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TREES WITH THE SAME PATH-TABLE

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As a generalization of isomorphisms of graphs, we consider path-congruences, that is maps which preserve the number of paths of any length. We construct families of pairs of non-isomorphic trees with the same path-table.

1. Introduction.

In this paper, graphs will be finite, labelled, undirected and simple. Let G_1, G_2 be two graphs. A *path-congruence* $\Phi : G_1 \to G_2$ is a bijection $V(G_1) \to V(G_2)$ such that, for every positive integer l, and every $v \in V(G_1)$, the number of paths of length l passing through v equals the number of paths of length l passing through $\Phi(v)$. Note that it is not required that the number of paths in G_1 of length l and containing v in a specified position (say, as an end point) be equal to the number of paths in G_2 of length l and containing $\Phi(v)$ in the same position. If there is a path-congruence $\Phi : G_1 \to G_2$, we say that G_1 and G_2 are *path-congruent*. The path-table $\mathbb{P}(G)$ of a graph G is defined as follows. It has $|V(G)|$ rows and k columns, where k is the maximum length of a path in G. To each vertex v is associated a row whose entry in column l is the number $p_l(v)$ of paths of length l *passing* through v. In the

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special case of a tree T, $k = \text{diam}(T)$. In this paper we shall only consider path-congruences between trees. The notion of path-congruence is similar to a notion introduced by Randić in [3]. We shall call *Randič-relation* between two trees T_1, T_2 a bijection $\sigma : V(T_1) \to V(T_2)$ such that for every vertex v of T_1 and any integer $l \geq 1$, the number of paths contained in T_1 of length l and *starting* at v, equals the number of paths contained in T_2 of length l and *starting* at $\sigma(v)$. T_1, T_2 will then be said *Randič-related*. The *Randič-table* $\mathbb{S}(T)$ of a tree T is the rectangular array having n rows and $\text{diam}(T)$ columns such that the (i, j)-entry is the number of paths in T of length j containing the vertex v_i as an end point. This notion is equivalent, in the case of trees, to the notions which appear in the literature, differently couched, under the names of *path layer matrix*, *path degree sequence* or *distance degree sequence* of T ([1], [2], [4]). Also, these coincide with the Atomic Path Code of a molecule ([3]). It is clear that two trees T_1, T_2 are path-congruent (respectively Randić-related) if and only if one can renumber the vertices of T_2 such that $\mathbb{P}(T_1) = \mathbb{P}(T_2)$ (resp. $\mathbb{S}(T_1) = \mathbb{S}(T_2)$). Randič conjectured that Randić-related trees are isomorphic ([3]). Slater has shown that it is not so. In ([4]) he has described an infinite set of example-pairs, and has conjectured that the unique smallest pair is that in Fig. 1 (see also [1] p. 180).

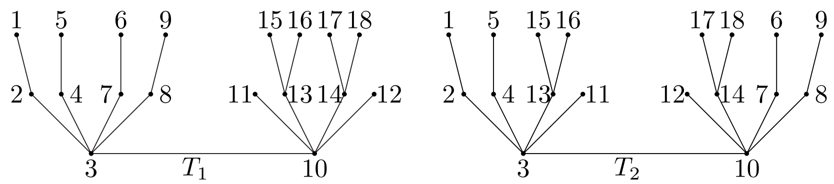

FIGURE 1. Smallest Slater pair of non-isomorphic trees with the same Randič-table.

In this note we follow an analogous idea to prove that path-congruent trees T_1, T_2 need not be isomorphic. We point out a canonical construction, and the smallest pair T_1, T_2 we obtained.

2. Construction of Pairs of Non-isomorphic Path-congruent Trees.

The pairs of graphs described by Slater do not have the same path-table. For example, the pair T_1, T_2 in Figure 1 gives the path-tables $\mathbb{P}(T_1)$ and $\mathbb{P}(T_2)$ in Table 1.

$\mathbb{S}(T_1) = \mathbb{S}(T_2)$

v\l	1	2	3	4	5
1	1	1	4	7	4
2	2	4	7	4	0
3	5	8	4	0	0
4	2	4	7	4	0
5	1	1	4	7	4
6	1	1	4	7	4
7	2	4	7	4	0
8	2	4	7	4	0
9	1	1	4	7	4
10	5	8	4	0	0
11	1	4	8	4	0
12	1	4	8	4	0
13	3	4	6	4	0
14	1	4	8	4	0
15	1	2	4	6	4
16	1	2	4	6	4
17	1	2	4	6	4
18	1	2	4	6	4

$\mathbb{P}(T_1)$

v\l	1	2	3	4	5
1	1	1	4	7	4
2	2	5	11	11	4
3	5	18	36	38	16
4	2	5	11	11	4
5	1	1	4	7	4
6	1	1	4	7	4
7	2	5	11	11	4
8	2	5	11	11	4
9	1	1	4	7	4
10	5	18	36	36	16
11	1	4	8	4	0
12	1	4	8	4	0
13	3	7	14	16	8
14	3	7	14	16	8
15	1	2	4	6	4
16	1	2	4	6	4
17	1	2	4	6	4
18	1	2	4	6	4

$\mathbb{P}(T_2)$

v\l	1	2	3	4	5
1	1	1	4	7	4
2	2	5	11	11	4
3	5	18	36	37	16
4	2	5	11	11	4
5	1	1	4	7	4
6	1	1	4	7	4
7	2	5	11	11	4
8	2	5	11	11	4
9	1	1	4	7	4
10	5	18	36	37	16
11	1	4	8	4	0
12	1	4	8	4	0
13	3	7	14	16	8
14	3	7	14	16	8
15	1	2	4	6	4
16	1	2	4	6	4
17	1	2	4	6	4
18	1	2	4	6	4

TABLE 1. The Randič-table $\mathbb{S}(T_1) = \mathbb{S}(T_2)$ and the path-tables $\mathbb{P}(T_1)$, $\mathbb{P}(T_2)$ of the smallest Slater pair in Fig. 1.

Therefore T_1 and T_2 are not path-congruent. Consequently, we are lead to the following problem.

Problem. Are path-congruent trees necessarily isomorphic?

We shall now give a negative answer. Indeed, by generalizing Slater's construction ([4] p. 90), we obtain a class of example-pairs of path-congruent non-isomorphic trees (as well as more examples of Randič-related non-isomorphic trees). Before discussing the general construction, we show in Figure 2 the smallest such pair U_1, U_2 (see Corollary 2). In Table 2 the path-table and the Randič-table of this pair are given.

The fact that U_1 is not isomorphic to U_2 is easily verified by noting that in U_1 there are 3 couples of vertices of degree 3 (the couples (8, 18), (3, 8) and (3, 17)) such that the vertices of each couple are at distance 3, whereas in U_2 there are only 2 such couples.

We now proceed to illustrate the general construction by which the pair of trees shown in Figure 2 has been obtained.

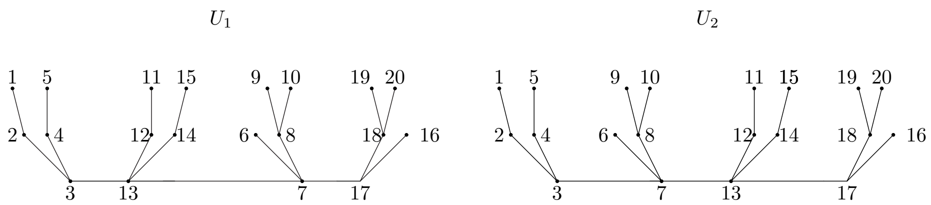

FIGURE 2. Smallest pair of non-isomorphic trees with the same path-table and Randič-table.

$\mathbb{S}(U_1) = \mathbb{S}(U_2)$

v\l	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1	1	1	2	4	5	4	2
2	2	2	4	5	4	2	0
3	3	5	5	4	2	0	0
4	2	2	4	5	4	2	0
5	1	1	2	4	5	4	2
6	1	3	7	6	2	0	0
7	4	7	6	2	0	0	0
8	3	3	5	6	2	0	0
9	1	2	3	5	6	2	0
10	1	2	3	5	6	2	0
11	1	1	3	6	6	2	0
12	2	3	6	6	2	0	0
13	4	7	6	2	0	0	0
14	2	3	6	6	2	0	0
15	1	1	3	6	6	2	0
16	1	2	5	5	4	2	0
17	3	5	5	4	2	0	0
18	3	2	3	5	4	2	0
19	1	2	2	3	5	4	2
20	1	2	2	3	5	4	2

$\mathbb{P}(U_1) = \mathbb{P}(U_2)$

v\l	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1	1	1	2	4	5	4	2
2	2	3	6	9	9	6	2
3	3	8	15	21	20	12	4
4	2	3	6	9	9	6	2
5	1	1	2	4	5	4	2
6	1	3	7	6	2	0	0
7	4	13	27	36	32	16	4
8	3	6	11	16	14	4	0
9	1	2	3	5	6	2	0
10	1	2	3	5	6	2	0
11	1	1	3	6	6	2	0
12	2	4	9	12	8	2	0
13	4	13	27	37	32	16	4
14	2	4	9	12	8	2	0
15	1	1	3	6	6	2	0
16	1	2	5	5	4	2	0
17	3	8	15	20	20	12	4
18	3	5	7	11	14	10	4
19	1	2	2	3	5	4	2
20	1	2	2	3	5	4	2

TABLE 2. The Randič-table and the path-table of the smallest pair in Fig. 2

Theorem 1. *There exist infinitely many pairs of non-isomorphic path-congruent trees. Moreover, these pairs are also Randič-related.*

Proof. Let A_1, A_2, A_3, A_4 be rooted trees, with roots r_1, ..., r_4, and let A'_1, A'_2, A'_3, A'_4 be (respectively) isomorphic to A_1, A_2, A_3, A_4 through isomorphisms

$\sigma_1, ..., \sigma_4$. Let H be a graph with four vertices $h_1, ..., h_4$ singled out. We construct a graph U_1 by identifying r_i with h_i and - given a permutation λ of $\{1, ..., 4\}$ - another graph U_2 by identifying $\sigma_i(r_i)$ with $h_{\lambda(i)}$ (see Fig. 3).

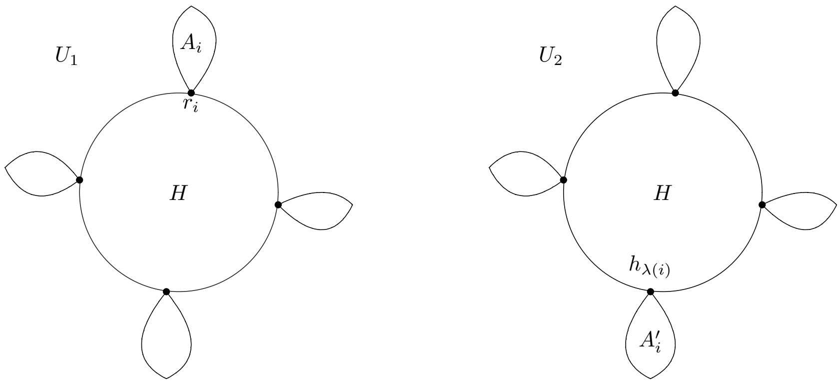

FIGURE 3. The general construction.

Define now the following map $\Phi : U_1 \to U_2$

$$\Phi(v) = \begin{cases} \sigma_i(v) \text{ if } v \in A_i \;\; i \in \{1, ..., 4\} \\ v \text{ if } v \in H \setminus \{h_1, ..., h_4\} \end{cases}$$

Note that Φ is a well-defined bijection. In order to make Φ into a global path-congruence, it is sufficient that

(1) For each $m \geq 1$ the number of paths of length m within A_i starting at the root r_i be independent of i.
(2) For each i there is a permutation θ of $\{h_j | j \neq i\}$ such that for any $k \geq 0$, for any $j \neq i$, the number of paths of length k within H with end-points h_i, h_j be equal to the number of paths of length k within H with end-points $h_i, h_{\theta(j)}$.

We can satisfy both conditions by taking, for example, the trees A_i as shown in Figure 4, H to be the path $\{h_1, h_2, h_3, h_4\}$, and $\theta = \begin{pmatrix} 1\,2\,3\,4 \\ 1\,3\,2\,4 \end{pmatrix}$.
For $n \in \mathbb{N}$ this yields infinitely many pairs of non-isomorphic path-congruent trees U_1, U_2. Also, by construction, it is easy to see that $\mathbb{S}(U_1) = \mathbb{S}(U_2)$, hence U_1 and U_2 are also Randič-related. □

Corollary 2. *The smallest number of vertices involved by the given construction is* 20.

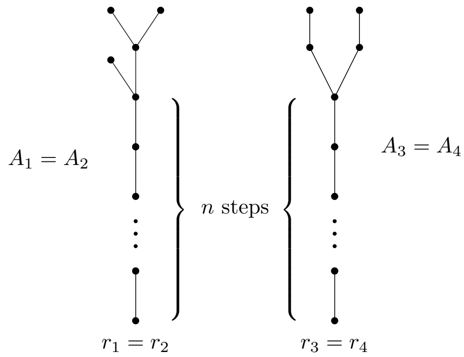

FIGURE 4. A canonical choice of the trees A_i in the construction.

Proof. With the same notation as in the proof of Theorem 1, suppose first that $|A_i| \leq 4$ for all $i \in \{1, ..., 4\}$. Then A_i is one of the eight rooted trees shown in Figure 5.

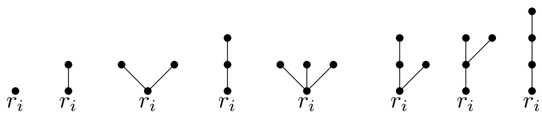

FIGURE 5. Rooted trees with $n \leq 4$ vertices.

In any case, condition (1) in the general construction implies that, for each pair $i, j \in \{1, ..., 4\}$, A_i is isomorphic to A_j, and consequently U_1 is isomorphic to U_2. Therefore, $|A_i| \geq 5$ for all $i \in \{1, ..., 4\}$, and we get

$$|U_1| = |U_2| = |\{\Phi(v) : v \in U_1\}| = |\{v \in H \setminus \{h_1, h_2, h_3, h_4\}\}|$$

$$+ \sum_{i=1}^{4} |\{\sigma_i(v) : v \in A_i \setminus \{r_i\}\}| =$$

$$= |\{v \in H \setminus \{h_1, h_2, h_3, h_4\}\}| + \sum_{i=1}^{4} |A_i \setminus \{r_i\}| \geq \sum_{i=1}^{4} |A_i| \geq 4 \cdot 5 = 20.$$

□

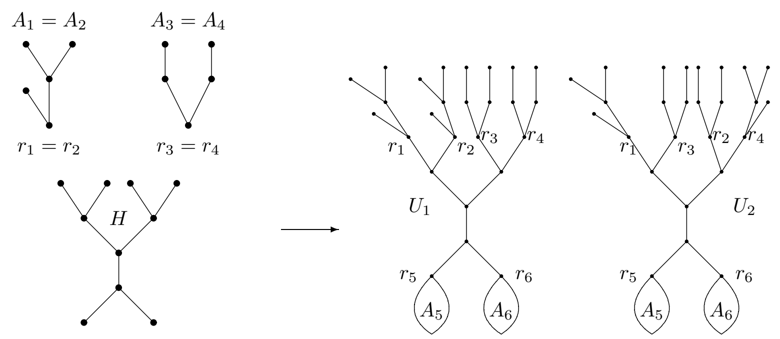

FIGURE 6. An extension of the construction in Theorem 1.

More general constructions are allowed by different choices of H and with more trees A_i to attach to it. See Figure 6 for such an example.

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