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CURVILINEAR SCHEMES AND MAXIMUM RANK OF FORMS

EDOARDO BALLICO - ALESSANDRA BERNARDI

We define the *curvilinear rank* of a degree d form P in $n+1$ variables as the minimum length of a curvilinear scheme, contained in the d-th Veronese embedding of $\mathbb{P}^n$, whose span contains the projective class of P. Then, we give a bound for rank of any homogenous polynomial, in dependance on its curvilinear rank.

1. Introduction

The *rank* $\mathrm{r}(P)$ of a homogeneous polynomial $P \in \mathbb{C}[x_0,\ldots,x_n]$ of degree d, is the minimum $r \in \mathbb{N}$ such that P can be written as sum of r pure powers of linear forms $L_1,\ldots,L_r \in \mathbb{C}[x_0,\ldots,x_n]$:

$$P = L_1^d + \cdots + L_r^d. \tag{1}$$

A very interesting open question is to determine the maximum possible value that the rank of a form (i.e. a homogeneous polynomial) of given degree in a certain number of variables can have.
At our knowledge, the best general achievement on this problem is due to J.M. Landsberg and Z. Teitler that in [16, Proposition 5.1] proved that the rank of a degree d form in $n+1$ variables is smaller than or equal to $\binom{n+d}{d} - n$. Unfortunately this bound is sharp only for $n=1$ if $d \geq 2$; in fact, for example, if

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$n = 2$ and $d = 3,4$, then the maximum ranks are 5 and 7 respectively (see [7, Theorem 40 and 44]). More recently G. Blekherman and Z. Teitler proved in [9] that the maximum rank is always smaller than or equal to twice the generic rank that is the rank of a generic polynomial, i.e. the minimum r s.t. the r-th secant variety to the Veronese variety fills up the ambient space (such a secant variety is classicaly defined to be the Zariski closure of the set of all r-th secant spaces to a Veronesean). In the celebrated Alexander and Hischowitz paper [1] they computed the dimensions of all such secant varieties, so the generic rank is nowadays considered a classical result. Cleary finding a bound for the rank of any polynomial given the number of variables and the degree is a very different and difficult problem.

Few more results were obtained by focusing the attention on limits of forms of given rank. When a form P is in the Zariski closure of the set of forms of rank s, it is said that P has *border rank* $\underline{\mathrm{r}}(P)$ equal to s. For example, the maximum rank of forms of border ranks 2, 3 and 4 are known (see [7, Theorems 32 and 37] and [3, Theorem 1]). In this context, in [2] we posed the following:

Question 1 ([2])**.** Is it true that $\mathrm{r}(P) \leq d(\underline{\mathrm{r}}(P)-1)$ for all degree d forms P? Moreover, does the equality hold if and only if the projective class of P belongs to the tangential variety of a Veronese variety?

The Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}}$, with $n \geq 1$, $d \geq 2$ and $N_{n,d} := \binom{n+d}{d} - 1$ is the image of the classical d-uple Veronese embedding $\nu_d : \mathbb{P}^n \to \mathbb{P}^{N_{n,d}}$ and parameterizes projective classes of degree d pure powers of linear forms in $n+1$ variables. Therefore the rank $\mathrm{r}(P)$ of $[P] \in \mathbb{P}^{N_{n,d}}$ is the minimum r for which there exists a length r smooth zero-dimensional scheme $Z \subset X_{n,d}$ whose span contains $[P]$ (with an abuse of notation we are extending the definition of rank of a form P given in (1) to its projective class $[P]$). More recently, other notions of polynomial rank have been introduced and widely discussed ([10], [17], [8], [6], [4]). They are all related to the minimal length of a certain zero-dimensional schemes embedded in $X_{n,d}$ whose span contains the given form. Here we recall only the notion of *smoothable rank* $\mathrm{smr}(P)$ of a form P with $[P] \in \mathbb{P}^{N_{n,d}}$ (see [6, 10]):

$$\mathrm{smr}(P) = \min\{\deg(Z) \mid Z \text{ limit of smooth schemes } Z_i, \deg(Z_i) = \deg(Z),$$

$$Z, Z_i \subset X_{n,d}, \dim_K Z = \dim_K Z_i = 0 \text{ and } [P] \in \langle Z \rangle\}.$$

With this definition, it seems more reasonable to state Question 1 as follows:

Question 2. Fix $[P] \in \mathbb{P}^{N_{n,d}}$. Is it true that $\mathrm{r}(P) \leq (\mathrm{smr}(P)-1)d$?

In this paper we want to deal with a more restrictive but easier to handle notion of rank, namely the “ curvilinear rank ”. We say that a scheme $Z \subset \mathbb{P}^N$

is *curvilinear* if it is a finite union of schemes of the form $\mathcal{O}_{\mathcal{C}_i,P_i}/\mathfrak{m}_{P_i}^{e_i}$ for smooth points P_i on reduced curves $\mathcal{C}_i \subset \mathbb{P}^N$, or equivalently that the tangent space at each connected component of Z supported at the P_i's has Zariski dimension ≤ 1. We define the *curvilinear rank* $\mathrm{Cr}(P)$ of a degree d form P in $n+1$ variables as:

$$\mathrm{Cr}(P) := \min\{\deg(Z) \mid Z \subset X_{n,d},\, Z \text{ curvilinear}, [P] \in \langle Z\rangle\}.$$

The main result of this paper is the following:

Theorem 1. *For any degree d form P we have that*

$$\mathrm{r}(P) \leq (\mathrm{Cr}(P)-1)d + 2 - \mathrm{Cr}(P).$$

Theorem 1 is sharp if $\mathrm{Cr}(P) = 2,3$ ([7, Theorem 32 and 37]).

Clearly if a scheme is curvilinear is also smoothable, so the next question will be to understand if Theorem 1 holds even though we substitute the curvilinear rank with the smoothable rank:

Question 3. Fix $[P] \in \mathbb{P}^{N_{n,d}}$. Is it true that $\mathrm{r}(P) \leq (\mathrm{smr}(P)-1)d + 2 - \mathrm{smr}(P)$?

This paper is organized as follows: Section 2 is entirely devoted to the proof of Theorem 1 with a lemma; in Section 3 we study the case of ternary forms and we prove that, in such a case, Question 2 has an affirmative answer.

We will always work with an algebraically closed field K of characteristic 0.

2. Proof of Theorem 1

Let us begin this section with a Lemma that will allow us to give a lean proof of the main theorem.

We say that an irreducible curve $\mathcal{T}$ is *rational* if its normalization is isomorphic to $\mathbb{P}^1$.

Lemma 2.1. *Let $Z \subset \mathbb{P}^r$, $r \geq 2$, be a zero-dimensional curvilinear scheme of degree k. Then there is an irreducible and rational curve $\mathcal{T} \subset \mathbb{P}^r$ such that* $\deg(\mathcal{T}) \leq k-1$ *and* $Z \subset \mathcal{T} \subseteq \langle Z\rangle$.

Proof. If the scheme Z is in linearly general position, namely $\langle Z\rangle \simeq \mathbb{P}^{k-1}$, then there always exists a rational normal curve of degree $k-1$ passing through it (this is a classical fact, see for instance [13, Theorem 1]). If Z is not in linearly general position, consider $\mathbb{P}(H^0(Z,\mathcal{O}_Z(1))) \simeq \mathbb{P}^{k-1}$. In such a $\mathbb{P}^{k-1}$ there exists a curvilinear scheme W of degree k in linearly general position such that the projection $\ell_V \colon \mathbb{P}^{k-1} \setminus V \to \langle Z\rangle$ from a $(k-\dim(\langle Z\rangle)-2)$-dimensional vector space V induces an isomorphism between W and Z. Consider now the degree $k-1$ rational normal curve $\mathcal{C} \subset \mathbb{P}^{k-1}$ passing through W, its projection $\ell_V(\mathcal{C})$ contains Z and it is irreducible and rational since $\mathcal{C}$ is irreducible and rational and, by construction, $\deg(\ell_V(\mathcal{C})) \leq \deg(\mathcal{C}) = k-1$. □

We do not claim that the curve $\mathcal{T}$ is smooth, because we only need that its normalization is $\mathbb{P}^1$.

Let $X \subset \mathbb{P}^r$ be an integral non-degenerate variety. For any $P \in \langle X \rangle$ the X-*rank* $r_X(P)$ is the minimal cardinality of a subset $S \subset X$ such that $P \in \langle S \rangle$.

We are now ready to prove the main theorem of this paper.

Proof of Theorem 1: Let $X_{n,d}$ be the Veronese image of $\mathbb{P}^n$ into $\mathbb{P}^{\binom{n+d}{d}-1}$ via $\mathcal{O}(d)$, let $Z \subset X_{n,d}$ be a minimal degree curvilinear scheme such that $P \in \langle Z \rangle$, and let $U \subset \mathbb{P}^n$ be the curvilinear scheme such that $\nu_d(U) = Z$. The minimality of Z gives $P \notin \langle Z' \rangle$ for any $Z' \subsetneq Z$. Say that $\mathrm{Cr}(P) = \deg(Z) = \deg(U) := k \geq 2$. If Z is reduced, then $\mathrm{r}(P) = k$ and the statement of the theorem in this case is trivial. Hence we may assume that Z is not reduced. By Lemma 2.1, there exists a rational curve $\mathcal{T} \subset \mathbb{P}^n$ such that $U \subset \mathcal{T}$ and $c := \deg(\mathcal{T}) \leq k-1$. Set $\mathcal{Y} := \nu_d(\mathcal{T})$:

$$\begin{array}{ccc} \mathbb{P}^n & \stackrel{\nu_d}{\hookrightarrow} & \mathbb{P}^{\binom{n+d}{d}-1} \\ U \subset \mathcal{T} & \mapsto & Z \subset \mathcal{Y} \end{array} .$$

The curve $\mathcal{Y} \subset \mathbb{P}^{N_{n,d}}$ has degree cd and $Z \subset \mathcal{Y}$. Hence $P \in \langle \mathcal{Y} \rangle$. Since $\mathcal{Y} \subset X_{n,d}$, we have $\mathrm{r}(P) \leq r_{\mathcal{Y}}(P)$. Hence it is sufficient to prove that $r_{\mathcal{Y}}(P) \leq d(k-1) + 2 - k$. Since the function $t \mapsto dt$ is increasing and $c \leq k-1$, it is sufficient to prove that $r_{\mathcal{Y}}(P) \leq dc + 2 - k$. Since $\mathcal{T}$ is a degree c rational curve, there are a rational normal curve $\mathcal{D} \subset \mathbb{P}^c$ such that $\mathcal{T}$ is obtained from $\mathcal{D}$ using the linear projection from a linear subspace $E \subset \mathbb{P}^c$ with $\dim(E) = c - \dim(\langle E \rangle) - 1$ and $E \cap \mathcal{D} = \emptyset$. We use the embedding ν_d also for any projective space. We need to use it for $\mathbb{P}^s$ with $s := \max\{n, c\}$. Now let $\mathcal{C} := \nu_d(\mathcal{D})$.

$$\begin{array}{ccc} \mathbb{P}^c & \stackrel{\nu_d}{\hookrightarrow} & \mathbb{P}^{\binom{c+d}{d}-1} \\ \mathcal{D} & \mapsto & \mathcal{C} \\ & & \\ \downarrow & & \downarrow \ell_M \\ & & \\ \mathbb{P}^n & \stackrel{\nu_d}{\hookrightarrow} & \mathbb{P}^{\binom{n+d}{d}-1} \\ \mathcal{T} & \mapsto & \mathcal{Y} \end{array} .$$

The curve $\mathcal{C}$ is a degree cd rational normal curve in its linear span $\langle \mathcal{C} \rangle \cong \mathbb{P}^{dc}$. Since $\mathcal{Y}$ is embedded in $\mathbb{P}^{N_{n,d}}$ by the restriction of the degree d forms, $\mathcal{Y}$ is a linear projection of $\mathcal{C}$ from a linear subspace $M \subset \mathbb{P}^{dc}$ such that $\mathcal{C} \cap M = \emptyset$ and $\dim(M) = cd - \dim(\langle \mathcal{Y} \rangle) - 1$ (we have $M \cap \mathcal{C} = \emptyset$, because $\deg(\mathcal{Y}) = cd$). Call $\ell_M \colon \mathbb{P}^{dc} \setminus M \to \langle \mathcal{Y} \rangle$ the linear projection from M. Since $\mathcal{C} \cap M = \emptyset$, the morphism ℓ_M is surjective. Since $M \cap \mathcal{C} = \emptyset$, the map $\ell_M|_{\mathcal{C}}$ is a degree one morphism $\ell : \mathcal{C} \to \mathcal{Y}$. Set $W := \ell^{-1}(Z)$ (scheme-theoretic counterimage). Since ℓ is proper and surjective, $\ell(W) = Z$ and hence $\deg(W) = k$.

$$\begin{array}{ccc} \mathbb{P}^c & \stackrel{\nu_d}{\hookrightarrow} & \mathbb{P}^{\binom{c+d}{d}-1} \\ \mathcal{D} & \mapsto & W \subset \mathcal{C} \\ & & \\ \downarrow & & \downarrow \ell_M \\ & & \\ \mathbb{P}^n & \stackrel{\nu_d}{\hookrightarrow} & \mathbb{P}^{\binom{n+d}{d}-1} \\ U \subset \mathcal{T} & \mapsto & Z \subset \mathcal{Y} \end{array} .$$

Set $\ell' := \ell_M|(\langle W \rangle \setminus M \cap \langle W \rangle)$ and notice that even though by construction we clearly have that $W \cap M = \emptyset$, we cannot assume that also $M \cap \langle W \rangle = \emptyset$. Since $\ell(W) = Z$ and ℓ_M is surjective, ℓ' is surjective. Fix $O \in \langle W \rangle \setminus M \cap \langle W \rangle$ such that $\ell'(O) = P$. Since $P \notin \langle Z' \rangle$ for each $Z' \subseteq Z$ and $W = \ell^{-1}(Z)$, then $O \notin \langle W' \rangle$ for any $W' \subsetneq W$.

(a) First assume $\deg(W) \leq \lfloor (dc+2)/2 \rfloor$. This implies that O has border rank $\deg(W)$ and that either $\mathrm{r}_{\mathcal{C}}(O) = \deg(W)$ or $\mathrm{r}_{\mathcal{C}}(O) = dc + 2 - \deg(W)$ ([12], [16, Theorem 4.1], [7, Theorem 23]). Take $S \subset \mathcal{C}$ evincing $\mathrm{r}_{\mathcal{C}}(O)$. Since $P = \ell_M(O)$, we have $P \in \langle \ell(S) \rangle$. Since $\sharp(\ell(S)) \leq \sharp(S) \leq cd + 2 - k$, we get $\mathrm{r}_{\mathcal{Y}}(P) \leq cd + 2 - k$.

(b) Now assume $\deg(W) > \lfloor (dc+2)/2 \rfloor$. A classical result attributed to JJ. Sylvester gives the relation between the length of two 0-dimensional subschemes contained in the rational normal curve and such that their spans contain the same point (see e.g. [7, 12]). If $P \in \langle A \rangle \cap \langle B \rangle$ with A, B two 0-dimensional schemes on the rational normal curve of degree d then the sum of the degrees $\deg(A) + \deg(B) = d + 2$. Since $P \notin \langle W' \rangle$ for any $W' \subsetneq W$ and any zero-dimensional subscheme of $\mathcal{C}$ with degree at most $dc + 2$ is linearly independent, Sylvester's theorem gives $\mathrm{r}_{\mathcal{C}}(O) \leq \deg(W)$. As in step (a) we get $\mathrm{r}_{\mathcal{Y}}(P) \leq k < d(k-1) + 2 - k$.

□

3. Superficial case

In this section we show that Question 2 has an affirmative answer in the case $n = 2$ of ternary forms and that the bound in Question 2 is seldom sharp in this case (for large $\mathrm{cr}(P)$ the upper bound in Question 2 is worst than the true one by [9]).

More precisely, we prove the following result.

Proposition 1. *Let P be a ternary form of degree d with $2 \le \mathrm{cr}(P) \le d$. If $\mathrm{cr}(P) \le d$, then* $\mathrm{r}(P) \le \binom{d+2}{2} - \binom{d-\mathrm{cr}(P)+1}{2} - 1$.

Before giving the proof of Proposition 1, we need the following result.

Proposition 2. *Let $Z \subset \mathbb{P}^2$ be a degree $k \ge 4$ zero-dimensional scheme.*

There is an integral curve $\mathcal{C} \subset \mathbb{P}^2$ such that $\deg(\mathcal{C}) = k-1$ *and $Z \subset \mathcal{C}$ if and only if Z is not contained in a line.*

Proof. First assume that Z is contained in a line $\mathcal{D}$. Bézout theorem gives that $\mathcal{D}$ is the only integral curve of degree $< k$ containing Z.

Now assume that Z is not contained in a line.

Claim 1. The linear system $|\mathcal{I}_Z(k-1)|$ has no base points outside Z_{red}.

Proof of Claim 1. Fix $P \in \mathbb{P}^2 \setminus Z_{\mathrm{red}}$. Since $\deg(Z \cup \{P\}) = k+1$, we have $h^1(\mathcal{I}_{Z\cup\{P\}}(k-1)) > 0$ if and only if there is a line $\mathcal{D}$ containing $Z \cup \{P\}$, but, since in our case Z is not contained in a line, we get $h^1(\mathcal{I}_{Z\cup\{P\}}(k-1)) = 0$. Hence $h^0(\mathcal{I}_{Z\cup\{P\}}(k-1)) = h^0(\mathcal{I}_Z(k-1)) - 1$, i.e. P is not a base point of $|\mathcal{I}_Z(k-1)|$.

By Claim 1, the linear system $|\mathcal{I}_Z(k-1)|$ induces a morphism $\psi\colon \mathbb{P}^2 \setminus Z_{\mathrm{red}} \to \mathbb{P}^x$.

Claim 2. We have $\dim(Im(\psi)) = 2$.

Proof of Claim 2. It is sufficient to prove that the differential $d\psi(Q)$ of ψ has rank 2 for a general $Q \in \mathbb{P}^2$. Assume that $d\psi(Q)$ has rank ≤ 1, i.e. assume the existence of a tangent vector $\mathbf{v}$ at Q in the kernel of the linear map $d\psi(Q)$. Since $h^1(\mathcal{I}_{Z\cup\{P\}}(k-1)) = 0$ (see proof of Claim 1), this is equivalent to $h^1(\mathcal{I}_{Z\cup\mathbf{v}}(k-1)) > 0$. Since $\deg(Z \cup \mathbf{v}) = k+2 \le 2(k-1)+1$, there is a line $\mathcal{D} \subset \mathbb{P}^2$ such that $\deg(\mathcal{D} \cap (Z \cup \mathbf{v})) \ge k+1$ ([7, Lemma 34]). Hence $\deg(Z \cap \mathcal{D}) \ge k-1$. Since $k \ge 4$ there are at most finitely many lines $\mathcal{D}_1, \dots, \mathcal{D}_s$ such that $\deg(\mathcal{D}_i \cap Z) \ge k-1$ for all i. If $Q \notin \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_s$, then $\deg(\mathcal{D} \cap (Z \cup \mathbf{v})) \le k$ for every line $\mathcal{D}$.

By Claim 2 and Bertini's second theorem ([15, Part 4 of Theorem 6.3]) a general $\mathcal{C} \in |\mathcal{I}_Z(k-1)|$ is irreducible. □

Any degree 2 zero-dimensional scheme $Z \subset \mathbb{P}^n$, $n \ge 2$ is contained in a unique line and hence it is contained in a unique irreducible curve of degree $2-1$. Now we check that in case our form has curvilinear rank equal to 3, then Proposition 2 fails in a unique case.

Remark 1. Let $Z \subset \mathbb{P}^2$ be a zero-dimensional scheme such that $\deg(Z) = 3$. Since $h^1(\mathcal{I}_Z(2)) = 0$ ([7], Lemma 34), we have $h^0(\mathcal{I}_Z(2)) = 3$. A dimensional count gives that Z is not contained in a smooth conic if and only if there is $P \in \mathbb{P}^2$ with $I_Z = I_P^2$ (in this case $|\mathcal{I}_Z(2)|$ is formed by the unions $\mathcal{R} \cup \mathcal{L}$ with $\mathcal{R}$ and $\mathcal{L}$ lines through P).

We conclude our paper with the Proof of Proposition 1.

Proof of Proposition 1. Let us recall that the *cactus rank* of a point $P \in \langle \nu_d(\mathbb{P}^n) \rangle$ is the minimum length of a 0-dimensional scheme $Z \subset \mathbb{P}^n$ such that $P \in \langle \nu_d(Z) \rangle$.
Take $Z \subset \mathbb{P}^2$ evincing the cactus rank. If Z is contained in a line $\mathcal{L}$, then $P \in \langle \nu_d(\mathcal{L}) \rangle$ and hence $\mathrm{r}(P) \leq d$ by a theorem of Sylvester that we have already recalled in item (b) of the proof of our Theorem 1 (see [7, 12] for modern and precise proof of Sylvester's theorem) or by [16, Proposition 5.1]. Now assume that Z is not contained in a line. Let $\mathcal{C} \subset \mathbb{P}^2$ be an integral curve of degree $\mathrm{cr}(P) - 1$ containing Z. We have $P \in \langle \nu_d(\mathcal{C}) \rangle$ and $\dim(\langle \nu_d(\mathcal{C}) \rangle) = \binom{d+2}{2} - \binom{d+1-Cr(P)}{2} - 1$. Apply [16, Proposition 5.1]. □

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EDOARDO BALLICO
Dept. of Mathematics
University of Trento
e-mail: edoardo.ballico@unitn.it

ALESSANDRA BERNARDI
Dept. of Mathematics
University of Trento
e-mail: alessandra.bernardi@unitn.it