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SQUARE ROOTS AND N^{TH} ROOTS IN PSEUDO$-$MICHAEL ALGEBRAS

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The purpose of this paper is to study the existence of square roots and more generally n^{th} roots in pseudo-Michael algebras. We examine several conditions that imply the existence of n^{th} roots and hermitian n^{th} roots. We also simplify the proofs of the results given by D.Sterbova.

1. Introduction

The existence of square roots plays an important role in the theory of Banach algebras and more generally in topological algebras. In [6], Gardner showed that an element of a unital Banach algebra, whose spectrum is contained in the complement of the non positive real numbers, has a unique square root with a spectrum contained in the right half complex plane. For the involutive case, Ford proved his lemma on the existence of hermitian square roots ([5]). Then Bonsall and Stirling ([3]) ameliore and simplify the proof of the lemma of Ford. In [11], Yood proved that a hermitian element possesing a positive spectrum has a hermitian n^{th} root with positive spectrum. In the more general case of locally multiplicatively convex algebras, D.Sterbova ([9]) extended the results

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of [3] and [5]. In [10], the same author studied the case of elements with an unbounded spectrum and precisely elements possesing a positive spectrum. Another study of square roots on topological algebras was published by Mart and Mati Abel([1]).

In this paper, we study the existence of square roots and more generally n^{th} roots in pseudo-Michael algebras, with or without involution. We examine several conditions that imply the existence of n^{th} roots and hermitian n^{th} roots (in the involutive case). We simplify the proofs of Sterbova's Theorems ([9],[10]) and extend them to the pseudo-convex case. We also prove that, in the Banach case, the results of Bonsall-Stirling ([3]) and Ford ([5]) can be deduced from Gardner's Theorem ([6]).

2. Preliminaries

A topological algebra A is said to be a locally multiplicatively pseudo-convex algebra if its topology can be given by mean of a family $(|.|_\lambda)_{\lambda\in\Lambda}$ of a submultiplicative p_λ-semi-norms, $0 < p_\lambda \leq 1$. In particular, if $p_\lambda = 1$, for all $\lambda \in \Lambda$, we obtain the classical locally multiplicatively convex algebras ([7]). A p-normed algebra is an algebra endowed with a submultiplicative p-norm $\|.\|_p$. A complete p-normed algebra is called a p-Banach algebra. All locally multiplicatively convex algebras and all p-normed algebras are locally multiplicatively pseudo-convex algebras. As in [2], we call a complete Hausdorff locally multiplicatively pseudo-convex algebra A as a pseudo-Michael algebra. It is known that a pseudo-Michael algebra $(A,|.|_\lambda)_{\lambda\in\Lambda}$ is the projective limit of the p_λ-Banach algebras $\widehat{A_\lambda}$, the completions of $A_\lambda = A/N_\lambda$, with $N_\lambda = \{x \in A \ / \ |x|_\lambda = 0\}$ and $\|\bar{x}\|_\lambda = |x|_\lambda$ ([2], Theorem 4.5.3). Moreover, we have $Sp_A(a) = \bigcup\limits_{\lambda} Sp_{\widehat{A_\lambda}}(a_\lambda)$ ([2], Corollary 4.5.7). Let a be an element of A. A n^{th} root of a is an element $x \in A$ with $x^n = a$. A square root of a is an element $x \in A$ with $x^2 = a$. A quasi-square root of a is an element $x \in A$ with $x \circ x = a$, where $x \circ x = 2x - x^2$. The spectum and the spectral radius of a will be denoted by $Sp(a)$ and $\rho(a)$. If $Sp(a) \subset \mathbb{R}_+^*$, we write $Sp(a) > 0$. We designate by $Rad(A)$ the Jacobson radical of A. If moreover A is endowed with an algebra involution $x \longmapsto x^*$, a is said to be hermitian (resp., normal) if $a = a^*$ (resp., $a^*a = aa^*$). We designate by $H(A)$ (resp., $N(A)$) the set of hermitian (resp., normal) elements of A. Denote by $Re(a) = \frac{1}{2}(a + a^*)$ the real part of a.

If A has a unit e and $a \in A$, then the maximal commutative subalgebra $C(a)$ of A containing a is a pseudo-Michael algebra such that

$$Sp(a) = Sp_{C(a)}(a).$$

This follows from Proposition1.7.26 and Proposition 4.4.12 ([2]). Moreover, we have $Rad(A) \neq A$ since $e \in A \setminus Rad(A)$. Hence $C(a)$ satisfies conditions of Theorem 7.2.21 ([2]). Consequently, $C(a)$ is t. spectrally Gelfand in the sense of [2], that is $Sp_{C(a)}(a) = \{\chi(a) : \chi$ continuous character on $C(a)\}$. This proves the following lemma.

Lemma 2.1. *Let A be a complex, unital pseudo-Michael algebra and $a \in A$. Then we have $Sp(a) = Sp_{C(a)}(a) = \{\chi(a) : \chi$ continuous character on $C(a)\}$, where $C(a)$ denotes the maximal commutative subalgebra of A containing a.*

Now we consider the case when A is a an involutive p-Banach algebra. By a similar proof to the case of Banach algebras ([8], Lemma 1.3), we show the following result.

Lemma 2.2. *Let $(A, \|.\|_p)$ be an involutive p-Banach algebra, $0 < p \leq 1$, and let $a \in N(a)$. Then there exists a closed, commutative and involutive subalgebra B of A containing a such that $Sp_B(x) = Sp_A(x), \forall x \in B$.*

For additional information about pseudo-Michael algebras and p-Banach algebras, see [2] and [12].

3. Existence of n^{th} roots

Throughout this section, A designates a complex, unital pseudo-Michael algebra with topology given by mean of a family $(|.|_\lambda)_{\lambda \in \Lambda}$ of submultiplicative p_λ-semi-norms, $0 < p_\lambda \leq 1$. We denote by e its unit element.

Proposition 3.1. *Let $a \in A$ with $Sp(a) > 0$. Then for every integer $n > 0$, there exists $b \in A$ such that $b^n = a$ and $Sp(b) > 0$.*

Proof. Let $a \in A$ with $Sp(a) > 0$ and n a strictly positive integer.
First, let's suppose that $\rho(e-a) < 1$. Thus the series for $\ln(a)$ converges absolutely ([2], 5.2.11) and the series

$$b = \exp(\frac{1}{n}\ln(a)) = \sum_{k=0}^{+\infty} \binom{\frac{1}{n}}{k} (a-e)^k.$$

converges in A, where $b^n = a$. Our aim is to show that $Sp(b) > 0$. Let χ be a continuous character on $C(a)$, where $C(a)$ denotes the maximal commutative subalgebra of A containing a. Then

$$\chi(b) = \sum_{k=0}^{+\infty} \binom{\frac{1}{n}}{k} (\chi(a) - 1)^k.$$

Since $Sp(a) > 0$, we have $\chi(a) > 0$ and therefore $\chi(b) = \sqrt[n]{\chi(a)} > 0$. It follows from Lemma 2.1 that $Sp_A(a) = Sp_{C(a)}(a) > 0$.
More generally, let's suppose now that we have only the condition $Sp(a) > 0$. Then $u = (e+a)^{-1}$ and $v = e - u$ exist in A and satisfy $\rho(e-u) < 1$ and $\rho(e-v) < 1$. According to the first case, there exist x and y in A such that $x^n = u$, $y^n = v$, $Sp(x) > 0$ and $Sp(y) > 0$. Thus x^{-1} and y^{-1} exist in A. Let $C(a,u)$ be the maximal commutative subalgebra of A containing a and u. Since $x, y \in C(a,u)$, we obtain $xy = yx$. Put $b = x^{-1}y$. Thus

$$(x^{-1}y)^n = (x^{-1})^n y^n = u^{-1}v = u^{-1}(e-u) = u^{-1} - e = a.$$

Moreover $Sp(b) > 0$. Indeed, if χ is a continuous character on $C(a,u)$, then we have $\chi(b) = \frac{\chi(y)}{\chi(x)} > 0$. This completes the proof. □

If now A is endowed with a continuous involution, then for every $\lambda \in \Lambda$, the following correspondence $(x + Ker|.|_\lambda)^* = x^* + Ker|.|_\lambda$ is a well defined involution on $A_\lambda = A/Ker|.|_\lambda$. In this way, the completion $\widehat{A_\lambda}$ of A_λ becomes an involutive p_λ-Banach algebra and A becomes a projective limit of the involutive p_λ-Banach algebras $\widehat{A_\lambda}$. We prove that, in several cases, the condition "all the $\widehat{A_\lambda}$'s are involutive" is sufficient to confirm the existence of hermitian n^{th} roots even when the involution in A is not necessarily continuous.

Proposition 3.2. *If A is involutive and every $\widehat{A_\lambda}$ is involutive according to the same involution of A, then for every $h \in H(A)$ with $\rho(e-h) < 1$ and every integer $n > 0$, there exists $b \in H(A)$ such that $b^n = h$.*

Proof. The algebra A is the projective limit of the involutive p_λ-Banach algebras $\widehat{A_\lambda}$. Moreover, the hermitian element h can be written

$$h = (h_\lambda)_\lambda = (\pi_\lambda(h))_\lambda,$$

where $\pi_\lambda : A \to A_\lambda$ is the canonical surjection. Since every h_λ is hermitian in

$\widehat{A_\lambda}$ and $\rho_{\widehat{A_\lambda}}(e-h_\lambda) < 1, \forall \lambda \in \Lambda$, we may assume without loss of generality that A is an involutive p-Banach algebra. So let's consider that $(A, \|.\|_p)$ is an involutive p-Banach algebra, $0 < p \leq 1$, and let $h \in H(A)$ with $\rho(e-h) < 1$. Since $\rho(e-h)^p = \lim_n \|(e-h)^n\|_p^{\frac{1}{n}} < 1$, then for every integer $n > 0$, the series

$$b = \sum_{k=0}^{+\infty} \binom{\frac{1}{n}}{k} (h-e)^k$$

converges in A, where $b^n = h$. It remains to prove that h is hermitian. By Lemma 2.2, let B denote the closed maximal commutative involutive subalgebra of A containing h. Since B is closed, it follows that $b \in B$ and hence B is normal. Let $s : A \to A/Rad(A)$ be the canonical surjection. By Theorem 9.4.3 of [2], the involution is continuous on the semi-simple p-Banach algebra $A/Rad(A)$. Therefore $s(b)^* = s(b)$ in $A/Rad(A)$, so that $b^* - b \in Rad(A)$. We can write $b = x+iy$, where $x, y \in H(A)$. Since $b \in B$, we see that $x; y \in B$ and hence $xy = yx$. By the binomial theorem, we have

$$b^n = (x+iy)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} (iy)^k.$$

As $b^n = h \in H(A)$, we obtain

$$\sum_{\substack{1 \leq k \leq n \\ k \text{ odd number}}} \binom{n}{k} x^{n-k} y^k = 0.$$

Then

$$\begin{aligned} & y \sum_{\substack{1 \leq k \leq n \\ k \text{ odd number}}} \binom{n}{k} x^{n-k} y^{k-1} \\ = \quad & nyx^{n-1} \left[e + (nx^{n-1})^{-1} \left(\sum_{\substack{3 \leq k \leq n \\ k \text{ odd number}}} \binom{n}{k} x^{n-k} y^{k-1} \right) \right] = 0. \qquad (1) \end{aligned}$$

On the other hand, h is invertible for $Sp(h) \subset \{z \in \mathbb{C} : |z-1| < 1\}$. Therefore b is invertible. Now since $b^{-1}x = b^{-1}(b-iy) = e - ib^{-1}y$ is invertible and $y = \frac{b-b^*}{2i} \in Rad(A)$, we obtain that x is invertible, and hence nx^{n-1} is invertible too. Moreover

$$\sum_{\substack{3 \leq k \leq n \\ k \text{ odd number}}} \binom{n}{k} x^{n-k} y^{k-1} \in Rad(A).$$

Thus, it follows from (1) that $nyx^{n-1} = 0$. Consequently $y = 0$ and hence $b \in H(A)$. $\square$

As a consequence, we obtain the following generalization of Yood's result ([11], Lemma 2.4)

Corollary 3.3. *If A is involutive and every $\widehat{A_\lambda}$ is involutive according to the same involution of A, then for every $h \in H(A)$ with $Sp(h) > 0$ and every integer $n > 0$, there exists $b \in H(A)$ such that $b^n = h$ and $Sp(b) > 0$.*

Proof. Let $u = (e+h)^{-1}$ and $v = e - u$. Then u and v are hermitians and satisfy $\rho(e-u) < 1$ and $\rho(e-v) < 1$. By Proposition 3.2, there exist x and y in $H(A)$ such that $x^n = u$ and $y^n = v$. Put $b = x^{-1}y$. Thus $b^n = h$ and $b \in H(A)$. Since x and y are given by the binomial expansion, we prove as in Proposition 3.1 that $Sp(b) > 0$. $\square$

Remark 3.4. Proposition 3.2 and Corollary 3.3 are valid for every unital involutive p-Banach algebra even when the involution is not continuous.

4. Existence of square roots

Let A be a complex pseudo-Michael algebra with topology given by mean of a family $(|.|_\lambda)_{\lambda\in\Lambda}$ of submultiplicative p_λ-semi-norms, $0 < p_\lambda \leq 1$. First, we prove the following generalization of a result of Gardner ([6]).

Proposition 4.1. *If A is a unital pseudo-Michael algebra and $a \in A$ with $Sp(a) \subset \mathbb{C}\setminus\mathbb{R}_-$, then there exists a unique square root x of a such that*

$Sp(x) \subset \{\lambda \in \mathbb{C} : Re(\lambda) > 0\}$. If moreover A is endowed with an algebra involution and a is hermitian, then the square root x of a is also hermitian.

Proof. By projective limit, we can write $a = (a_\lambda)_\lambda$, where $a_\lambda \in \widehat{A_\lambda}$. Then $Sp_{\widehat{A_\lambda}}(a_\lambda) \subset \mathbb{C}\setminus\mathbb{R}_-$, for $Sp_A(a) = \bigcup\limits_\lambda Sp_{\widehat{A_\lambda}}(a_\lambda)$. Put $\Omega = \{z \in \mathbb{C} : z \notin \mathbb{R}_-\}$. There is a holomorphic function f in Ω such that $f^2(z) = z$ and $f(1) = 1$. Let Γ be a closed curve such that $Sp(a)$ is contained in its interior $int\Gamma$ and $int\Gamma \cup \Gamma$ is contained in Ω. By holomorphic functional calculus in each p_λ-Banach algebra $\widehat{A_\lambda}$ ([12]), put $x_\lambda = \frac{1}{2\pi i}\int_\Gamma f(z)(z-a_\lambda)^{-1}dz$. Then $x_\lambda^2 = a_\lambda$ and $Sp(x_\lambda) \subset \{\lambda \in \mathbb{C} : Re(\lambda) > 0\}$. Using the fact that, in any p-Banach algebra $\left(B, \|.\|_p\right)$, we have

$$\rho(x)^p = \lim_n \|a^n\|_p^{\frac{1}{n}} \text{ and } \|\alpha x\|_p = |\alpha|^p \|x\|_p,\ \forall x \in B, \forall \alpha \in \mathbb{C},$$

we prove, as in ([6]), that every x_λ is unique. Thus $x = (x_\lambda)_\lambda$ is the required square root.
If moreover A is endowed with an algebra involution and a is hermitian, then $(x^*)^2 = a$ and $Sp(x^*) = \left\{\overline{\lambda} \in \mathbb{C} : \lambda \in Sp(x)\right\} \subset \{\lambda \in \mathbb{C} : Re(\lambda) > 0\}$. The unicity of x implies that $x^* = x$. $\square$

As a consequence, we obtain an extension of the essential result of [10]. Here the unicity of the square root is more ensured.

Proposition 4.2. *If A is a unital pseudo-Michael algebra and $a \in A$ with $Sp(a) > 0$, then there exists a unique square root x of a such that $Sp(x) > 0$. If moreover A is endowed with an algebra involution and a is hermitian, then x is also hermitian.*

Proof. By Proposition 4.1, there exists a unique square root x of a such that $Sp(x) \subset \{\lambda \in \mathbb{C} : Re(\lambda) > 0\}$. To show that $Sp(x) > 0$, let $\lambda \in Sp(x)$. Then $\lambda \in \mathbb{R}^*$, for $\lambda^2 > 0$ and hence $\lambda = Re(\lambda) > 0$. $\square$

Proposition 4.3. *If A is a unital pseudo-Michael algebra and $a \in A$ with $\rho(e - a) < 1$, then there exists a unique square root x of a such that $\rho(e - x) < 1$. If moreover A is involutive and a is hermitian, then x is also hermitian.*

Proof. Since $Sp(e-a) = \{1 - \lambda \in \mathbb{C} : \lambda \in Sp(a)\}$, it is easily seen that $Sp(a) \subset \mathbb{C} \setminus \mathbb{R}_-$. Thus, by Proposition 4.1, there exists a unique square root x of a such that $Sp(x) \subset \{\lambda \in \mathbb{C} : Re(\lambda) > 0\}$. It remains to prove that $\rho(e-x) < 1$. Suppose that $\rho(e-x) \geq 1$ and let r with $\rho(e-a) < r < 1$. This implies the existence of a suitable sequence $(\chi_n)_{n\geq 1}$ of continuous characters on $C(x)$ such that $\lim_n |\chi_n(e-x)| = \alpha$, $\alpha \geq 1$, where $C(x)$ denotes the maximal commutative subalgebra of A containing x (Lemma 2.1). We have $Re(\chi_n(x)) > 0$, for every n. Thus $|1+\chi_n(x)| \geq |1+Re(\chi_n(x))| > 1$ and hence

$$|\chi_n(e-x)| < |1-\chi_n(x)|\,|1+\chi_n(x)| = |\chi_n(e-a)| < r < 1.$$

By tending n to infinity, we get a contradiction that $1 \leq \alpha \leq r < 1$.
If A is involutive and a is hermitian, the unicity of x shows that it is hermitian. $\square$

If the algebra has not necessarily a unit, we obtain as a consequence the following generalization of Ford's Lemma ([5]) given, in the $l.m.c.a.$ case, by D. Sterbova ([9]).

Proposition 4.4. *If A is a pseudo-Michael algebra and $a \in A$ with $\rho(a) < 1$, then there exists a unique quasi-square root x of a such that $\rho(x) < 1$. If moreover A is involutive and a is hermitian, then x is also hermitian.*

Proof. If A has a unit, it is sufficient to apply Proposition 4.3 to the element $e-a$. Suppose that A has not a unit and denote by e the unit element of the algebra $A^1 = A+\mathbb{C}$ obtained by adjoining a unit to A. The unitization A^1 is again a pseudo-Michael algebra (see [2], Proposition 4.4.13). Since $\rho_{A^1}(a) < 1$,

there exists a unique quasi-square root x of a, in A^1, such that $\rho_{A^1}(x) < 1$. It remains to prove that $x \in A$. Let $x = b+\alpha e$, where $b \in A$, $\alpha \in \mathbb{C}$. Suppose that $\alpha \neq 0$. Thus $\alpha = 2$ since $a = \left(2b-b^2-2\alpha b\right)+\left(2\alpha-\alpha^2\right)e$. But $0 \in Sp(b)$ for A has not a unit. Hence $2 \in Sp(x)$, which contradicts the fact that $\rho(x) < 1$. $\square$

Remark 4.5.

1 In the Banach case, the same proofs of Proposition 4.3 and Proposition 4.4 permit to deduce the results of Bonsall-Stirling ([3]) and Ford ([5]) from Gardner's Theorem ([6]).

2 In [1], the authors showed some results on the existence of square roots on some topological algebras using the radius of boundedness. Since the spectral radius and the radius of boundedness are equal in a pseudo-Micheal algebra, Proposition 4.3 and Proposition 4.4 can be deduced from Corollary 2.2 and Corollary 2.5 of [1]. The approaches and methods are carried out differently.

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