As a generalization of isomorphisms of graphs, we consider path-congruences, that is maps which preserve the number of paths of any length. We construct families of pairs of non-isomorphic trees with the same path-table.

1. Introduction.

In this paper, graphs will be finite, labelled, undirected and simple. Let $G_1, G_2$ be two graphs. A path-congruence $\Phi : G_1 \rightarrow G_2$ is a bijection $V(G_1) \rightarrow V(G_2)$ such that, for every positive integer $l$, and every $v \in V(G_1)$, the number of paths of length $l$ passing through $v$ equals the number of paths of length $l$ passing through $\Phi(v)$. Note that it is not required that the number of paths in $G_1$ of length $l$ and containing $v$ in a specified position (say, as an end point) be equal to the number of paths in $G_2$ of length $l$ and containing $\Phi(v)$ in the same position. If there is a path-congruence $\Phi : G_1 \rightarrow G_2$, we say that $G_1$ and $G_2$ are path-congruent. The path-table $P(G)$ of a graph $G$ is defined as follows. It has $|V(G)|$ rows and $k$ columns, where $k$ is the maximum length of a path in $G$. To each vertex $v$ is associated a row whose entry in column $l$ is the number $p_l(v)$ of paths of length $l$ passing through $v$. In the

2000 Mathematics Subject Classification. 05C60.
Key words and phrases. Isomorphisms between trees, Path-congruence. Path-table matrix.
special case of a tree \( T \), \( k = \text{diam}(T) \). In this paper we shall only consider path-congruences between trees. The notion of path-congruence is similar to a notion introduced by Randić in [3]. We shall call Randić-relation between two trees \( T_1, T_2 \) a bijection \( \sigma : V(T_1) \to V(T_2) \) such that for every vertex \( v \) of \( T_1 \) and any integer \( l \geq 1 \), the number of paths contained in \( T_1 \) of length \( l \) and starting at \( v \), equals the number of paths contained in \( T_2 \) of length \( l \) and starting at \( \sigma(v) \). \( T_1, T_2 \) will then be said Randić-related. The Randić-table \( S(T) \) of a tree \( T \) is the rectangular array having \( n \) rows and \( \text{diam}(T) \) columns such that the \((i, j)\)-entry is the number of paths in \( T \) of length \( j \) containing the vertex \( v_i \) as an end point. This notion is equivalent, in the case of trees, to the notions which appear in the literature, differently couched, under the names of path layer matrix, path degree sequence or distance degree sequence of \( T \) ([1], [2], [4]). Also, these coincide with the Atomic Path Code of a molecule ([3]). It is clear that two trees \( T_1, T_2 \) are path-congruent (respectively Randić-related) if and only if one can renumber the vertices of \( T_2 \) such that \( S(T_1) = S(T_2) \) (resp. \( S(T_1) = P(T_2) \)). Randić conjectured that Randić-related trees are isomorphic ([3]). Slater has shown that it is not so. In ([4]) he has described an infinite set of example-pairs, and has conjectured that the unique smallest pair is that in Fig. 1 (see also [1] p. 180).

\[
\begin{array}{ccccccccccc}
1 & 5 & 6 & 9 & 15\ldots & 18 & 1 & 5 & 16 & 17\ldots & 18 \\
2 & 4 & 17 & 11\ldots & 12 & 2 & 4 & 13 & 11 & 12 & 4 & 17 & 8 \\
3 & 10 & & & & & & & & & & &
\end{array}
\]

FIGURE 1. Smallest Slater pair of non-isomorphic trees with the same Randić-table.

In this note we follow an analogous idea to prove that path-congruent trees \( T_1, T_2 \) need not be isomorphic. We point out a canonical construction, and the smallest pair \( T_1, T_2 \) we obtained.


The pairs of graphs described by Slater do not have the same path-table. For example, the pair \( T_1, T_2 \) in Figure 1 gives the path-tables \( P(T_1) \) and \( P(T_2) \) in Table 1.
Therefore $T_1$ and $T_2$ are not path-congruent. Consequently, we are led to the following problem.

**Problem.** Are path-congruent trees necessarily isomorphic?

We shall now give a negative answer. Indeed, by generalizing Slater’s construction ([4] p. 90), we obtain a class of example-pairs of path-congruent non-isomorphic trees (as well as more examples of Randić-related non-isomorphic trees). Before discussing the general construction, we show in Figure 2 the smallest such pair $U_1, U_2$ (see Corollary 2). In Table 2 the path-table and the Randić-table of this pair are given.

The fact that $U_1$ is not isomorphic to $U_2$ is easily verified by noting that in $U_1$ there are 3 couples of vertices of degree 3 (the couples $(8, 18)$, $(3, 8)$ and $(3, 17)$) such that the vertices of each couple are at distance 3, whereas in $U_2$ there are only 2 such couples.

We now proceed to illustrate the general construction by which the pair of trees shown in Figure 2 has been obtained.
FIGURE 2. Smallest pair of non-isomorphic trees with the same path-table and Randić-table.

\[
S(U_1) = S(U_2) \quad \text{and} \quad P(U_1) = P(U_2)
\]

TABLE 2. The Randić-table and the path-table of the smallest pair in Fig. 2

**Theorem 1.** There exist infinitely many pairs of non-isomorphic path-congruent trees. Moreover, these pairs are also Randić-related.

**Proof.** Let \( A_1, A_2, A_3, A_4 \) be rooted trees, with roots \( r_1, \ldots, r_4 \), and let \( A'_1, A'_2, A'_3, A'_4 \) be (respectively) isomorphic to \( A_1, A_2, A_3, A_4 \) through isomorphisms
Let $H$ be a graph with four vertices $h_1, ..., h_4$ singled out. We construct a graph $U_1$ by identifying $r_i$ with $h_i$ and another graph $U_2$ by identifying $\sigma_i(r_i)$ with $h_{\lambda(i)}$ (see Fig. 3).

Define now the following map $\Phi : U_1 \rightarrow U_2$

$$\Phi(v) = \begin{cases} 
\sigma_i(v) & \text{if } v \in A_i, i \in \{1, ..., 4\} \\
v & \text{if } v \in H \setminus \{h_1, ..., h_4\}
\end{cases}$$

Note that $\Phi$ is a well-defined bijection. In order to make $\Phi$ into a global path-congruence, it is sufficient that

1. For each $m \geq 1$ the number of paths of length $m$ within $A_i$ starting at the root $r_i$ be independent of $i$.
2. For each $i$ there is a permutation $\theta$ of $\{h_j | j \neq i\}$ such that for any $k \geq 0$, for any $j \neq i$, the number of paths of length $k$ within $H$ with end-points $h_i, h_j$ be equal to the number of paths of length $k$ within $H$ with end-points $h_i, h_{\theta(j)}$.

We can satisfy both conditions by taking, for example, the trees $A_i$ as shown in Figure 4, $H$ to be the path $\{h_1, h_2, h_3, h_4\}$, and $\theta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$.

For $n \in \mathbb{N}$ this yields infinitely many pairs of non-isomorphic path-congruent trees $U_1, U_2$. Also, by construction, it is easy to see that $S(U_1) = S(U_2)$, hence $U_1$ and $U_2$ are also Randić-related. □

**Corollary 2.** The smallest number of vertices involved by the given construction is 20.
Proof. With the same notation as in the proof of Theorem 1, suppose first that $|A_i| \leq 4$ for all $i \in \{1, ..., 4\}$. Then $A_i$ is one of the eight rooted trees shown in Figure 5.

In any case, condition (1) in the general construction implies that, for each pair $i, j \in \{1, ..., 4\}$, $A_i$ is isomorphic to $A_j$, and consequently $U_1$ is isomorphic to $U_2$. Therefore, $|A_i| \geq 5$ for all $i \in \{1, ..., 4\}$, and we get

$$|U_1| = |U_2| = |\{v \in U_1\}| = |\{v \in H \setminus \{h_1, h_2, h_3, h_4\}\}|$$

$$+ \sum_{i=1}^{4} |\{v \in A_i \setminus \{r_i\}\}| =$$

$$= |\{v \in H \setminus \{h_1, h_2, h_3, h_4\}\}| + \sum_{i=1}^{4} |A_i \setminus \{r_i\}| \geq \sum_{i=1}^{4} |A_i| \geq 4 \cdot 5 = 20.$$
More general constructions are allowed by different choices of $H$ and with more trees $A_i$ to attach to it. See Figure 6 for such an example.

REFERENCES


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