$ be a monomorphism of $R$-modules. Then we have the following.*

*(a) If $N$ is a strongly 2-absorbing secondary submodule of $M$, then $f(N)$ is a strongly 2-absorbing secondary submodule of $\acute{M}$.*

*(b) If $\acute{N}$ is a strongly 2-absorbing secondary submodule of $\acute{M}$ and $\acute{N} \subseteq f(M)$, then $f^{-1}(\acute{N})$ is a strongly 2-absorbing secondary submodule of $M$.*

*Proof.* (a) Since $N \neq 0$ and $f$ is a monomorphism, we have $f(N) \neq 0$. Let $a, b \in R$, $\acute{K}$ be a submodule of $\acute{M}$, and $abf(N) \subseteq \acute{K}$. Then $abN \subseteq f^{-1}(\acute{K})$. As $N$ is strongly 2-absorbing secondary submodule, $a(sec(N)) \subseteq f^{-1}(\acute{K})$ or $b(secN)) \subseteq f^{-1}(\acute{K})$ or $abN = 0$. Therefore, by Lemma 2.19 (a),

$$a(sec(f(N))) = a(f(sec(N))) \subseteq f(f^{-1}(\acute{K})) = f(M) \cap \acute{K} \subseteq \acute{K}$$

or

$$b(sec(f(N))) = b(f(sec(N))) \subseteq f(f^{-1}(\acute{K})) = f(M) \cap \acute{K} \subseteq \acute{K}$$

or $abf(N) = 0$, as needed.

(b) If $f^{-1}(\acute{N}) = 0$, then $f(M) \cap \acute{N} = f(f^{-1}(\acute{N})) = f(0) = 0$. Thus $\acute{N} = 0$, a contradiction. Therefore, $f^{-1}(\acute{N}) \neq 0$. Now let $a, b \in R$, $K$ be a submodule of $M$, and $abf^{-1}(\acute{N}) \subseteq K$. Then

$$ab\acute{N} = ab(f(M) \cap \acute{N}) = abff^{-1}(\acute{N}) \subseteq f(K).$$

As $\acute{N}$ is strongly 2-absorbing secondary submodule, we have $a(sec(\acute{N}) \subseteq f(K)$ or $b(sec(\acute{N}) \subseteq f(K)$ or $ab\acute{N} = 0$. Hence by Lemma 2.19 (b),

$$a(sec(f^{-1}(\acute{N}))) = af^{-1}(sec(\acute{N})) \subseteq f^{-1}(f(K)) = K$$

or

$$b(sec(f^{-1}(\acute{N}))) = bf^{-1}(sec(\acute{N})) \subseteq f^{-1}(f(K)) = K$$

or $abf^{-1}(\acute{N}) = 0$, as desired. □

**Corollary 2.21.** *Let $M$ be an $R$-module and let $N \subseteq K$ be two submodules of $M$. Then $N$ is a strongly 2-absorbing secondary submodule of $K$ if and only if $N$ is a strongly 2-absorbing secondary submodule of $M$.*

*Proof.* This follows from Theorem 2.20 by using the natural monomorphism $K \to M$. □

**Proposition 2.22.** *Let $M$ be a cocyclic $R$-module with minimal submodule $K$ and $N$ be a submodule of $M$ such that $rN \neq K$ for each $r \in R$. If $N/K$ is a strongly 2-absorbing secondary submodule of $M/K$, then $N$ is a strongly 2-absorbing secondary submodule of $M$.*

*Proof.* Let $a, b \in R$ and $H$ be a submodule of $M$ such that $abN \subseteq H$. Then $ab(N/K) \subseteq H/K$ implies that $a(sec(N/K)) \subseteq H/K$ or $b(sec(N/K)) \subseteq H/K$ or $ab(N/K) = 0$. If $ab(N/K) = 0$, then $abN = 0$ because $rN \neq K$ for each $r \in R$. Otherwise, since $a(sec(N))/K \subseteq sec(N/K)$, we have $a(sec(N)) \subseteq H$ or $b(sec(N)) \subseteq H$ as required. □

Let $R_i$ be a commutative ring with identity and $M_i$ be an $R_i$-module, for $i = 1,2$. Let $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is an $R$-module and each submodule of $M$ is in the form of $N = N_1 \times N_2$ for some submodules $N_1$ of $M_1$ and $N_2$ of $M_2$. In addition, $M_i$ is a comultiplication $R_i$-module, for $i = 1,2$ if and only if $M$ is a comultiplication $R$-module by [19, 2.1].

**Lemma 2.23.** *Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$, where $M_1$ is an $R_1$-module and $M_2$ is an $R_2$-module. If $N = N_1 \times N_2$ is a submodule of $M$, then we have the following.*

*(a) $N$ is a second submodule of $M$ if and only if $N = S_1 \times 0$ or $N = S_2 \times 0$, where $S_1$ is a second submodule of $N_1$ and $S_2$ is a second submodule of $M_2$.*

*(b) $sec(N) = sec(N_1) \times sec(N_2)$.*

*Proof.* (a) This is straightforward.

(b) This follows from part (a). □

**Theorem 2.24.** *Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$, where $M_1$ is a comultiplication $R_1$-module and $M_2$ is a comultiplication $R_2$-module. Then we have the following.*

*(a) If $M_1$ be a finitely generated $R_1$-module, then a non-zero submodule $K_1$ of $M_1$ is a strongly 2-absorbing secondary submodule if and only if $N = K_1 \times 0$ is a strongly 2-absorbing secondary submodule of $M$.*

*(b) If $M_2$ be a finitely generated $R_2$-module, then a non-zero submodule $K_2$ of $M_2$ is a strongly 2-absorbing secondary submodule if and only if $N = 0 \times K_2$ is a strongly 2-absorbing secondary submodule of $M$.*

*(c) If $K_1$ is a secondary submodule of $M_1$ and $K_2$ is a secondary submodule of $M_2$, then $N = K_1 \times K_2$ is a strongly 2-absorbing secondary submodule of $M$.*

*Proof.* (a) Let $K_1$ be a strongly 2-absorbing secondary submodule of $M_1$. Then $Ann_{R_1}(K_1)$ is a 2-absorbing primary ideal of $R_1$ by Theorem 2.11. Now since $Ann_R(N) = Ann_{R_1}(K_1) \times R_2$, we have $Ann_R(N)$ is a 2-absorbing primary ideal of $R$ by [9, 2.23]. Thus the result follows from Theorem 2.12. Conversely, let $N = K_1 \times 0$ be a strongly 2-absorbing secondary submodule of $M$. Then $Ann_R(N) = Ann_{R_1}(K_1) \times R_2$ is a primary ideal of $R$ by Theorem 2.11. Thus $Ann_{R_1}(K_1)$ is a primary ideal of $R_1$ by [9, 2.23]. Thus by Theorem 2.12, $K_1$ be a strongly 2-absorbing secondary submodule of $M_1$.

(b) We have similar arguments as in part (a).

(c) Let $K_1$ be a secondary submodule of $M_1$ and $K_2$ be a secondary submodule of $M_2$. Then $Ann_{R_1}(K_1)$ and $Ann_{R_2}(K_2)$ are primary ideals of $R_1$ and $R_2$, respectively. Now since $Ann_R(N) = Ann_{R_1}(K_1) \times Ann_{R_2}(K_2)$, we have $Ann_R(N)$ is a 2-absorbing primary ideal of $R$ by [9, 2.23]. Thus the result follows from Theorem 2.12. □

**Lemma 2.25.** *Let $N$ be a submodule of a comultiplication $R$-module $M$. Then $N$ is a secondary module if and only if $Ann_R(N)$ be a primary ideal of $R$.*

*Proof.* The necessity is clear. For converse, let $r \in R$. As $M$ is a comultiplication module, $rN = (0 :_M I)$ for some ideal $I$ of $R$. Now $rI \subseteq Ann_R(N)$ implies that $I \subseteq Ann_R(N)$ or $r^t \in Ann_R(N)$ for some positive integer $t$. Thus as $M$ is a comultiplication $R$-module, $N = rN$ or $r^t N = 0$ for some positive integer $t$. □

**Theorem 2.26.** *Let $R = R_1 \times R_2$ be a decomposable ring and $M = M_1 \times M_2$ be a finitely generated comultiplication $R$-module, where $M_1$ is an $R_1$-module and $M_2$ is an $R_2$-module. Suppose that $N = N_1 \times N_2$ is a non-zero submodule of $M$. Then the following conditions are equivalent:*

*(a) $N$ is a strongly 2-absorbing secondary submodule of $M$;*

*(b) Either $N_1 = 0$ and $N_2$ is a strongly 2-absorbing secondary submodule of $M_2$ or $N_2 = 0$ and $N_1$ is a strongly 2-absorbing secondary submodule of $M_1$ or $N_1$, $N_2$ are secondary submodules of $M_1$, $M_2$, respectively.*

*Proof.* $(a) \Rightarrow (b)$. Let $N = N_1 \times N_2$ be a strongly 2-absorbing secondary submodule of $M$. Then $Ann_R(N) = Ann_{R_1}(N_1) \times Ann_{R_2}(N_2)$ is a 2-absorbing primary ideal of $R$ by Theorem 2.11. By [9, 2.23], we have $Ann_{R_1}(N_1) = R_1$ and $Ann_{R_2}(N_2)$ is a 2-absorbing primary ideal of $R_2$ or $Ann_{R_2}(N_2) = R_2$ and $Ann_{R_1}(N_1)$ is a 2-absorbing primary ideal of $R_1$ or $Ann_{R_1}(N_1)$ and $Ann_{R_2}(N_2)$ are primary ideals of $R_1$ and $R_2$, respectively. Suppose that $Ann_{R_1}(N_1) = R_1$ and $Ann_{R_2}(N_2)$ is a 2-absorbing primary ideal of $R_2$. Then $N_1 = 0$ and $N_2$ is a strongly 2-absorbing secondary submodule of $M_2$ by Theorem 2.12. Similarly, if $Ann_{R_2}(N_2) = R_2$ and $Ann_{R_1}(N_1)$ is a 2-absorbing primary ideal of $R_1$, then $N_2 = 0$ and $N_1$ is a strongly 2-absorbing secondary submodule of $M_1$. If the last case hold, then as $M_1$ (resp. $M_2$) is a comultiplication $R_1$ (resp. $R_2$) module, $N_1$ (resp. $N_2$) is a secondary submodule of $M_1$ (resp. $M_2$) by Lemma 2.25.

$(b) \Rightarrow (a)$. This follows from Theorem 2.25. □

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