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EXPRESSING THE GENERALIZED FIBONACCI POLYNOMIALS IN TERMS OF A TRIDIAGONAL DETERMINANT

FENG QI - BAI-NI GUO

In the paper, the authors express the generalized Fibonacci polynomials in terms of a tridiagonal determinant. Consequently, they also express the Fibonacci numbers and polynomials in terms of tridiagonal determinants.

1. Main results

A tridiagonal matrix is a square matrix with nonzero elements only on the diagonal and slots horizontally or vertically adjacent the diagonal. In other words, a square matrix $H = (h_{ij})_{n\times n}$ is called a tridiagonal matrix if $h_{ij} = 0$ for all pairs (i, j) such that $|i-j| > 1$. A matrix $H = (h_{ij})_{n\times n}$ is called a lower (or an upper, respectively) Hessenberg matrix if $h_{ij} = 0$ for all pairs (i, j) such that $i+1 < j$ (or $j+1 < i$, respectively). See the papers [4, 5] and closely-related references therein.

The well-known Fibonacci numbers

$$F_n = \frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{2^n\sqrt{5}}$$

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for $n \in \mathbb{N}$ form a sequence of integers and satisfy the linear recurrence relation

$$F_n = F_{n-1} + F_{n-2} \tag{1}$$

with $F_1 = F_2 = 1$. The first fourteen Fibonacci numbers F_n for $1 \le n \le 14$ are

$$1, \quad 1, \quad 2, \quad 3, \quad 5, \quad 8, \quad 13, \quad 21, \quad 34, \quad 55, \quad 89, \quad 144, \quad 233, \quad 377.$$

The Fibonacci numbers F_n can be viewed as a particular case $F_n(1)$ of the Fibonacci polynomials

$$F_n(s) = \frac{1}{2^n} \frac{\left(s + \sqrt{4+s^2}\right)^n - \left(s - \sqrt{4+s^2}\right)^n}{\sqrt{4+s^2}}$$

which can be generated by

$$\frac{z}{1 - sz - z^2} = \sum_{n=1}^{\infty} F_n(s) z^n = z + sz^2 + \left(s^2+1\right)z^3 + \left(s^3+2s\right)z^4 + \cdots.$$

The generalized Fibonacci polynomials $F_n(s,t)$ are defined by $F_0(s,t) = 0$, $F_1(s,t) = 1$, and the recurrence relation

$$F_n(s,t) = sF_{n-1}(s,t) + tF_{n-2}(s,t), \quad n \ge 2. \tag{2}$$

It is easy to deduce that

$$F_2(s,t) = s, \quad F_3(s,t) = s^2 + t, \quad F_4(s,t) = s^3 + 2st, \quad F_5(s,t) = s^4 + 3s^2t + t^2.$$

The generalized Fibonacci polynomials $F_n(s,t)$ can be generalized by

$$\frac{z}{1 - sz - tz^2} = \sum_{n=0}^{\infty} F_n(s,t) z^n. \tag{3}$$

It is clear that $F_n(s,1) = F_n(s)$ and $F_n(1,1) = F_n$ for $n \in \mathbb{N}$.

Formulas for the Fibonacci numbers and its generalizations are classical, starting with the well-known Cassini formulas for F_n. Some of these formulas for the Fibonacci numbers F_n, the Fibonacci polynomials $F_n(s)$, and the generalized Fibonacci polynomials $F_n(s,t)$ can be found in the papers [1, 8] and the monograph [3].

In this paper, the authors will express the generalized Fibonacci polynomials $F_n(s,t)$ in terms of a tridiagonal determinant. Consequently, they will also express the Fibonacci numbers F_n and the Fibonacci polynomials in terms of tridiagonal determinants.

Our main results can be stated as the following theorem.

Theorem 1.1. *For $n \in \{0\} \cup \mathbb{N}$, the generalized Fibonacci polynomials $F_n(s,t)$ can be expressed as*

$$F_n(s,t) = \frac{1}{n!}\begin{vmatrix} 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & \binom{1}{0}s & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2\binom{2}{0}t & \binom{2}{1}s & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 2\binom{3}{1}t & \binom{3}{2}s & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \binom{n-2}{n-3}s & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 2\binom{n-1}{n-3}t & \binom{n-1}{n-2}s & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 2\binom{n}{n-2}t & \binom{n}{n-1}s \end{vmatrix}. \tag{4}$$

Consequently, for $n \in \mathbb{N}$, the Fibonacci polynomials $F_n(s)$ and the Fibonacci numbers F_n can be expressed respectively as

$$F_n(s) = \frac{1}{n!}\begin{vmatrix} 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & \binom{1}{0}s & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2\binom{2}{0} & \binom{2}{1}s & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 2\binom{3}{1} & \binom{3}{2}s & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \binom{n-2}{n-3}s & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 2\binom{n-1}{n-3} & \binom{n-1}{n-2}s & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 2\binom{n}{n-2} & \binom{n}{n-1}s \end{vmatrix} \tag{5}$$

and

$$F_n = \frac{1}{n!}\begin{vmatrix} 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & \binom{1}{0} & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2\binom{2}{0} & \binom{2}{1} & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 2\binom{3}{1} & \binom{3}{2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \binom{n-2}{n-3} & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 2\binom{n-1}{n-3} & \binom{n-1}{n-2} & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 2\binom{n}{n-2} & \binom{n}{n-1} \end{vmatrix}. \tag{6}$$

2. A lemma

In order to prove Theorem 1.1, we need the following lemma which is a reformulation of [2, p. 40, Exercise 5)] in the papers [6, Section 2.2, p. 849], [7, p. 94], and [9, Lemma 2.1].

Lemma 2.1. *Let $u(x)$ and $v(x) \neq 0$ be differentiable functions, let $U_{(n+1)\times 1}(x)$ be an $(n+1)\times 1$ matrix whose elements $u_{k,1}(x) = u^{(k-1)}(x)$ for $1 \le k \le n+1$, let $V_{(n+1)\times n}(x)$ be an $(n+1)\times n$ matrix whose elements*

$$v_{i,j}(x) = \begin{cases} \binom{i-1}{j-1} v^{(i-j)}(x), & i-j \ge 0 \\ 0, & i-j < 0 \end{cases}$$

for $1 \le i \le n+1$ and $1 \le j \le n$, and let $|W_{(n+1)\times(n+1)}(x)|$ denote the lower Hessenberg determinant of the $(n+1)\times(n+1)$ lower Hessenberg matrix

$$W_{(n+1)\times(n+1)}(x) = \begin{bmatrix} U_{(n+1)\times 1}(x) & V_{(n+1)\times n}(x) \end{bmatrix}.$$

Then the nth derivative of the ratio $\frac{u(x)}{v(x)}$ can be computed by

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\frac{u(x)}{v(x)}\right] = (-1)^n \frac{\left|W_{(n+1)\times(n+1)}(x)\right|}{v^{n+1}(x)}. \tag{7}$$

3. Proof of Theorem 1.1

Applying $u(z) = z$ and $v(z) = 1 - sz - tz^2$ to the formula (7) yields

$$
\begin{aligned}
\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left(\frac{z}{1-sz-tz^2}\right) &= \frac{(-1)^n}{(1-sz-tz^2)^{n+1}} \\
&\times \begin{vmatrix}
z & 1-sz-tz^2 & 0 & 0 & \cdots & 0 & 0 & 0 \\
1 & -(s+2tz)\binom{1}{0} & 1-sz-tz^2 & 0 & \cdots & 0 & 0 & 0 \\
0 & -2\binom{2}{0}t & -(s+2tz)\binom{2}{1} & 1-sz-tz^2 & \cdots & 0 & 0 & 0 \\
0 & 0 & -2\binom{3}{1}t & -(s+2tz)\binom{3}{2} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & -(s+2tz)\binom{n-2}{n-3} & 1-sz-tz^2 & 0 \\
0 & 0 & 0 & 0 & \cdots & -2\binom{n-1}{n-3}t & -(s+2tz)\binom{n-1}{n-2} & 1-sz-tz^2 \\
0 & 0 & 0 & 0 & \cdots & 0 & -2\binom{n}{n-2}t & -(s+2tz)\binom{n}{n-1}
\end{vmatrix} \\
&\to (-1)^n \begin{vmatrix}
0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
1 & -\binom{1}{0}s & 1 & 0 & \cdots & 0 & 0 \\
0 & -2\binom{2}{0}t & -\binom{2}{1}s & 1 & \cdots & 0 & 0 \\
0 & 0 & -2\binom{3}{1}t & -\binom{3}{2}s & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & 0 & \cdots & -\binom{n-1}{n-2}s & 1 \\
0 & 0 & 0 & 0 & \cdots & -2\binom{n}{n-2}t & -\binom{n}{n-1}s
\end{vmatrix}
\end{aligned}
$$

as $z \to 0$ for $n \in \mathbb{N}$. By the generating function in (3), we obtain that

$$F_n(s,t) = \frac{1}{n!}\lim_{z\to 0}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left(\frac{z}{1-sz-tz^2}\right)$$
$$= \frac{(-1)^n}{n!}\begin{vmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -\binom{1}{0}s & 1 & 0 & \cdots & 0 & 0 \\ 0 & -2\binom{2}{0}t & -\binom{2}{1}s & 1 & \cdots & 0 & 0 \\ 0 & 0 & -2\binom{3}{1}t & -\binom{3}{2}s & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -\binom{n-1}{n-2}s & 1 \\ 0 & 0 & 0 & 0 & \cdots & -2\binom{n}{n-2}t & -\binom{n}{n-1}s \end{vmatrix}$$

which can be rewritten as the expression (4).

The expression (5) follows from taking the limit $t \to 1$ on both sides of the expression (4).

The expression (6) can be derived from either letting $(s,t) \to (1,1)$ in (4) or letting $s \to 1$ in (5). The proof of Theorem 1.1 is complete.

4. Remarks

Remark 4.1. The expressions (4), (5), and (6) can be rearranged as

$$F_n(s,t) = \frac{1}{n!}\begin{vmatrix} \binom{2}{1}s & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 2\binom{3}{1}t & \binom{3}{2}s & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2\binom{4}{2}t & \binom{4}{3}s & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 2\binom{5}{3}t & \binom{5}{4}s & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \binom{n-2}{n-3}s & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 2\binom{n-1}{n-3}t & \binom{n-1}{n-2}s & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 2\binom{n}{n-2}t & \binom{n}{n-1}s \end{vmatrix},$$

$$F_n(s) = \frac{1}{n!}\begin{vmatrix} \binom{2}{1}s & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 2\binom{3}{1} & \binom{3}{2}s & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2\binom{4}{2} & \binom{4}{3}s & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \binom{n-2}{n-3}s & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 2\binom{n-1}{n-3} & \binom{n-1}{n-2}s & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 2\binom{n}{n-2} & \binom{n}{n-1}s \end{vmatrix},$$

and

$$F_n = \frac{1}{n!}\begin{vmatrix} \binom{2}{1} & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 2\binom{3}{1} & \binom{3}{2} & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2\binom{4}{2} & \binom{4}{3} & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 2\binom{5}{3} & \binom{5}{4} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \binom{n-2}{n-3} & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 2\binom{n-1}{n-3} & \binom{n-1}{n-2} & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 2\binom{n}{n-2} & \binom{n}{n-1} \end{vmatrix}$$

for $n \geq 2$.

Remark 4.2. It is easy to see that, from the expressions (4), (5), and (6), we can recover the recurrence relation

$$F_n(s) = sF_{n-1}(s) + F_{n-2}(s)$$

and the recurrence relations (1) and (2) for $n \geq 3$.

Remark 4.3. It is worthwhile to mentioning that some determinantal and permanental representations of the generalized Fibonacci polynomials in terms of various Hessenberg matrices were given in the preprint [8]. These results are general form of determinantal and permanental representations of k sequences of the generalized order-k Fibonacci and Pell numbers.

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FENG QI
Institute of Mathematics
Henan Polytechnic University
Jiaozuo City
Henan Province, 454010
China
College of Mathematics
Inner Mongolia University for Nationalities
Tongliao City
Inner Mongolia Autonomous Region, 028043
China
Department of Mathematics
College of Science
Tianjin Polytechnic University
Tianjin City, 300387
China
e-mail: qifeng618@gmail.com
URL: *https://qifeng618.wordpress.com*

BAI-NI GUO
School of Mathematics and Informatics
Henan Polytechnic University
Jiaozuo City
Henan Province, 454010
China
e-mail: bai.ni.guo@hotmail.com