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MANIFOLDS WITH INTEGRABLE AFFINE SHAPE OPERATOR

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This work establishes the conditions for the existence of vector fields with the property that theirs covariant derivative, with respect to the affine normal connection, be the affine shape operator$S$ in hypersurfaces. Some results are obtained from this property and, in particular, for some kind of affine decomposable hypersurfaces we explicitely get the actual vector fields.

1. Introduction.

When making an analysis of the affine hypersurfaces with constant Pick invariant that had been studied in [1], [2] we run into the existence of a special type of vector fields V, (tensors of type $(1, 0)$), for which $\nabla V = S$, where ∇ is the connection induced by the affine normal and S is the affine shape operator (the Weingarten operator).

We also noticed that this was possible and obtained explicit expression for the affine hypersurfaces of decomposable type studied in [4].

Is the existence of these vector firlds possible in the general case of affine hypersurfaces? And if this is so, which are the necessary conditions for that to happen?

These matters are treated in the present work, after developing briefly the necessary basic theory and establishing the notation to be used. With this

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purpose, we took into consideration the exposition made in K. Nomizu and T. Sasaki book. [5].

Consider an n-dimensional manifold M of class C^∞ and $F : M \to \mathbb{R}^{n+1}$ an immersion of class C^∞, where we assume the affine space $\mathbb{R}^{n+1}$ endowed with its usual flat connection D and a fixed volume form ω.

The Weingarten's structural equation $D_X\xi = -SX$ establish the existence of a tensor S which is of type (1, 1), called the shape operator and the structural Gauss' equation $D_X Y = \nabla_X Y + h(X, Y)\xi$ give us ∇, the so called normal connection.

Considering the vector fields as tensors of type (1, 0), if we apply them the covariant derivative, they become tensors of type (1, 1).

We begin establishing the basic theory of affine immersions geometry in Section 2.

Section 3 is devoted to establishing the conditions for the existence of the mentioned vector fields.

In Section 4 we give some results derivated from the existence of these vector fields and in the fifth and last section we apply the theory developed for the particular case of affines hypersurfaces of decomposable type with some special properties.

2. Geometry of affine immersions.

Let M be an n-dimensional manifold of class C^∞ and $F : M \to \mathbb{R}^{n+1}$ an immersion of class C^∞. We assume the affine space $\mathbb{R}^{n+1}$ with its usual flat affine connection D and a fixed parallel volume element ω.

A differentiable vector field η it said transversal to $F(M)$ if at each point $p \in M$ and for any referential $(X_1, \ldots, X_n)$ the vectors $(F_*)_p(X_1)$, $(F_*)_p(X_2), \ldots, (F_*)_p(X_n)$, η_p form a basis of $T_{F(p)}(\mathbb{R}^{n+1}) \cong \mathbb{R}^{n+1}$. Obviously, this condition is equivalent to $\omega(X_1, ..., X_n, \eta) \neq 0$. For the sake of simplicity, we shall identify $F_*(X)$ with X for each $X \in I_0^1(M)$.

For an arbitrary transversal vector field η we have the following structures:

A non trivial volumes form θ given by

$$\theta(X_1, ..., X_n) = \omega(X_1, ..., X_n, \eta) \tag{2.1}$$

A tensor S of type (1, 1) and a 1-form τ by means of the Weingarten's structural equation

$$D_X\eta = -SX + \tau(X)\eta \quad \text{(Weingarten equation)} \tag{2.2}$$

A bilinear form h and a connection ∇ for the Gauss formula

$$D_X Y = \nabla_X Y + h(X, Y)\eta \quad \text{(Gauss equation)} \tag{2.3}$$

The symmetric bilinear form h is called the affine fundamental form relative to the transversal vector field η.

We are interested in verifying if the couple (∇, θ) defines an affine unimodular structure, that is, if $\nabla\theta = 0$. Since $\nabla\theta = \theta \otimes \tau$, the condition $\nabla\theta = 0$ is equivalent to $\tau = 0$. [5].

If the affine fundamental form h is nondegenerate we have a volume form ω_h defined by

$$\omega_h(X_1, ..., X_n) = |\det(h_{ij})|^{1/2}, \text{ where } h_{ij} = h(X_i, X_j) \text{ and } \theta(X_1, ..., X_n) = 1$$

If we choos an arbitrary transversal vector field η, then we obtain on M the affine fundamental form h, the induced connection ∇ and the induced volume element θ. We want to achieve, by an appropriate choice of η, the following two goals:

(I) $\nabla\theta = 0$

(II) $\theta = \omega_h$

For each point $p \in M$, there is a transversal vector field ξ defined in a neighborhood of p satisfying the conditions (I) and (II) above [5]. Such a transversal vector field is unique up to sign.

This transversal vector field is called the affine normal field and the induced connection ∇, the affine fundamental form h, and the affine shape operator S make up the so called Blaschke structure (∇, h, S) on the hypersurface M. The induced connection ∇ is independent of the choice of the sign of ξ and is called the Blaschke connection.

For an immersion of this type we have the following equations:

$$(\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z) \qquad \text{Codazzi equation for } h \tag{2.4}$$

$$(\nabla_X S)Y = (\nabla_Y S)X \qquad \text{Codazzi equation for } S \tag{2.5}$$

$$h(SX, Y) = h(X, SY) \qquad \text{Ricci equation} \tag{2.6}$$

$$\nabla\theta = 0 \qquad \text{Equiaffine condition} \tag{2.7}$$

$$\theta = \omega_h \qquad \text{Volume condition} \tag{2.8}$$

$$\nabla \omega_h = 0 \qquad \text{Apolarity condition} \tag{2.9}$$

Definition 2.10. The torsion tensor field T, associated with a given affine connection ∇, which is a tensor of type $(1, 2)$ is given by:

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]. \tag{2.11}$$

For the connection ∇ induced by the affine normal field, the torsion tensor field is zero. We say in this case that ∇ is torsion free.

Definition 2.12. The curvature tensor field R, associated with a connection ∇, which is of type $(1, 3)$, is given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{2.13}$$

If the connection ∇ is torsion free, the tensor field R hold the first and the second Bianchi identities:

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0. \tag{2.14}$$

$$\begin{aligned}(\nabla_X R)(Y, Z, W) + (\nabla_Y R)(Z, W, X) + (\nabla_Z R)(W, X, Y) +\\ + (\nabla_W R)(X, Y, Z) = 0.\end{aligned} \tag{2.15}$$

Definition 2.16. The Ricci tensor field, which is of type $(0, 2)$ is given by

$$\text{Ric}(Y, Z) = \text{ trace } \{X \to R(X, Y)Z\}. \tag{2.17}$$

The tensor field R satisfies the fundamental equation

$$R(X, Y)Z = h(Y, Z)SX - h(X, Z)SY \qquad \text{(Gauss equation)} \tag{2.18}$$

from which we obtain

$$\text{Ric}(Y, Z) = \text{Tr}(S)h(Y, Z) - h(SY, Z) \tag{2.19}$$

where $Tr(S)$ is the trace of the shape operator S.

From Codazzi equation (2.4) it is seen that the cubic form C given by

$$C(X, Y, Z) = (\nabla h)(X, Y, Z) \tag{2.20}$$

is symmetric on X and Z.

If we denote $\widetilde{\nabla}$ the Levi-Civita connection of h, we can consider the tensor field K given by the difference between the connections ∇ and $\widetilde{\nabla}$. This tensor field, which is of type (1, 2), called difference tensor is given by

$$K(X, Y) = \nabla_X Y - \widetilde{\nabla}_X Y$$

Since ∇ and $\widetilde{\nabla}$ are free torsion we have $K(X, Y) = K(Y, X)$. Moreover, it can be seen that $C(X, Y, Z) = -2h(K(X, Y), Z)$, therefore, the cubic form C is symmetric in all three arguments. [5].

3. The form vector field.

Let M be a C^∞ manifold of dimension n, connected and 1-connected, and $F : M \to \mathbb{R}^{n+1}$ an immersion of class C^∞.

For each $p \in M$ the set

$$Q_p = \{Z_p \in T_p(M) | R_p(X_p, Y_p)Z_p = 0 \text{ for all } X_p, Y_p \in T_p(M)\}$$

is a subspace of $T_p(M)$, thus, the assignement $p \to Q_p$ for each $p \in M$ defines a distribution Q in M. We assume that $\dim(Q) \geq 1$.

Let V be a vector field and $(U, x = (x^1, ..., x^n))$ a coordinate neighborhood around a point $p \in M$. If we put $X_i = \partial/\partial x^i$, we can write the vector field V as $V = \sum_k a^k X_k$, where $a^k \in C^\infty(U)$.

The vector field is a tensor field of type $(1, 0)$, therefore ∇V is a tensor field of type (1, 1) and we make ourselves the following question: Is it possible that $\nabla V = S$ for some vector field V? The answer is affirmative if $V \in Q$.

The equality $\nabla V = S$ is equivalent to $(\nabla V)X = SX$ or

$$\nabla_X V = SX \quad \text{for all} \quad X \in \mathcal{X}(U). \tag{3.1}$$

If we consider (3.1) for $X = X_i$, $i = 1, 2, ..., n$ we obtain

$$\partial_i a^k = S_i^k - \sum_m \Gamma_{im}^k a^m, \; i, k = 1, 2, ..., n \tag{3.2}$$

where S_i^k are the components of the tensor field S with respect to the local coordinate system $x = (x^1, ..., x^n)$ and Γ_{im}^k are the Christoffel's symbols of the normal connection ∇.

To use the Frobenius' theorem that guaranteed the existence of solution for the system (3.2) we must check the integrability conditions given by

$$\partial_j\left(S_i^k - \sum_k \Gamma_{im}^k a^m\right) - \partial_i\left(S_j^k - \sum_k \Gamma_{jm}^k a^m\right) = 0 \tag{3.3}$$

Calling A the first term in the left side of (3.3) we have

$$A = \partial_j S_i^k - \sum_m (\partial_j \Gamma_{im}^k) a^m - \sum_m \Gamma_{im}^k (\partial_j a^m) - \partial_i S_j^k + \sum_m (\partial_i \Gamma_{jm}^k) a^m +$$
$$+ \sum_m \Gamma_{jm}^k (\partial_i a^m)$$

Replacing $\partial_j a^m$ and $\partial_i a^m$ obtained from (3.2) and reordering we have

$$A = \partial_j S_i^k - \sum_m \Gamma_{im}^k \left(S_j^m - \sum_l \Gamma_{jl}^m a^l\right) - \partial_i S_j^k + \sum_m \Gamma_{jm}^k \left(S_i^m - \sum_l \Gamma_{il}^m a^l\right) +$$
$$+ \sum_m \left\{\partial_i \Gamma_{jm}^k - \partial_j \Gamma_{im}^k\right\} a^m$$
$$= \partial_j S_i^k - \sum_m \Gamma_{im}^k S_j^m + \sum_{l,m} \Gamma_{im}^k \Gamma_{jl}^m a^l - \partial_i S_j^k + \sum_m \Gamma_{jm}^k S_i^m - \sum_l \Gamma_{jm}^k \Gamma_{il}^m a^l +$$
$$+ \sum_m \left\{\partial_i \Gamma_{jm}^k - \partial_j \Gamma_{im}^k\right\} a^m$$
$$= \left[\partial_j S_i^k - \partial_i S_j^k + \sum_m \left\{\Gamma_{jm}^k S_i^m - \Gamma_{im}^k S_j^m\right\}\right] + \left[\sum_m \Big\{\partial_i \Gamma_{jm}^k - \partial_j \Gamma_{im}^k + \right.$$
$$\left. + \sum_l \left(\Gamma_{il}^k \Gamma_{jm}^l - \Gamma_{jl}^k \Gamma_{im}^l\right)\Big\} a^m\right]$$

Now, we see that every one of the brackets in the above expression are zero.

For the first, by codazzi equation for S we have

$$0 = \omega^k\Big(\big(\nabla_{X_j} S\big) X_i - \big(\nabla_{X_i} S\big) X_j\Big)$$

$$= \omega^k\left(\sum_l (\partial_j S_i^l) X_l + \sum_{l,m} \Gamma_{jm}^l S_i^m X_l - \sum_{l,m} \Gamma_{ji}^m S_m^l X_l\right) + \omega^k - \sum_l (\partial_i S_j^l) X_l -$$

$$- \sum_{l,m} \Gamma_{im}^l S_j^m X_l + \sum_{l,m} \Gamma_{ji}^m S_m^l X_l \Bigg)$$

$$= \omega^k \left(\sum_l (\partial_j S_i^l) X_l + \sum_{l,m} \Gamma_{jm}^l S_i^m X_l - \sum_l (\partial_i S_j^l) X_l - \sum_{l,m} \Gamma_{im}^l S_j^m X_l \right)$$

$$= \partial_j S_i^k - \partial_i S_j^k + \sum_m \left(\Gamma_{jm}^k S_i^m - \Gamma_{im}^k S_j^m \right)$$

where $\omega^k = dx^k$, which shows that the first bracket is zero.
For the second we proceed as follows:

the term $\partial_i \Gamma_{jm}^k - \partial_j \Gamma_{im}^k + \sum_l \left(\Gamma_{il}^k \Gamma_{jm}^l - \Gamma_{jl}^k \Gamma_{im}^l \right)$ is the component R_{jim}^k of the curvature tensor R respect to the local coordinate system $x = (x^1, ..., x^n)$, and since $V \in Q$ we have

$$0 = R(X_j, X_i)V = R(X_j, X_i) \left(\sum_m a^m X_m \right)$$

$$= \sum_m a^m R(X_j, X_i) X_m = \sum_{k,m} R_{jim}^k a^m X_k$$

and hence, $\sum_m R_{jim}^k a^m = 0$, which shows the second bracket is zero too.

Since the integrability conditions are held, the Frobenius' theorem guarantee us the existence of solution for the system (3.2) and moreover, it can be seen that the system does not suffer any alterations by coordinate changes in such manner that V can be extended globally.

The next proposition shows that the existence of nonzero vector field V such that $\nabla V = S$ guarantee us that $\dim(Q) \geq 1$.

Proposition 3.4. *Let $V \in \mathfrak{X}(M)$ be such that $\nabla V = S$, and $V \neq 0$. Then $R(X, Y)V = 0$ for all $X, Y \in \mathfrak{X}(M)$.*

Proof. Let $X, Y \in \mathfrak{X}(M)$, then

$$\begin{aligned} R(X, Y)V &= \nabla_X \nabla_Y V - \nabla_Y \nabla_X V - \nabla_{[X,Y]} V \\ &= \nabla_X (SY) - \nabla_Y (SX) - S[X, Y] \end{aligned} \tag{3.5}$$

From Codazzi's equation we have

$$0 = \left(\nabla_X S\right)Y - \left(\nabla_Y S\right)X = \nabla_X(SY) - S\left(\nabla_X Y\right) - \nabla_Y(SX) + S\left(\nabla_Y X\right) \tag{3.6}$$

from which

$$\nabla_X(SY) - \nabla_Y(SX) = S\big(\nabla_X Y\big) - S\big(\nabla_Y X\big) = S\big(\nabla_X Y - \nabla_Y X\big). \tag{3.7}$$

Replacing in (3.5)

$$\begin{aligned} R(X,Y)V &= S(\nabla_X Y - \nabla_Y X) - S[X,Y] \\ &= S(\nabla_X Y - \nabla_Y X - [X,Y]) \\ &= S(T(X,Y)) \\ &= 0 \end{aligned}$$

since ∇ is torsion free.

4. Some results.

In this section we prove some results based in the existence of the vector field V with the property $\nabla V = S$.

An immediate consequence of this condition is that the trace of S is the divergence of V. In fact, remembering that the divergence $\mathrm{div}(X)$ of a vector field X (relative to a connection ∇) is given by

$$\mathrm{div}(X) = \mathrm{trace}\{Z \to \nabla_Z X\}$$

we have

$$\mathrm{div}(V) = \mathrm{trace}\{Z \to \nabla_Z V\} = \mathrm{trace}\{Z \to SZ\} = \mathrm{Tr}(S).$$

Proposition 4.1. *Let $V \in \mathcal{X}(M)$ be such that $\nabla V = S$, then the covariant derivative and the Lie derivative of S respect to V are the same, that is, $L_V S = \nabla_V S$.*

Proof. Let $Y \in \mathcal{X}(M)$ be, then

$$\begin{aligned} (\nabla_V S)Y &= \nabla_V(SY) - S(\nabla_V Y) \\ &= \nabla_{SY} V + [V, SY] - S(\nabla_Y V + [V,Y]) \\ &= S(SY) + [V, SY] - S(SY) - S[V,Y] \\ &= [V, SY] - S[V,Y] \\ &= L_V(SY) - S(L_V Y) \\ &= (L_V S)Y \end{aligned}$$

from which $L_V S = \nabla_V S$ since Y was arbitrary.

Next, we give a relation that involves the cubic form C, the Lie derivative of the first fundamental form h respect to V and the tensor S, whenever V is one of the arguments of the cubic form.

Proposition 4.2. *Let $V \in \mathfrak{X}(M)$ be such that $\nabla V = S$, then*

$$C(X, Y, V) = (L_V h)(X, Y) - 2h(SX, Y)$$

for all $X, Y \in \mathfrak{X}(M)$.

Proof. Let $X, Y \in \mathfrak{X}(M)$ be, then

$$\begin{aligned} C(X, Y, V) &= (\nabla h)(X, Y, V) = (\nabla_V h)(X, Y) \\ &= V(h(X, Y)) - h(\nabla_V X, Y) - h(X, \nabla_V Y) \\ &= V(h(X, Y)) - h(\nabla_X V + [V, X], Y) - h(X, \nabla_V Y + [V, Y]) \\ &= V(h(X, Y)) - h([V, X], Y) - h(X, [V, Y]) - h(SX, Y) - h(X, SY) \end{aligned}$$

Now, since S is selfadjoint with respect to h (Ricci's equation), we have, $h(SX, Y) = h(X, SY)$, therefore

$$C(X, Y, V) = (L_V h)(X, Y) - 2h(SX, Y).$$

Proposition 4.3. *Let $V \in \mathfrak{X}(M)$ be such that $\nabla V = S$ and A a tensor field of type $(1, r)$, then*

$$SA = (\nabla_V - L_V)A$$

where SA is given by

$$(SA)(Y_1, ..., Y_r) = S(A(Y_1, ..., Y_r)) - \sum_{k=1}^{r} A(Y_1, ..., SY_k, ..., Y_r)$$

Proof. Let $Y_1, ..., Y_r \in \mathfrak{X}(M)$ be arbitrary vector fields on M, then

$$(\nabla_V A)(Y_1, ..., Y_r) = \nabla_V(A(Y_1, ..., Y_r)) - \sum_{k=1}^{r} A(Y_1, ..., \nabla_V Y_k, ..., Y_r)$$

$$= \nabla_{A(Y_1,...,Y_r)} V + [V, A(Y_1, ..., Y_r)] - \sum_{k=1}^{r} A(Y_1, ..., \nabla_{Y_k} V + [V, Y_k], ..., Y_r)$$

$$= S(A(Y_1, ..., Y_r)) + L_V(A(Y_1, ..., Y_r)) - \sum_{k=1}^{r} A(Y_1, ..., SY_k, ..., Y_r) -$$

$$- \sum_{k=1}^{r} A(Y_1, ..., L_V Y_k, ..., Y_r)$$

$$= (SA)(Y_1, ..., Y_r) + (L_V A)(Y_1, ..., Y_r)$$

that is, $\nabla_V A = SA + L_V A$, and hence

$$SA = (\nabla_V - L_V)A.$$

Proposition 4.4. *Let V be a vector field on M such that $\nabla V = S$ and let $\mathcal{X}_V(M) = \{Y \in \mathcal{X}(M) | h(V, Y) = 0\}$, that is, $\mathcal{X}_V(M)$ is the submodule of $\mathcal{X}(M)$ make up of all orthogonal fields to V with respect to h. Then $\mathcal{X}_V(M)$ is invariant by S.*

Proof. Let $Y \in \mathcal{X}(M)$ be an arbitrary vector field, then

$$\mathrm{Ric}(Y, V) = \mathrm{trace}\{Z \to R(Z, Y)V\} = 0 \tag{4.5}$$

since $R(Z, Y)V = 0$ by proposition (3.4).

On the other hand, (2.19) give

$$\mathrm{Ric}(Y, V) = \mathrm{Tr}(S)h(Y, V) - h(SY, V) \tag{4.6}$$

Combining (4.5) and (4.6) we obtain

$$h(SY, V) = \mathrm{Tr}(S)h(Y, V) \tag{4.7}$$

from which is immediate that $h(SY, V) = 0$ if $h(Y, V) = 0$, that is, $SY \in \mathcal{X}_V(M)$ if $Y \in \mathcal{X}_V(M)$. We conclude that $\mathcal{X}_V(M)$ is invariant for S.

5. The decomposable case.

In this section we develop the theory for the affine hypersurfaces of decomposable type, some of which, have been studied in [1], [3], [4].

We start with a special case. In [4] we have a hypersurface M of decomposable type for which $\nabla K = \lambda R$, with $\lambda = -\frac{1}{2}$. It is parametrized by

$$(t_1, ..., t_n) \to \left(t_1, ..., t_n, \frac{t_1^{-n}}{n(n+1)} + \frac{t_2^2}{2} + \cdots + \frac{t_n^2}{2}\right), \qquad (t_1 > 0).$$

In this case, the expressions for S_i^k and Γ_{ij}^k in (3.2) are

$$S_1^1 = -nt_1^{n-1} \text{ and } S_j^i = 0 \text{ in all other cases.}$$

$\Gamma_{11}^{1} = -t_1^{-1}$, $\Gamma_{kk}^{1} = -t_1^{n+1}$ if $k > 1$ and $\Gamma_{ij}^{k} = 0$ in all other cases.

The system (3.2) becomes

$$\partial_1 a^1 = -nt_1^{n-1} + t_1^{-1} a^1 \tag{5.1}$$

$$\partial_k a^1 = t_1^{n+1} a^k \qquad k = 2, 3, ..., n \tag{5.2}$$

$$\partial_k a^j = 0 \qquad k = 1, 2, ..., n. \qquad j = 2, 3, ..., n. \tag{5.3}$$

From (5.3) we see that $a^2, ..., a^n$ are constant functions. Now, we verify the integrability conditions for (5.1) and (5.2).

For $j, k \geq 2$, the conditions in (5.2) are $\partial_j(t_1^{n+1}a^k) - \partial_k(t_1^{n+1}a^j) = 0$ which are satisfied trivially since a^k is constant for $k \geq 2$.

If $k \geq 2$, the conditions in (5.1) are $\partial_k(-nt_1^{n-1} + t_1^{-1}a^1) - \partial_1(t_1^{n+1}a^k) = 0$, that is,

$$t_1^{-1}\partial_k a^1 - (n+1)t_1^n a^k = t_1^n a^k - (n+1)t_1^n a^k = -nt_1^n a^k = 0$$

which implies that $a^k = 0$ for $k \geq 2$ and from (5.2) we see that a^1 only depends on the variable t_1.

On the other hand, from (5.1) the function a^1 holds with the ordinary differential equation

$$\partial_1 a^1 - t_1^{-1} a^1 = -nt_1^{n-1} \tag{5.4}$$

whose solution (with normalized constants) is

$$a^1(t_1) = -\frac{n}{n-1} t_1^n$$

The vector field V is therefore

$$V = -\frac{n}{n-1} t_1^n \left(1, 0, ..., 0, -\frac{t_1^{-n-1}}{n+1} \right) \tag{5.5}$$

Now, we make us the following question: What happens if as above, we have $n-1$ functions of parabolic type, but the remaining one is an arbitrary function?

For the sake of simplicity we take $f_1(t_1) = f(t_1)$ and $f_i(t_i) = \frac{t_i^2}{2}$ for $i = 2, 3, ..., n$.

On one hand, $S_1^1 = \frac{1}{(n+2)^2}(f'')^{1/(n+2)}[(n+2)h - (2n+3)g]$ and $S_j^i = 0$ for all other cases, where $h = \frac{f^{(4)}}{(f'')^2}$ and $g = \frac{(f''')^2}{(f'')^3}$. [3].

On the other hand, $\Gamma_{11}^1 = \frac{1}{n+2}\frac{f'''}{f''}$, $\Gamma_{kk}^1 = \frac{1}{n+2}\frac{f'''}{(f'')^2}$ if $k \geq 2$ and $\Gamma_{ij}^k = 0$ for all other cases.

Moreover, note that $\Gamma_{kk}^1 = \frac{\Gamma_{11}^1}{f''}$.
In this case, the system (3.2) becames

$$\partial_1 a^1 = S_1^1 - \Gamma_{11}^1 a^1 \tag{5.6}$$

$$\partial_k a^1 = -\Gamma_{kk}^1 a^k \qquad k = 2, 3, ..., n \tag{5.7}$$

$$\partial_k a^j = 0 \qquad k = 1, 2, ..., n, \qquad j = 2, 3, ..., n \tag{5.8}$$

From (5.8) we see that the functions $a^2, ..., a^n$ are constant. For (5.7) the integrability conditions for $j, k \geq 2$ are

$$\partial_j(-\Gamma_{kk}^1 a^k) - \partial_k(-\Gamma_{jj}^1 a^j) = 0$$

which are trivially satisfied since a^k is constant and Γ_{kk}^1 only depend on the variable t_1 for $k \geq 2$.

For the remaining ones, with $k \geq 2$, we have that

$$\partial_k(S_1^1 - \Gamma_{11}^1 a^1) - \partial_1(-\Gamma_{kk}^1 a^k) = 0 \tag{5.9}$$

or equivalently

$$(\Gamma_{11}^1\Gamma_{kk}^1 + \partial_1\Gamma_{kk}^1)a^k = 0 \qquad k = 2, 3, ..., n \tag{5.10}$$

So, we have two possible cases:

$$(i) \quad \Gamma_{11}^1\Gamma_{kk}^1 + \partial_1\Gamma_{kk}^1 \neq 0 \qquad \text{or} \qquad (ii) \quad \Gamma_{11}^1\Gamma_{kk}^1 + \partial_1\Gamma_{kk}^1 = 0$$

(i) $\Gamma_{11}^1\Gamma_{kk}^1 + \partial_1\Gamma_{kk}^1 \neq 0$
This implies that $a^k = 0$ for $k = 2, 3, ..., n$, therefore, from (5.7) we see that the function a^1 only depends on the variable t_1 and from (5.6), a^1 is solution for the ordinary differential equation of first order

$$y' + \Gamma_{11}^1 y = S_1^1. \tag{5.11}$$

Moreover, it is immediate to see that

$$\Gamma_{11}^1\Gamma_{kk}^1 + \partial_1\Gamma_{kk}^1 = (f'')^{1/(n+2)}S_1^1 \tag{5.12}$$

therefore $S_1^1 = 0$.

The solution of (5.11), (after normalizing constants) is given by

$$a^1 = (f'')^{-1/(n+2)}\int (f'')^{1/(n+2)}S_1^1 \tag{5.13}$$

We can summarize this case in the following theorem:

Theorem 5.13. *Let M be an affine hypersurface of decomposable type parametrized by*

$$(t_1, ..., t_n) \to (t_1, ..., t_n, f(t_1) + \frac{t_2^2}{2} + \cdots + \frac{t_n^2}{2})$$

If the non parabolic component function f satisfies $(n+2)h - (2n+3)g \neq 0$, where $h = \frac{f^{(4)}}{(f'')^2}$ and $g = \frac{(f''')^2}{(f'')^3}$, then there exists a vector field V such that $\nabla V = S$. This vector field is given by $V = a^1 X_1$, where

$$a^1 = (f'')^{-1/(n+2)} \int (f'')^{1/(n+2)} S_1^1$$

and

$$X_1 = F_*(\partial_1)$$

with S_1^1 given by

$$S_1^1 = \frac{1}{(n+2)^2}(f'')^{1/(n+2)}[(n+2)h - (2n+3)g].$$

(ii) $\Gamma_{11}^1\Gamma_{kk}^1 + \partial_1\Gamma_{kk}^1 = 0$

This implies that the a^k are either constants and for (5.12) we have $S_1^1 = 0$, therefore $S = 0$, so M is an improper affine hypersphere.

From $S_1^1 = 0$ we obtain $h = \frac{2n+3}{n+2}g$, from which $f(t_1) = t^{n/(n+1)}$ (normalizing constants). In this case, the equations (5.6) and (5.7) are:

$$\partial_1 a^1 = \frac{1}{n+1} t_1^{-1} a^1 \tag{5.15}$$

$$\partial_k a^1 = -\frac{n+1}{n} t_1^{1/(n+1)} a^k, \qquad k \geq 2, \quad a^k \qquad \text{constant.} \tag{5.16}$$

From (5.16) we see that a^1 is

$$a^1(t_1, ..., t_n) = \beta(t_1) - \frac{n+1}{n} t_1^{1/(n+1)} \sum_{k=2}^{n} a^k t_k \tag{5.17}$$

Using the equation (5.15) we obtain $\alpha(t_1)$ since

$$\alpha'(t_1) - \frac{1}{n} t_1^{-n/(n+1)} \sum_{k=2}^{n} a^k t_k = \frac{1}{n+1} t_1^{-1} \left[\alpha(t_1) - \frac{n+1}{n} t_1^{1/(n+1)} \sum_{k=2}^{n} a^k t_k \right]$$

from which, rearranging and simplifying we have

$$\alpha'(t_1) = \frac{1}{n+1} t_1^{-1} \beta(t_1).$$

Hence,

$$\alpha(t_1) = t_1^{1/(n+1)}$$

Finally, the function a^1 is given by

$$\begin{aligned} a^1(t_1, ..., t_n) &= t_1^{1/(n+1)} - \frac{n+1}{n} t_1^{1/(n+1)} \sum_{k=2}^{n} a^k t_k \\ &= t_1^{1/(n+1)} \left[1 - \frac{n+1}{n} \sum_{k=2}^{n} a^k t_k \right]. \end{aligned}$$

We summarize this case in the following theorem:

Theorem 5.18. *Let M be an affine hypersurface of decomposable type parametrized by*

$$(t_1, ..., t_n) \to \left(t_1, ..., t_n, f(t_1) + \frac{t_2^2}{2} + \cdots + \frac{t_n^2}{2} \right)$$

where the non parabolic component function f satisfies $(n+2)h - (2n+3)g = 0$ with h and g as in the Theorem 5.13. Then M is an improper affine hypersphere, f is of the type $t \to t^{n/(n+1)}$ and there exists a vector field V such that $\nabla V = S$, which is given by

$$V = a^1 X_1 + \sum_{k=2}^{n} a^k X_k$$

where $a^1(t_1, ..., t_n) = t_1^{1/(n+1)} \left[1 - \frac{n+1}{n} \sum_{k=2}^{n} a^k t_k \right]$, with a^k constant for $k \geq 2$ and $X_j = F_*(\partial_j)$ for $j = 1, 2, ..., n$.

As a final illustration we apply the above development for the affine hypersurface of decomposable type whose unimodular Pick invariant is a nonzero constant [1].

For this hypersurface,

$$f(t_1) = \frac{(n+1)^2}{2(n+3)} t_1^{-2/(n+1)}$$

therefore

$$g(t_1) = \frac{4(n+2)^2}{(n+1)^2} t_1^{2/(n+1)}$$

$$h(t_1) = \frac{2(n+2)(3n+5)}{(n+1)^2} t_1^{2/(n+1)}$$

and using these, $S_1^1 = -\frac{2}{n+1}$. Since $S_1^1 \neq 0$ we apply the Theorem 5.14. From (5.13) we have

$$a^1(t_1) = \frac{2}{n-1} t_1$$

so that, the vector field V is given by

$$V = \frac{2}{n-1} t_1 \left(1, 0, 0, ..., 0, -\frac{n+1}{n+3} t_1^{-(n+3)/(n+1)} \right).$$

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