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EXISTENCE OF PERIODIC SOLUTIONS FOR A SECOND-ORDER NONLINEAR NEUTRAL DIFFERENTIAL EQUATION BY THE KRASNOSELSKII'S FIXED POINT TECHNIQUE

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The objective of this work is the application of Krasnoselskii's fixed point technique to prove the existence of periodic solutions of the second-order nonlinear neutral differential equation

$$\frac{d^2}{dt^2}x(t)+p(t)\frac{d}{dt}x(t)+q(t)x(t)$$
$$=\frac{d^2}{dt^2}g(t,x(t-\tau(t)))+f(t,x(t),x(t-\tau(t))).$$

The idea of this technique is based on the inverting of the considered equation into an integral equation whose solution is recourse to Krasnoselskii's fixed point theorem. In addition, by application of the Banach principle on the inverted integral equation and under certain specified constraints we proved the uniqueness of the periodic solution.

1. Introduction

Delay differential equations are often more realistic in describing natural phenomena compared to those without delay (EDO). They model many natural

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phenomena and appear in many fields such as physics, chemistry, biology, dynamics of populations, medicine, ... etc.

For these reasons, this type of equations was given a great importance in the work of many researchers. There has been recently many activities concerning the existence, uniqueness, stability and positivity of solutions for delay differential equations, see [1]–[20], [22] and references therein.

But it is often difficult to prove the existence of such solutions because there is no specific way to solve this kind of problems. Where some researchers used the theory of differential equations while others used the fixed point theory, ... etc.

We know that a slight change perturbation in the delay term may lead to completely change the modeled phenomena. So we can represent several phenomena using a single equation model with the proviso that we make some changes in the delay term. In this context and using Krasnoselskii's fixed point theorem we find several works concerning the existence and the uniqueness of the periodic solutions of delay second-order differential equations as it is shown in the work of Wang, Lian and Ge [22] on the equation

$$\begin{aligned}&\frac{d^2}{dt^2}x(t)+p(t)\frac{d}{dt}x(t)+q(t)x(t)\\&=r(t)x'(t-\tau(t))+f(t,x(t),x(t-\tau(t))),\end{aligned}$$

and the work of Ardjouni and Djoudi [3] on the equation

$$\begin{aligned}&\frac{d^2}{dt^2}x(t)+p(t)\frac{d}{dt}x(t)+q(t)x(t)\\&=\frac{d}{dt}g(t,x(t-\tau(t)))+f(t,x(t),x(t-\tau(t))).\end{aligned}$$

In this work, we concentrate on the existence and uniqueness of periodic solutions for the second order nonlinear neutral differential equation

$$\begin{aligned}&\frac{d^2}{dt^2}x(t)+p(t)\frac{d}{dt}x(t)+q(t)x(t)\\&=\frac{d^2}{dt^2}g(t,x(t-\tau(t)))+f(t,x(t),x(t-\tau(t))),\end{aligned}\tag{1}$$

where p and q are positive continuous real-valued functions. The functions $g:\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ and $f:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ are continuous with respect to its arguments. To attain our desired end we have to transform (1) into an integral equation and then use Krasnoselskii's fixed point theorem to show the existence of periodic solutions. The obtained integral equation is the sum of two mappings, one is a contraction and the other is compact. Also, the transformation of equation (1) enables us to show the uniqueness of the periodic solution by invoking the contraction mapping principle.

We note that the study of this one is more complicated compared to the equations considered in [3] and [22] since the delay here, further it's nonlinear and variable it also contains a differentiability of second order. As a consequence, our analysis is different from that in [3].

The organization of this article is as follows. In the first, we introduce some notations and lemmas, and state some preliminary results needed later. After we give the Green's function of our equation which plays an important role in this paper. Also, we present the inversion of (1) and state Krasnoselskii's fixed point theorem. For details on Krasnoselskii theorem we refer the reader to [21]. Finally, we present our main results on existence and uniqueness.

2. Preliminaries

For $T > 0$, let P_T be the set of all continuous scalar functions x, periodic in t of period T. Then $(P_T, \|.\|)$ is a Banach space with the supremum norm

$$\|x\| = \sup_{t\in\mathbb{R}} |x(t)| = \sup_{t\in[0,T]} |x(t)| .$$

Since we are searching for the existence of periodic solutions for equation (1), it is natural to assume that

$$p(t+T) = p(t), \ q(t+T) = q(t), \ \tau(t+T) = \tau(t), \tag{2}$$

with τ being scalar function, continuous, and $\tau(t) \geq \tau^* > 0$. Also, we assume

$$\int_0^T p(s)\, ds > 0, \ \int_0^T q(s)\, ds > 0. \tag{3}$$

Functions $g(t,x)$ and $f(t,x,y)$ are periodic in t of period T. They are also supposed to be globally Lipschitz continuous in x and in x and y, respectively. That is

$$g(t+T,x) = g(t,x), \ f(t+T,x,y) = f(t,x,y), \tag{4}$$

and there are positive constants k_1, k_2, k_3 such that

$$|g(t,x) - g(t,y)| \leq k_1 \|x-y\|, \tag{5}$$

and

$$|f(t,x,y) - f(t,z,w)| \leq k_2 \|x-z\| + k_3 \|y-w\| . \tag{6}$$

Lemma 2.1 ([16]). *Suppose that (2) and (3) hold and*

$$\frac{R_1\left[\exp\left(\int_0^T p(u)\,du\right)-1\right]}{Q_1 T}\geq 1, \tag{7}$$

where

$$R_1=\max_{t\in[0,T]}\left|\int_t^{t+T}\frac{\exp\left(\int_t^s p(u)\,du\right)}{\exp\left(\int_0^T p(u)\,du\right)-1}q(s)\,ds\right|,$$

$$Q_1=\left(1+\exp\left(\int_0^T p(u)\,du\right)\right)^2 R_1^2.$$

Then there are continuous T-periodic functions a and b such that $b(t)>0$, $\int_0^T a(u)\,du>0$ and

$$a(t)+b(t)=p(t),\ \frac{d}{dt}b(t)+a(t)b(t)=q(t),\ \textit{for } t\in\mathbb{R}.$$

Lemma 2.2 ([22]). *Suppose the conditions of Lemma 2.1 hold and $\phi\in P_T$. Then the equation*

$$\frac{d^2}{dt^2}x(t)+p(t)\frac{d}{dt}x(t)+q(t)x(t)=\phi(t),$$

has a T-periodic solution. Moreover, the periodic solution can be expressed by

$$x(t)=\int_t^{t+T}G(t,s)\phi(s)\,ds,$$

where

$$G(t,s)=\frac{\int_t^s\exp\left[\int_t^u b(v)\,dv+\int_u^s a(v)\,dv\right]du}{\left[\exp\left(\int_0^T a(u)\,du\right)-1\right]\left[\exp\left(\int_0^T b(u)\,du\right)-1\right]}$$
$$+\frac{\int_s^{t+T}\exp\left[\int_t^u b(v)\,dv+\int_u^{s+T} a(v)\,dv\right]du}{\left[\exp\left(\int_0^T a(u)\,du\right)-1\right]\left[\exp\left(\int_0^T b(u)\,du\right)-1\right]}.$$

Corollary 2.3 ([22]). *Green's function G satisfies the following properties*

$$
\begin{aligned}
&G(t,t+T) = G(t,t),\ G(t+T,s+T) = G(t,s),\\
&\frac{\partial}{\partial s}G(t,s) = a(s)\,G(t,s) - \frac{\exp\left(\int_t^s b(v)\,dv\right)}{\exp\left(\int_0^T b(v)\,dv\right)-1},\\
&\frac{\partial}{\partial t}G(t,s) = -b(t)\,G(t,s) + \frac{\exp\left(\int_t^s a(v)\,dv\right)}{\exp\left(\int_0^T a(v)\,dv\right)-1},\\
&\frac{\partial^2}{\partial s^2}G(t,s) = \left(a'(s)+a^2(s)\right)G(t,s) - (a(s)+b(s))\,\frac{\exp\left(\int_t^s b(v)\,dv\right)}{\exp\left(\int_0^T b(v)\,dv\right)-1}.
\end{aligned}
$$

The following lemma is essential to our results.

Lemma 2.4. *Suppose (2)–(4) and (7) hold. If $x \in P_T$, then x is a solution of equation (1) if and only if*

$$
\begin{aligned}
x(t) = &\int_t^{t+T} \left\{ \left(a'(s)+a^2(s)\right) g(s,x(s-\tau(s))) \right.\\
&\left. + f(s,x(s),x(s-\tau(s))) \right\} G(t,s)ds\\
&+ g(t,x(t-\tau(t))) - \int_t^{t+T} p(s)\,E(t,s)\,g(s,x(s-\tau(s)))ds \qquad (8)
\end{aligned}
$$

where

$$
E(t,s) = \frac{\exp\left(\int_t^s b(v)\,dv\right)}{\exp\left(\int_0^T b(v)\,dv\right)-1}. \qquad (9)
$$

Proof. Let $x \in P_T$ be a solution of (1). From Lemma 2.2, we have

$$
x(t) = \int_t^{t+T} G(t,s)\left[\frac{\partial^2}{\partial s^2} g(s,x(s-\tau(s))) + f(s,x(s),x(s-\tau(s)))\right] ds. \qquad (10)
$$

Using the twice integration by parts, we have

$$
\begin{aligned}
&\int_t^{t+T} G(t,s)\,\frac{\partial^2}{\partial s^2} g(s,x(s-\tau(s)))\,ds\\
&= -\int_t^{t+T} \left(\frac{\partial}{\partial s}G(t,s)\right)\left(\frac{\partial}{\partial s} g(s,x(s-\tau(s)))\right) ds\\
&- \left[\left(\frac{\partial}{\partial s}G(t,s)\right) g(s,x(s-\tau(s)))\right]_t^{t+T}\\
&+ \int_t^{t+T} \left(\frac{\partial^2}{\partial s^2}G(t,s)\right) g(s,x(s-\tau(s)))\,ds.
\end{aligned}
$$

Since

$$
\begin{aligned}
&-\left[\left(\frac{\partial}{\partial s}G(t,s)\right)g(s,x(s-\tau(s)))\right]_t^{t+T}\\
&=-\left[\left(a(s)G(t,s)-\frac{\exp\left(\int_t^s b(v)\,dv\right)}{\exp\left(\int_0^T b(v)\,dv\right)-1}\right)g(s,x(s-\tau(s)))\right]_t^{t+T}\\
&=-\left(a(t+T)G(t,t+T)-\frac{\exp\left(\int_t^{t+T} b(v)\,dv\right)}{\exp\left(\int_0^T b(v)\,dv\right)-1}\right)\\
&\times g(t+T,x(t+T-\tau(t+T)))\\
&+\left(a(t)G(t,t)-\frac{\exp\left(\int_t^t b(v)\,dv\right)}{\exp\left(\int_0^T b(v)\,dv\right)-1}\right)g(t,x(t-\tau(t)))\\
&=-\left(a(t)G(t,t)-\frac{\exp\left(\int_0^T b(v)\,dv\right)}{\exp\left(\int_0^T b(v)\,dv\right)-1}\right)g(t,x(t-\tau(t)))\\
&+\left(a(t)G(t,t)-\frac{1}{\exp\left(\int_0^T b(v)\,dv\right)-1}\right)g(t,x(t-\tau(t)))\\
&=g(t,x(t-\tau(t))),
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_t^{t+T}\left(\frac{\partial^2}{\partial s^2}G(t,s)\right)g(s,x(s-\tau(s)))ds\\
&=\int_t^{t+T}\left\{\left(a'(s)+a^2(s)\right)g(s,x(s-\tau(s)))G(t,s)\right.\\
&\left.-p(s)E(t,s)g(s,x(s-\tau(s)))\right\}ds,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\int_t^{t+T}G(t,s)\frac{\partial^2}{\partial s^2}g(s,x(s-\tau(s)))ds\\
&=g(t,x(t-\tau(t)))\\
&+\int_t^{t+T}\left\{\left(a'(s)+a^2(s)\right)g(s,x(s-\tau(s)))G(t,s)\right.\\
&\left.-p(s)E(t,s)g(s,x(s-\tau(s)))\right\}ds, \qquad (11)
\end{aligned}
$$

where E is given by (9). Then substituting (11) in (10) completes the proof. □

Lemma 2.5 ([22]). *Let* $A = \int_0^T p(u)\,du$, $B = T^2 \exp\left(\frac{1}{T}\int_0^T \ln(q(u))\,du\right)$. *If*

$$A^2 \geq 4B, \tag{12}$$

then we have

$$\min\left\{\int_0^T a(u)\,du, \int_0^T b(u)\,du\right\} \geq \frac{1}{2}\left(A - \sqrt{A^2 - 4B}\right) := l,$$
$$\max\left\{\int_0^T a(u)\,du, \int_0^T b(u)\,du\right\} \leq \frac{1}{2}\left(A + \sqrt{A^2 - 4B}\right) := m.$$

Corollary 2.6 ([22]). *Functions G and E satisfy*

$$\frac{T}{(e^m - 1)^2} \leq G(t,s) \leq \frac{T\exp\left(\int_0^T p(u)\,du\right)}{(e^l - 1)^2},\ E(t,s) \leq \frac{e^m}{e^l - 1}.$$

Lastly in this section, we state Krasnoselskii's fixed point theorem which enables us to prove the existence of periodic solutions to equation (1). For its proof we refer the reader to [21].

Theorem 2.7 (Krasnoselskii). *Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $(\mathbb{B}, \|.\|)$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ map $\mathbb{M}$ into $\mathbb{B}$ such that*

(i) $x, y \in \mathbb{M}$, *implies* $\mathcal{A}x + \mathcal{B}y \in \mathbb{M}$,

(ii) $\mathcal{A}$ *is compact and continuous,*

(iii) $\mathcal{B}$ *is a contraction mapping.*

Then there exists $z \in \mathbb{M}$ with $z = \mathcal{A}z + \mathcal{B}z$.

3. Main results

We present our existence results in this section. To this end, we first define the operator $H : P_T \to P_T$ by

$$\begin{aligned}(H\varphi)(t) = &\int_t^{t+T} G(t,s)\left\{\left(a'(s) + a^2(s)\right) g(s, \varphi(s - \tau(s)))\right.\\ &\left.+ f(s, \varphi(s), \varphi(s - \tau(s)))\right\} ds\\ &+ g(t, \varphi(t - \tau(t))) - \int_t^{t+T} p(s) E(t,s) g(s, \varphi(s - \tau(s)))\,ds. \end{aligned} \tag{13}$$

From Lemma 2.4, we see that fixed points of H are solutions of (1) and vice versa. In order to employ Theorem 2.7 we need to express the operator H as the sum of two operators, one of which is compact and the other of which is a contraction. Let $(H\varphi)(t) = (\mathcal{A}\varphi)(t) + (\mathcal{B}\varphi)(t)$ where

$$(\mathcal{A}\varphi)(t) = \int_t^{t+T} G(t,s)\left\{\left(a'(s)+a^2(s)\right)g(s,\varphi(s-\tau(s))) + f(s,\varphi(s),\varphi(s-\tau(s)))\right\}ds, \tag{14}$$

and

$$(\mathcal{B}\varphi)(t) = g(t,\varphi(t-\tau(t))) - \int_t^{t+T} p(s)E(t,s)g(s,\varphi(s-\tau(s)))\,ds. \tag{15}$$

To simplify notations, we introduce the following constants

$$\alpha = \frac{T\exp\left(\int_0^T p(u)\,du\right)}{\left(e^l-1\right)^2},\ \beta = \frac{e^m}{e^l-1},\ \gamma = \max_{t\in[0,T]}|a(t)|,$$
$$\gamma' = \max_{t\in[0,T]}\left|a'(t)\right|,\ \lambda = \max_{t\in[0,T]}\{b(t)\},\ \theta = \max_{t\in[0,T]}\{p(t)\}. \tag{16}$$

Lemma 3.1. *Suppose that conditions (2)–(7) and (12) hold. Then $\mathcal{A}: P_T \to P_T$ is compact.*

Proof. Let $\mathcal{A}$ be defined by (14). Obviously, $\mathcal{A}\varphi$ is continuous and it is easy to show that $(\mathcal{A}\varphi)(t+T) = (\mathcal{A}\varphi)(t)$. To see that $\mathcal{A}$ is continuous, we let $\varphi,\psi \in P_T$. Given $\varepsilon > 0$, take $\eta = \varepsilon/N$ with $N = \alpha T\left(\left(\gamma'+\gamma^2\right)k_1 + k_2 + k_3\right)$ where k_1, k_2 and k_3 are given by (5) and (6). Now, for $\|\varphi-\psi\| < \eta$, we obtain

$$\begin{aligned}\|\mathcal{A}\varphi - \mathcal{A}\psi\| &\le \alpha\int_t^{t+T}\left[\left(\gamma'+\gamma^2\right)k_1\|\varphi-\psi\| + (k_2+k_3)\|\varphi-\psi\|\right]ds \\ &\le N\|\varphi-\psi\| < \varepsilon.\end{aligned}$$

This proves that $\mathcal{A}$ is continuous. To show that the image of $\mathcal{A}$ is contained in a compact set, we consider $\mathbb{D} = \{\varphi \in P_T : \|\varphi\| \le L\}$, where L is a fixed positive constant. Let $\varphi_n \in \mathbb{D}$, where n is a positive integer. Observe that in view of (5) and (6) we have

$$\begin{aligned}|g(t,x)| &= |g(t,x) - g(t,0) + g(t,0)| \\ &\le |g(t,x) - g(t,0)| + |g(t,0)| \\ &\le k_1\|x\| + \rho_1,\end{aligned}$$

similarly,

$$\begin{aligned}|f(t,x,y)| &= |f(t,x,y) - f(t,0,0) + f(t,0,0)| \\ &\le |f(t,x,y) - f(t,0,0)| + |f(t,0,0)| \\ &\le k_2\|x\| + k_3\|y\| + \rho_2,\end{aligned}$$

where $\rho_1 = \max_{t\in[0,T]} |g(t,0)|$ and $\rho_2 = \max_{t\in[0,T]} |f(t,0,0)|$. Hence, if $\mathcal{A}$ is given by (14) we obtain that

$$\|\mathcal{A}\varphi_n\| \le D,$$

for some positive constant D. Next we calculate $\frac{d}{dt}(\mathcal{A}\varphi_n)(t)$ and show that it is uniformly bounded. By making use of (2), (3) and (4) we obtain by taking the derivative in (14) that

$$\begin{aligned}
&\frac{d}{dt}(\mathcal{A}\varphi_n)(t) \\
&= \int_t^{t+T} \left[-b(t)G(t,s) + \frac{\exp\left(\int_t^s a(v)\,dv\right)}{\exp\left(\int_0^T a(v)\,dv\right) - 1} \right] \\
&\times \left\{ \left(a'(s) + a^2(s)\right) g(s,\varphi_n(s-\tau(s))) + f(s,\varphi_n(s),\varphi_n(s-\tau(s))) \right\} ds.
\end{aligned}$$

Consequently, by invoking (5), (6) and (16), we obtain

$$\begin{aligned}
\left|\frac{d}{dt}(\mathcal{A}\varphi_n)(t)\right| &\le T(\lambda\alpha+\beta)\left[\left(\gamma'+\gamma^2\right)(k_1L+\rho_1) + (k_2+k_3)L+\rho_2\right] \\
&\le M,
\end{aligned}$$

for some positive constant M. Hence the sequence $(\mathcal{A}\varphi_n)$ is uniformly bounded and equicontinuous. The Ascoli-Arzela theorem implies that a subsequence $(\mathcal{A}\varphi_{n_k})$ of $(\mathcal{A}\varphi_n)$ converges uniformly to a continuous T-periodic function. Thus $\mathcal{A}$ is continuous and $\mathcal{A}(\mathbb{D})$ is contained in a compact subset of P_T. □

Lemma 3.2. *If $\mathcal{B}$ is given by (15) with*

$$k_1(1+\theta\beta T) < 1, \tag{17}$$

then $\mathcal{B}: P_T \to P_T$ is a contraction.

Proof. Let $\mathcal{B}$ be defined by (15). It is easy to show that $(\mathcal{B}\varphi)(t+T) = (\mathcal{B}\varphi)(t)$. To see that $\mathcal{B}$ is a contraction. Let $\varphi,\psi \in P_T$ we have

$$\begin{aligned}
&|(\mathcal{B}\varphi)(t) - (\mathcal{B}\psi)(t)| \\
&\le |g(t,\varphi(t-\tau(t))) - g(t,\psi(t-\tau(t)))| \\
&+ \int_t^{t+T} p(s)E(t,s)\,|g(s,\varphi(s-\tau(s))) - g(s,\psi(s-\tau(s)))|\,ds \\
&\le k_1(1+\theta\beta T)\|\varphi-\psi\|.
\end{aligned}$$

Then, we get

$$\|\mathcal{B}\varphi - \mathcal{B}\psi\| \le k_1(1+\theta\beta T)\|\varphi-\psi\|.$$

Thus $\mathcal{B}: P_T \to P_T$ is a contraction by (17). □

Theorem 3.3. *Let α, β, γ and γ' be given by (16). Suppose that conditions (2)–(7), (12) and (17) hold. Suppose there exists a positive constant J satisfying the inequality*

$$\begin{aligned}&\left\{\left[\alpha\left(\gamma'+\gamma^2\right)+\theta\beta\right]\rho_1+\alpha\rho_2\right\}T+\rho_1\\&+\left\{\left[\left(\alpha\left(\gamma'+\gamma^2\right)+\theta\beta\right)k_1+\alpha\left(k_2+k_3\right)\right]T+k_1\right\}J\ \le\ J.\end{aligned}$$

Then (1) has a solution $x\in P_T$ such that $\|x\|\le J$.

Proof. Define $\mathbb{M}=\{\varphi\in P_T:\|\varphi\|\le J\}$. By Lemma 3.1, the operator $\mathcal{A}:\mathbb{M}\to P_T$ is compact and continuous. Also, from Lemma 3.2, the operator $\mathcal{B}:\mathbb{M}\to P_T$ is a contraction. Conditions (ii) and (iii) of Theorem 2.7 are satisfied. We need to show that condition (i) is fulfilled. To this end, let $\varphi,\psi\in\mathbb{M}$. Then

$$\begin{aligned}&\left|\left(\mathcal{A}\varphi\right)(t)+\left(\mathcal{B}\psi\right)(t)\right|\\&\le\alpha\int_t^{t+T}\left[\left(\gamma'+\gamma^2\right)\left(k_1\|\varphi\|+\rho_1\right)+\left(k_2+k_3\right)\|\varphi\|+\rho_2\right]ds\\&+k_1\|\psi\|+\rho_1+\theta\beta\int_t^{t+T}\left(k_1\|\psi\|+\rho_1\right)ds\\&\le\left\{\left[\alpha\left(\gamma'+\gamma^2\right)+\theta\beta\right]\rho_1+\alpha\rho_2\right\}T+\rho_1\\&+\left\{\left[\left(\alpha\left(\gamma'+\gamma^2\right)+\theta\beta\right)k_1+\alpha\left(k_2+k_3\right)\right]T+k_1\right\}J\ \le\ J.\end{aligned}$$

Thus $\|\mathcal{A}\varphi+\mathcal{B}\psi\|\le J$ and so $\mathcal{A}\varphi+\mathcal{B}\psi\in\mathbb{M}$. All the conditions of Theorem 2.7 are satisfied and consequently the operator H defined in (13) has a fixed point in $\mathbb{M}$. By Lemma 2.4 this fixed point is a solution of (1) and the proof is complete. □

Theorem 3.4. *Let α, β, γ and γ' be given by (16). Suppose that conditions (2)–(7), (12) and (17) hold. If*

$$\left[\left(\alpha\left(\gamma'+\gamma^2\right)+\theta\beta\right)k_1+\alpha\left(k_2+k_3\right)\right]T+k_1<1,$$

then (1) has a unique T-periodic solution.

Proof. Let the mapping H be given by (13). For $\varphi,\psi\in P_T$, we have

$$\begin{aligned}&\left|\left(H\varphi\right)(t)-\left(H\psi\right)(t)\right|\\&\le\alpha\int_t^{t+T}\left[\left(\gamma'+\gamma^2\right)k_1\|\varphi-\psi\|+\left(k_2+k_3\right)\|\varphi-\psi\|\right]ds\\&+k_1\|\varphi-\psi\|+\theta\beta\int_t^{t+T}k_1\|\varphi-\psi\|\,ds.\end{aligned}$$

Hence,

$$\|H\varphi - H\psi\| \leq \left\{ \left[\left(\alpha \left(\gamma' + \gamma^2 \right) + \theta\beta \right) k_1 + \alpha \left(k_2 + k_3 \right) \right] T + k_1 \right\} \|\varphi - \psi\| .$$

By the contraction mapping principle, H has a fixed point in P_T and by Lemma 2.4, this fixed point is a solution of (1). The proof is complete. □

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