LE MATEMATICHE
Vol. LX (2005) – Fasc. I, pp. 41–58

MIXED TYPE OF FREDHOLM-VOLTERRA INTEGRAL EQUATION

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In this paper, under certain conditions, the solution of mixed type of Fredholm-Volterra integral equation is discussed and obtained in the space $L_2(-1, 1) \times C[0, T]$, $T < \infty$. Here, the singular part of kernel of Fredholm-Volterra integral term is established in a logarithmic form, while the kernel of Fredholm-Volterra integral term is a positive continuous function in time and belongs to the class $C[0, T]$, $T < \infty$. The solution, when the mixed type integral, takes a system form of Fredholm integral equation of the first or second kind are discussed.

1. Introduction.

Many problem of mathematical physics, engineering and contact problems in the theory of elasticity lead to the integral equation of the first kind, see [1], [2]. Mkhitarian and Abdou, using Krein's method, obtained the spectral relationships for the FIE with logarithmic kernel and Carleman kernel, see [3], [4], respectively. The importance of Carleman kernel came from the work of Arytiunian [5] who has shown that, the contact problem of nonlinear theory of plasticity, in its first approximation reduce to a FIE of the first kind with Carleman kernel. Using potential theory method [6] Abdou and Hassan, in [7],

Entrato in redazione il 4 Marzo 2004.

Keywords: Fredholm-Volterra integral equation (F-VIE), Contact Problem, Chebyshev Polynomial, Logarithmic kernel, Potential theory method.

obtained the spectral relationships for the FIE of the first kind with logarithmic kernel. Also, in [8] the eigenvalue and eigenfunction are obtained for the FIE of the first kind with Carleman kernel. Abdou, in [9], [10], using potential theory method, obtained the spectral relationships for the FIE of the first kind with generalized potential kernel and Macdonold kernel, respectively.

In this work, and the following work, we will consider the mixed integral equation

$$\int_0^t \iint_\Omega F(t,\tau)k(x,y)\phi(y,\tau)\,dy d\tau + \int_0^t G(t,\tau)\phi(x,\tau)\,d\tau = f(x,t), \tag{1.1}$$

$$x = \bar{x}(x_1, x_2, x_3), \quad y = \bar{y}(y_1, y_2, y_3), \quad (x, y) \in \Omega, \; t, \tau \in [0, T], T < \infty$$

under the condition

$$\int_\Omega \phi(x,t)\,dx = P(t). \tag{1.2}$$

Here, the two given functions $F(t,\tau)$ and $G(t,\tau)$ which represent the kernels of Volterra integral term, are positive and continuous in the class $C[0,T]$, for all values of the time $t, \tau \in [0,T]$, $T < \infty$. The function $k(x,y)$, which has a term behavied badly in the domain Ω, is called the kernel of Fredholm integral term. The given function $f(x,t)$ is continuous with its partial dervatives with respect to position and time and belongs to the space $L_2(\Omega) \times C[0,T]$. The unknown function $\phi(x,t)$ is called the potential function of the mixed integral equations, and its result will be discussed in the space $L_2(\Omega) \times C[0,T]$.

In order to guarantee the existance of unique solutions of (1.1), under the condition (1.2), we assume the following conditions:

(i) The kernel of position $k(x,y) \in C([\Omega] \times [\Omega])$, and satisfies the following

$$\left\{ \iint_\Omega k^2(x,y)dxdy \right\}^{1/2} = A, \qquad A \text{ is a constant},$$

where $x = \bar{x}(x_1, x_2, x_3)$, $\quad y = \bar{y}(y_1, y_2, y_3)$.

(ii) For all values of $t, \tau \in [0,T]$ the functions $F(t,\tau)$ and $G(t,\tau)$ belong to $C([0,T] \times [0,T])$ and satisfy $|F(t,\tau)| < B$, $|G(t,\tau)| < D$ where B and D are constants.

(iii) The function $f(x,t) \in L_2(\Omega) \times C[0,T]$.

(iv) The unknown function $\phi(x,t)$ will satisfy Hölder condition with respect to time

$$|\phi(x,t_1) - \phi(x,t_2)| \le E(x)|t_1 - t_2|^\alpha, \qquad (0 < \alpha < 1)$$

and Lipschitz condition with respect to position

$$|\phi(x_1, t) - \phi(x_2, t)| \leq H(t)|x_1 - x_2|,$$

where $E(x)$ and $H(t)$ are continuous functions in x and t respectively.

2. Formulation of the problem.

Consider the integral equation

$$\int_0^t \int_{-1}^1 F(t,\tau)k(\frac{x-y}{\lambda})\phi(y,\tau)\,dyd\tau + \tag{2.1}$$

$$+\int_0^t G(t,\tau)\phi(x,\tau)\,d\tau = [\gamma(t) - f(x)] = f(x,t), \quad \lambda \in (0,\infty)$$

$$k(v) = \left(\frac{1}{2}\right)\int_{-\infty}^{\infty} \frac{\tanh u}{u} e^{iuv}\,du, \qquad i = \sqrt{-1} \tag{2.2}$$

under the condition

$$\int_{-1}^1 \phi(x,t)\,dx = P(t) \tag{2.3}$$

As in [3], p. 32, the kernel of (2.2) can be written in the form

$$k(v) = 1/2\int\limits_{-\infty}^{\infty} \frac{\tanh u}{u} e^{iuv} du = -\ln|\tanh\frac{\pi v}{4}|,\ v = \frac{x-y}{\lambda},\ \lambda \in (0,\infty) \tag{2.4}$$

If $\lambda \longrightarrow \infty$ and $(x-y)$ is very small, so that the condition $\tanh z \simeq z$, then we have

$$\ln|\tanh\frac{\pi v}{4}| = \ln|v| - d, \qquad d = \ln\frac{4\lambda}{\pi}, \tag{2.5}$$

In this case, the integral equation (2.1) will take the form

$$-\int_0^t \int_{-1}^1 F(t,\tau)(\ln|x-y| - d)\phi(y,\tau)\,dyd\tau + \tag{2.6}$$

$$+\int_0^t G(t,\tau)\phi(x,\tau)\,d\tau = f(x,t),$$

The integral equation (2.6) can be investigated from the contact problem of a rigid surface (G, ν) having an elastic material where G is the displacement magnitude, ν is Poisson's cofficient. If the stamp of length 2 unit, and its surface is describing by the formula $g(x)$, is impressed into an elastic layer surface of a strip by a variable force $P(t)$, $0 \le t \le T < \infty$, whose eccentricity of application $e(t)$, that cases rigid displacement $\gamma(t)$. If the function $F(t,\tau)$ represents the resistance force of material in the domain contact through the time $t \in [0, T]$, while $G(t,\tau)$ is the external force of resistance, which is supplied through the contact domain.

3. Method of solution.

To obtain the solution of (2.6) under the condition (2.3) we divide the interval $[0, T]$, $0 \le t \le T < \infty$ as $0 = t_0 \le t_1 < ... < t_N = T$, where $t = t_l$, $l = 1, 2, ..., N$, to get

$$-\int_0^{t_l}\int_{-1}^{1} F(t_l,\tau)(\ln|x-y|-d)\phi(y,\tau)\,dy d\tau + \tag{3.1}$$

$$+\int_0^{t_l} G(t_l,\tau)\phi(x,\tau)\,d\tau = f(x,t_l).$$

Hence, we have

$$-\sum_{j=0}^{l} u_j F(t_l,t_j)\int_{-1}^{1}(\ln|x-y|-d)\phi(y,t_j)\,dy+ \tag{3.2}$$

$$+\,O(h_l^{P+1}) + \sum_{j=0}^{l} v_j G(t_l,t_j)\phi(x,t_j) + O(h_l^{P+1}) = f(x,t_l),$$

$$(h_l^{P+1} \longrightarrow 0,\, h_l^{\tilde{P}+1} \longrightarrow 0,\, P > 0,\, \tilde{P} > 0)$$

where $h_{\bar{l}}, h_{\bar{l}} = Max_{0\le j\le l, 0\le j\le \bar{l}} h_j$; $\qquad h_j = t_{j+1} - t_j$.

The values of u_j ,P ,v_j and $\tilde{P}$ are depending on the number of drivatives of $F(t,\tau)$ and $G(t,\tau)$ with respect to t. For example, if $F(t,\tau) \in C^4[0, T]$, then we have $P = 4$, $l \simeq 4$ in the first term of (3.2), so we get $u_0 = \frac{1}{2}h_0$, $u_4 = \frac{1}{2}h_4$, $u_i = h_i$,$i = 1, 2, 3$. While, if $G(t,\tau) \in C^3[0, T]$, then we have $\tilde{P} = 3$, $\acute{l}$ and

$v_0 = \frac{1}{2}h_0$, $v_3 = \frac{1}{2}h_3$, $v_1 = h_1$, $v_2 = h_2$. More information for the characteristic points and the quadrature cofficients are found in [11], [12].

Using the following notations

$$\begin{aligned} &F(t_l, t_j) = F_{l,j}, \qquad G(t_l, t_j = G_{l,j}, \\ &\phi(y, t_j) = \phi_j(y) \quad \text{and} \quad f(x, t_l) = f_l(x), l = 0, 1, ...N; 0 \le j \le l, \end{aligned} \tag{3.3}$$

the formula (3.2) can be adapted in the form

$$\sum_{j=0}^{l} v_j G_{j,l}\phi_j(x) - \sum_{j=0}^{l} u_j F_{j,l} \int_{-1}^{1} (\ln|x - y| - d)\phi_j(y)\,dy = f_l(x). \tag{3.4}$$

Also, the boundary condition of (2.3) becomes

$$\int_{-1}^{1} \phi_l(x)dx = P_l. \tag{3.5}$$

The formula (3.4) represents a linear system of FIE of the second kind or of the first kind according to the relations between the number of derivatives of $F(t, \tau)$, $G(t, \tau)$ with respect to t for all values of $\tau \in [0, T]$. So we must study the following cases:

Case (1): If $G(t, \tau)$ has i derivatives, $i < l$, we have

$$-\sum_{j=i+1}^{l} u_j F_{j,l} \int_{-1}^{1} (\ln|x - y| - d)\phi_j(y)\,dy = f_l(x) - \sum_{j=0}^{i} \mu_j \phi_j(x) \tag{3.6}$$

The formula (3.6) represents a linear system of FIE of the first kind, where μ_i, $0 \le i < l$ are constants and $\phi_j(x)$, $0 \le j \le i$ can be obtained from the following integral equation

$$\sum_{j=0}^{i} u_j G_{j,l}\phi_j(x) - \sum_{j=0}^{i} u_j F_{j,l} \int_{-1}^{1} (\ln|x - y| - d)\phi_j(y)\,dy = f_i(x) \tag{3.7}$$

The formula (3.7) represents a linear system of FIE of the second kind.

Case (2): If the number of derivatives of $F(t, \tau)$ and $G(t, \tau)$ is equal, the integral equation (3.4) for all values of j, $0 \le j \le l$, represents a FIE of the second kind.

Case (3): If $F(t,\tau)$ has i derivatives, $i<l$, we have the following system

$$\sum_{j=0}^{i} v_j G_{j,l}\phi_j(x) == f_{i+1}(x) - \sum_{j=0}^{i} \mu_j^* \phi_j(x) f_i(x) \tag{3.8}$$

where μ_j^* ,$0 \le j \le l$, are constants and $\psi_j(x)$, $0 \le j \le l$ is the solution of the integral equation

$$\sum_{j=0}^{i} v_j G_{j,l}\psi_j(x) - \sum_{j=0}^{i} u_j F_{j,l} \int_{-1}^{1} (\ln|x-y| - d)\psi_j(y)\,dy = f_i(x) \tag{3.9}$$

4. Fredholm integral equation of the second kind.

To obtain the solution of (3.4) for all values of j, when the two functions $F(t,\tau)$ and $G(t,\tau)$ have the same number of derivatives, we write it in the following

$$\mu_l\phi_l(x) - \acute{\mu}_l \int_{-1}^{1} (\ln|x-y| - d)\phi_l(y)\,dy = f_l(x) - \sum_{j=0}^{l-1} v_j G_{l,j}\phi_j(x) \tag{4.1}$$

$$+ \sum_{j=0}^{l-1} u_j F_{j,l} \int_{-1}^{1} (\ln|x-y| - d)\phi_j(y)\,dy$$

$$(\mu_l = \frac{h_l}{2} G_{l,l}, \quad \acute{\mu}_l = \frac{h_l}{2} F_{l,l}, \quad G_{l,l} \neq 0, \quad F_{l,l} \neq 0)$$

where $\bar{h}_l, \bar{h}_{\acute{l}} = Max_{0\le j\le l, 0\le j\le \acute{l}} h_j$; $\qquad h_j = t_{j+1} - t_j$

The solution of (4.1) can be obtained by the recurrence relation. For $l=0$, we have the integral equation

$$\mu_0\phi_0(x) - \acute{\mu}_0 \int_{-1}^{1} (\ln|x-y| - d)\phi_0(y)dy = f_0(x) \tag{4.2}$$

Differentiating (4.2) with respect to x , we get

(4.3)

$$\mu\frac{d\phi_0(x)}{dx} - \int_{-1}^{1} \frac{\phi_0(y)}{x-y}\,dy = g_0(x), \ (\mu = \frac{\mu_0}{\acute{\mu}_0}, g_0(x) = \frac{1}{\acute{\mu}_0}\frac{df_0(x)}{dx}, \acute{\mu}_0 \neq 0)$$

The formula (4.3) represents a FIE of the second kind with Cauchy kernel. Using, in (4.3), the substitution $y = 2u - 1$, $x = 2v - 1$, we have

$$\frac{d\Theta}{dx} - \lambda \int_0^1 \frac{\Theta(u)}{u - v}\, du = h(u), \quad (\lambda = \frac{2}{\mu}, h(u) = \frac{2}{\mu} g_0(2u - 1)) \tag{4.4}$$

Under the boundary conditions $\Theta(0) = \Theta(1) = 0$, Frankel in his work [13], obtained the solution of (4.4) in the form of Chebyshev polynomials. Equation (4.4) has appeared in both combined infrared gaseous radiation and molecular conduction. A numerical method, Toeplitz matrices, is used, in [14], to obtain the solution of (4.2) and the error estimate is calculated. A series form of Legendre polynomial is used, in [15] to obtain the solution of (4.2) and the result is used to obtain the solution of the F-VIE of the second kind with logarithmic kernel with respect to Fredholm integral term and continuous function with respect to Volterra integral term.

To obtain the solution of (4.1), we will use a series in the Chebyshev polynomials form. For this, set the function $R(x)$ which charactrizes the singular behaviour of $\phi_l(x)$ in the form

$$R(x) = (1 + x)^{-\frac{1}{2}+\alpha}(1 - x)^{\frac{1}{2}+\beta}$$

where $\alpha, \beta = -1, 0, 1$ such that $-1 < -\frac{1}{2} + \alpha < 1$, $-1 < \frac{1}{2} + \beta < 1$. Let $\alpha = 0$, $\beta = -1$, then $R(x) = (1 - x^2)^{-\frac{1}{2}}$, which is called the weight function of the Chebyshev polynomials $T_n(x)$, $n = 0, 1, 2, ...$ Introduce new unknown functions $G_l(x)$, $0 \le l \le N$, where $\phi_l(x)$ will behave like

$$\phi_l(x) = R(x)\, G_l(x), \qquad R(x) = (1 - x^2)^{-\frac{1}{2}} \tag{4.6}$$

Now, for the numerical solution of (4.1), we express $G_l(x)$ as

$$G_l(x) = \sum_{n_l=0}^{\infty} a_{n_l} T_{n_l}(x) \tag{4.7}$$

Hence, we have

$$\phi_l(x) = \sum_{n_l=0}^{\infty} a_{n_l} \frac{T_{n_l}(x)}{\sqrt{(1 - x^2)}} \tag{4.8}$$

which can be truncated to

$$\phi_l^M(x) = \sum_{n_l=0}^{M} a_{n_l} \frac{T_{n_l}(x)}{\sqrt{(1 - x^2)}} \tag{4.9}$$

So, (4.1) tends to

$$\mu_l \sum_{n_l=0}^{M} a_{n_l} \frac{T_{n_l}(x)}{\sqrt{(1-x^2)}} - \acute{\mu}_l \begin{cases} \pi a_{0_i} \,(\ln 2 - d) & n_l = 0 \\ \pi \sum_{n_l=1}^{M} a_{n_l} \frac{T_{n_l}(x)}{n_l} & n_l \geq 1 \end{cases} \tag{4.10}$$

$$= \sum_{n_l=0}^{M} f_{n_l} \frac{T_{n_l}(x)}{\sqrt{(1-x^2)}} - \sum_{j=0}^{l-1} v_j G_{l,j} \sum_{n_l=0}^{M} a_{n_l} \frac{T_{n_l}(x)}{\sqrt{(1-x^2)}}$$

$$+ \sum_{j=0}^{l-1} u_j F_{j,l} \begin{cases} \pi a_{0_j} \,(\ln 2 - d) & n_j = 0 \\ \pi \sum_{n_j=1}^{M} a_{n_j} \frac{T_{n_j}(x)}{n_j} & n_j \geq 1 \end{cases}$$

where

$$f_{n_l} = \frac{2}{\pi} \int_{-1}^{1} \frac{f_l(x)\, T_{n_l}(x)}{\sqrt{(1-x^2)}}\, dx, \qquad n_l \geq 1 \tag{4.11a}$$

and

$$f_{n_l} = \frac{1}{\pi} \int_{-1}^{1} \frac{f_l(x)}{\sqrt{(1-x^2)}}\, dx \tag{4.11b}$$

The previous results of (4.10) is obtained after using the following spectral relation [16]

$$\int_{-1}^{1} (\ln|x-y| - d) \frac{T_n(y)}{\sqrt{(1-y^2)}}\, dy = \begin{cases} \pi \,(\ln 2 - d) & n = 0 \\ \frac{\pi}{n} T_n(x) & n \geq 1 \end{cases} \tag{4.12}$$

The formula (4.10) leads us to discuss the following cases:

Case (i): For $n_l = 0$, $l = 1, 2, \ldots$, we have

$$\frac{\mu_l a_{0_l}}{\sqrt{(1-x^2)}} - \acute{\mu}_l \pi a_{0_l} \,(\ln 2 - d) = \frac{f_{0_l}}{\sqrt{(1-x^2)}} \tag{4.13}$$

$$+ \pi \,(\ln 2 - d) \sum_{j=0}^{l-1} a_{0_j} u_j F_{j,l} - \sum_{j=0}^{l-1} v_j G_{l,j} \frac{a_{0_l}}{\sqrt{(1-x^2)}}, \quad 0 \leq l \leq N.$$

Integrating (4.13) with respect to x, we get

$$a_{0_l} = \frac{f_{0_l} + 2(\ln 2 - d) \sum\limits_{j=0}^{l-1} a_{0_j} u_j F_{j,l} - \sum\limits_{j=0}^{l-1} v_j G_{l,j} a_{0_j}}{\mu_l - 2(\ln 2 - d)\acute{\mu}_l}, \; 0 \leq l \leq N. \tag{4.14}$$

Eq. (4.14) represents the zero level of the eigenvalue of the potential function $\phi_l(x)$, $0 \leq l \leq N$, and must satisfy the relations for all values of l

$$\frac{\mu_l}{\acute{\mu}_l} \neq 2(\ln 2 - d) \tag{4.15}$$

a) when $G(t, \tau)$ and $F(t, \tau)$ have the same order of derivatives, the formula (4.14) becomes

$$a_{0_l} = \frac{f_{0_l} + \sum_{j=0}^{l-1} b_j H_{j,l} a_{0_j}}{h_l G_{l,l}(1 - 2(\ln 2 - d))}, \quad d \neq \frac{1}{2}[1 - 2\ln 2] \tag{4.16}$$

where $b_j H_{j,l} = 2(\ln 2 - d) u_j F_{j,l} - v_j G_{l,j}$.

b) If $G(t, \tau) = 0$ for all values of $t, \tau \in [0, T]$, we get

$$a_{0_l} = \frac{f_{0_l} + 2(\ln 2 - d) \sum_{j=0}^{l-1} a_{0_j} u_j F_{j,l}}{-h_l(\ln 2 - d) F_{l,l}}, \quad h_l = t_{l+1} - t_l \tag{4.17}$$

c) If $F(t, \tau) = 1$, $G(t, \tau) = 0$, we have

$$a_{0_l} = \frac{f_{0_l} + 2(\ln 2 - d) \sum_{j=0}^{l-1} H_j}{-h_l(\ln 2 - d)}, \quad d \neq \frac{1}{2}[1 - 2\ln 2] \tag{4.18}$$

Case (ii): For $n_l = 0$, $l = 0$, the formula (4.13) becomes

$$\frac{\mu_l a_{0_0}}{\sqrt{(1 - x^2)}} - \acute{\mu}_l \pi a_{0_0} (\ln 2 - d) = \frac{f_{0_l}}{\sqrt{(1 - x^2)}} \tag{4.19}$$

Integrating (4.19) with respect to x, finally, we have

$$a_{0_0} = \frac{f_{0_0}}{\mu_0 - 2(\ln 2 - d)\acute{\mu}_0}, \tag{4.20}$$

Case (iii): For $n_l \geq 1$, $l = 0, 1, ..., N$, the formula (4.10) becomes

$$\mu_l \sum_{n_l=0}^{M} a_{n_l} \frac{T_{n_l}(x)}{\sqrt{(1 - x^2)}} - \acute{\mu}_l \pi \sum_{n_l=1}^{M} a_{n_l} \frac{T_{n_l}(x)}{n_l} \tag{4.21}$$

$$= \sum_{n_l=0}^{M} f_{n_l} \frac{T_{n_l}(x)}{\sqrt{(1 - x^2)}} - \sum_{j=0}^{l-1} \sum_{n_l=0}^{M} v_j G_{l,j} a_{n_l} \frac{T_{n_l}(x)}{\sqrt{(1 - x^2)}}$$

$$+\pi \sum_{j=0}^{l-1} \sum_{n_j=1}^{M} u_j F_{j,l} a_{n_j} \frac{T_{n_j}(x)}{n_j}$$

Multiplying both sides of (4.21) by the term $T_{n_j}(x)dx$, and integrating with respect to x from -1 to 1, then using the following [17]

$$T_m(x)\, T_n(x) = \frac{1}{2}[T_{m+n}(x) + T_{m-n}(x)], \tag{4.22}$$

$$\int_{-1}^{1} T_n(x)\, dx = \begin{cases} \frac{2}{1-n^2}, & n = 0, 2, 4, ... \\ 0 & n = 1, 3, 5, ... \end{cases}$$

we get

$$\mu_l a_{m_l} - 2\acute{\mu}_l \sum_{n_l=0}^{M} \frac{A_{n_l,m_l}}{n_l} a_{n_l} = H_{m_l}, \tag{4.23}$$

where

$$H_{m_l} = f_{m_l} - \sum_{j=0}^{l-1} \upsilon_j G_{l,j} a_{m_l} + 2 \sum_{j=0}^{l-1} \sum_{n_j=1}^{M} u_j F_{j,l} \frac{A_{n_l,m_l}}{n_l} a_{n_j}, \tag{4.24}$$

$$m_l, n_l = 1, 2, ..., M; \quad 0 \le l \le N$$

and

$$A_{n_l,m_l} = \begin{cases} \frac{1}{1-(n_l+m_l)^2} + \frac{1}{1-(n_l-m_l)^2} & n_l + m_l \quad \text{even} \\ 0 & n_l + m_l \quad \text{odd} \end{cases} \tag{4.25}$$

Abdou and Bassen, in [18] proved that, for the system (4.23), for $l = 0$, is bounded and has a unique solution. By following the same way of Abdou and Bassen, in [18], we can prove that the system of (4.23), for all values of $0 \le l \le N$, is bounded, when $M \longrightarrow \infty$. To prove that the system (4.26) has a unique solution for $M \longrightarrow \infty$, we write it in the form

$$a_{m_l} = L_{m_l} + \lambda_l \sum_{n_l=1}^{\infty} R_{n_l,m_l} a_{n_l} \tag{4.26}$$

where

$$L_{m_l} = \frac{1}{\mu_l} H_{m_l}, \quad \lambda_l = \frac{2\acute{\mu}_l}{\mu_l}, \quad R_{n_l,m_l} = \frac{1}{n_l} A_{n_l,m_l}$$

Assume

$$\acute{S}_m = \acute{\lambda}_l \sum_{n=1}^{\infty} |R_{n_l,m_l}| = \lambda_l \sum_{n_l=1}^{\infty} \frac{1}{n_l} |A_{n_l,m_l}|$$

Applying Caudy- Minkowski inequality, we get

$$S_m \leq \lambda_l [\sum_{n=1}^{\infty} \frac{1}{n^2}]^{\frac{1}{2}} \sum_{n_l=1}^{\infty} [A^2_{n_l,m_l}]^{\frac{1}{2}} \tag{4.27}$$

Finally, we get

$$\acute{\mu}_l \leq 0.3 g \mu_l, \tag{4.28}$$

which represents the condition to have a unique solution, and the values of $|a_{m_l}|$ must satisfy the inequality

$$|a_{m_l}| \leq \frac{\mu_l |L_{m_l}|}{\mu_l - 2\acute{\mu}_l} \tag{4.29}$$

As special case, when $n_l = m_l$, $0 \leq l \leq M$, using uniqueness condition $\frac{\acute{\mu}_l}{\mu_l} < \frac{1}{\|k\|}$ to determine $\| k \|$, we get

$$|\frac{\acute{\mu}_l}{\mu_l}| < \frac{n_l(4n_l^2 - 1)}{4(n_l^2 - 1)} \tag{4.30}$$

Therefore

$$\| k \| = \max \begin{cases} \frac{4(2n_l^2-1)}{n_l(4n_l^2-1)}, & n \geq 1, 0 \leq l \leq M \\ \frac{1}{\ln 2 - d}, & n = 0 \end{cases} \tag{4.31}$$

Lemma 4.1. *This method is said to be convergent of order r in [-1, 1], if and only if for N sufficiently large, there exist a constant $D > 0$ independent of N such that*

$$\| \Phi(x) - \Phi_N(x) \| \leq DN^{-r} \tag{4.32}$$

Also, the error term E_N can be given by the relation

$$E_N = |\Phi - \Phi_N|$$

where $\Phi_N \longrightarrow \Phi$ as $N \longrightarrow \infty$.

5. Fredholm Integral Equation of the first kind..

According to case (1), in Section 3, when the derivatives of $G(t, \tau)$, will respect to t, for all $\tau \in [0, T]$ is less than the derivatives of $G(t, \tau)$ with respect to the same argument, we have the integral equation

$$\sum_{j=i+1}^{l} u_j F_{j,l} \int_{-1}^{1} [\ln \frac{1}{|x-y|} + d]\phi_j(y)\, dy = H_l(x) \qquad l \geq i+1 \tag{5.1}$$

where

$$H_l = f_l(x) - \sum_{j=0}^{i} \mu_i \Psi_i(x)$$

and $\Psi_i(x)$ represent the potential function of the integral equation (3.7).

The integral equation (5.1) represents a FIE of the first kind with logarithmic kernel, and its solution will be discussed, using potential theory method, under the condition

$$\int_{-1}^{1} \phi_l(x)\, dx = P_l, \qquad 0 \leq l \leq N \tag{5.2}$$

Introduce the logarithmic potential function

$$U_l(x, z) = \sum_{j=i+1}^{l} u_j F_{j,l} \int_{-1}^{1} [\ln \frac{1}{\sqrt{(x-y)^2 + z^2}} + d]\phi_j(y)\, dy \tag{5.3}$$

Eq. (5.3), under the condition (5.2) reduces to the Dirichlet boundary value problem

$$\begin{aligned}
&\Delta U_l(x, z) = 0, \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}, \qquad (x, z) \notin (-1, 1) \\
&U_l(x, z)|_{z=0} = H_l(x) \qquad x \in (-1, 1) \\
&U_l(x, z) \simeq P_l(\ln \frac{1}{r} + d), \qquad r = \sqrt{x^2 + z^2} \\
&P_l(\ln \frac{1}{r} + d) \longrightarrow \text{ finite term,} \quad \text{as } r \longrightarrow \infty
\end{aligned} \tag{5.4}$$

The solution of the integral equation (5.1) is equivalent to the solution of the Dirichlet problem (5.4). After constructing the functions $U_l(x, z)$, the potential functions $\phi_j(x)$, $i+1 \leq j \leq l$, will be determined from the formula

$$\phi_l(x) = -\frac{1}{\pi} \lim_{z \longrightarrow 0} sgn\, z \, \frac{\partial U_l(x, z)}{\partial z}, \qquad x \in (-1, 1) \tag{5.5}$$

Assume the density source functions

$$W_l(x,z) = U_l(x,z) - P_l(\ln\frac{1}{r} + d) \tag{5.6}$$

So, the boundary value problem (5.4) becomes

$$\begin{aligned} &\Delta W_l(x,z) = 0, \qquad (x,z) \not\in (-1,1) \\ &W_l(x,z)|_{z=0} = H_l(x) - P_l(\ln r - d) \\ &W_l(x,z) \longrightarrow 0, \qquad \text{as } r \longrightarrow \infty \end{aligned} \tag{5.7}$$

Consequently, Eq. (5.5) takes the form

$$\phi_l(x) = -\frac{1}{\pi} sgn_{z\longrightarrow 0}\ \{z[\frac{\partial W_l(x,z)}{\partial z} - P_l\ \delta(x)]\}, \qquad x \in (-1,1) \tag{5.8}$$

where $\delta(x)$ is the Dirac-delta function.

The boundary value problem (5.7) can be constructed by the method of conformal mapping, see [9], that transforms a given complicated region into a simpler one. For this aim, we use the mapping function,

$$v = \frac{1}{2}\omega(\zeta) = \frac{1}{2}(\zeta + \zeta^{-1}),\ v = x + iy, \tag{5.9}$$

$$i = \sqrt{-1},\ v = \rho e^{i\theta},\ 0 \leq \theta \leq 2\pi,$$

which maps the region in (x, y) plane into the region outside the unit circle γ, such that $\frac{d\omega(\zeta)}{d\zeta}$ does not vanish or becomes infinite outside γ. The mapping function (5.9) maps the upper and the lower half-plane $(x,z) \in (-1,1)$ into the lower and the upper of the semi-circle $\rho = 1$, respectively. Moreover the point $v \longrightarrow \infty$ will be mapped into the point $\zeta = 0$. Using the mapping (5.9), the function of Eq. (5.6) takes the following form

$$N_l(\rho,\theta) = M_l(\rho,\theta) - P_l(2\rho + d) \tag{5.10}$$

where $N_l(\rho,\theta) = W_l(x,y) = W_l(\frac{1}{2}(\rho + \frac{1}{\rho})\cos\theta,\ \frac{1}{2}(\rho - \frac{1}{\rho})\sin\theta)$ and $M_l(\rho,\theta) = U_l(\frac{1}{2}(\rho + \frac{1}{\rho})\cos\theta,\ \frac{1}{2}(\rho - \frac{1}{\rho})\sin\theta)$.

In view of Eq. (5.10), the boundary value problems of (5.7) will be transformed to

$$\begin{aligned} &\frac{\partial^2 N_l}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial N_l}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 N_l}{\partial \theta^2}, \qquad (\rho \leq 1, \quad -\pi \leq \theta \leq \pi) \\ &N_l(0,\theta) = 0, \\ &N_l(1,\theta)| = \tilde{H}_l(\theta) - P_l(\ln 2 + d) \\ &\tilde{H}_l(\theta) = H_l(x) = H_l(\rho\cos\theta), \qquad \rho = 1. \end{aligned} \tag{5.11}$$

Consequently, after using the chain rule, Eq. (5.8) is transformed to

(5.12) $$\phi_l(\cos\theta) = (\pi|\sin\theta|)^{-1}[P_l + \frac{\partial N_l}{\partial\rho}]_{\rho=1}$$

To solve the Dirichlet problem of (5.11), we use the Fourier series method, see [7]

$$N_l(\rho,\theta) = \sum_{n_l=0}^{\infty} a_{n_l}\rho^{n_l}\cos n_l\theta, \qquad (0 \le l \le N)$$

where

(5.13)

$$a_{0_l} = \frac{1}{2\pi}\int_{-\pi}^{\pi} N_l(\theta)\ d\theta,\ n_l = 0\ a_{0_l} = \frac{1}{2\pi}\int_{-\pi}^{\pi} N_l(\theta)\cos n_l\theta\, d\theta,\ n_l \ge 0.$$

substitute (5.13) in (5.11) after setting the given function $H_l(x) = \nu_{n_l}T_{n_l}(x)$, where ν_{n_l} are known constants, $T_{n_l}(x)$ are the Chebyshev polynomials of the first type of order n; $n = 0, 1, 2, ...$, then use the result in (5.12), we get

(5.14) $$\phi_l(\cos\theta) = \begin{cases} n_l\nu_{n_l}\cos n_l\theta(\pi\ \sin\theta)^{-1}, & n_l = 1, 2, ... \\ \nu_{n_l}P_l(\pi\ \sin\theta)^{-1} & n_l = 0 \end{cases} \qquad (0 \le l \le N)$$

and

$$P_l = [\pi(\ln 2 + d)]^{-1}\int_{-\pi}^{\pi} H_l(\cos\theta)\, d\theta, \quad n_l = 0$$

. Using (5.14) in (5.1), where, $\theta = \cos^{-1}x$, we have the following spectral relationships

(5.15) $$\sum_{j=i+1}^{l} u_j F_{j,l}\int_{-1}^{1}\frac{1}{\sqrt{1-y^2}}[\ln\frac{1}{|x-y|} + d]T_{n_l}(y)\, dy =$$

$$= \begin{cases} \pi(\ln 2 + d),\ n_l = 0 \\ \frac{\pi}{n_l}T_{n_l}(x) \end{cases}$$

which represents the spectral relationships for the Fredholm integral equations of the first kind with logarithmic kernl.

Numerical Results. To consider the behaviour of the solution function $\phi_l(x)$ which is represented numerically by $\phi_l^M(x)$ in Eqn (4.9) with Eqns (4.14), (4.21) and (4.24). The general behaviour can be described in figure 1.

Little difference may be occurred for changing the parameter values.

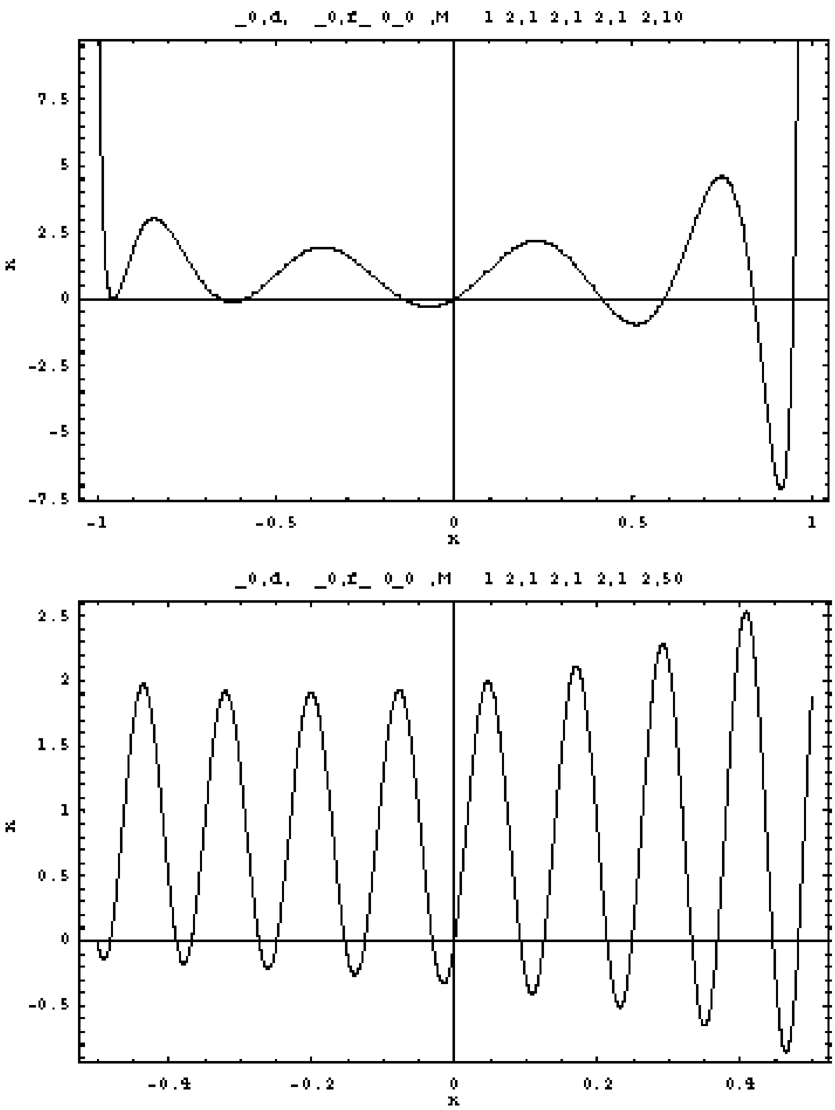

FIGURE 1.

Conclusion.

From the above results and discussions, the following may be concluded:

(1) The contact problem of a rigid surface having an elastic material, when a stamp of length 2 unit is impressed into an elastic layer surface of a strip by a variable force $P(t)$, $0 \leq t \leq T < \infty$, whose eccentricity of application $e(t)$, represents a Fredholm-Volterra integral equation of mixed type.

(2) The numerical method used gives us a system of Fredholm integral equation, which it's solution can be obtained using the recurrence relations.

(3) The kind of the system of Fredholm integral equation depends on the relation between the number of derivatives of $F(t, \tau)$, which represents the resistance force of material in the domain contact through the time $t \in [0, T]$ and $G(t, \tau)$, which represents the external force of resistance supplied through the material of contact domain.

(4) When the value of $G(t, \tau) \longrightarrow 0$, $t \in [0, T]$, $T < \infty$, i.e., their is no external force of resistance, the integral equation of mixed type takes a form of a system of Fredholm integral equation of the first kind.

(5) The displacement problems of antiplane deformation of an infinite rigid strip with width 2 unit, putting on an elastic layer of thickness h is considered as a special case of this work, when

$$G(t, \tau) = 0, \qquad F(t, \tau) = 1, \qquad t = 1, \qquad F(x, t) = H.$$

where H represents the displacement magnitude and $\phi(x, 1) = \psi(x)$ is the unknown displacement stress.

(6) The problems of infinite rigid strip, with width 2 unit immpresed in a viscous liquid layer of thickness h, when the strip has a velocity resulting from the impulsive force $v - v_0 e^{-i\omega t}$, $i = \sqrt{-1}$, where v_0 is the constant velocity, ω is the angular velocity resulting from rotating the strip about z-axis, are considered as special case of our work, when $G(t, \tau) = 0$, $F(t, \tau) =$contant.

In the discussion (5) and (6), we note that, when $h \longrightarrow \infty$, this means the depth of the liquid or the thickness of the elastic material becomes an infinite

.

(7) The three kinds of the displacement problems, in the theory of elasticity and mixed contact problem which discussed in [1] are considered special case of this work.

(8) The potential function method has a large application in mathematical physics problems, where the integral equation becomes equivalent to a boundary value problem, which can be soved easily. Also the potential function method enables us to discuss the spectral relation of the integral equation.

(9) The conformal mapping, in the theory of elasticity, or in the applied sciences transforms a given complicated region into a simpler one.

(10) The mixed integral equation with Carleman kernel can be established from this work, using the following relation

$$\ln|x - y| = h(x, y)|x - y|^{-\nu}, \qquad 0 \leq \nu < 1,$$

where $h(x, y) = |x - y|^{\nu} \ln|x - y|$ is a smooth function.

(11) The Fredholm integral equation of the first or second kind with logarithmic and Carleman kernels, are considered, now, as special cases of this work.

(12) Many spectral relations, that has a large applications in mathematical physics, were established from this work.

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