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FIXED POINTS OF THE BARNSLEY-HUTCHINSON OPERATORS INDUCED BY HYPER-CONDENSING MAPS

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The aim of this note is to show that the Barnsley-Hutchinson operator, induced by either single h-bounded hyper-condensing map or a compact infinite family of such maps, admits a maximal fixed point and global attractor. Under suitable continuity conditions both fixed point and attractor coincide. This improves and extends several earlier results.

To obtain these results some new properties of the Hausdorff measure of noncompactness on the hyperspace are established.

1. The hyperspace of closed sets.

Throughout the paper (X, d) will stand for a complete metric space. It is customary to allow distances to take infinite values ([13] chap. 1 sect. 1.1). We shall denote by $B(x, r)$ the open r-ball with center in $x \in X$, by $\mathcal{O}_r(A) = \bigcup_{a\in A} B(a, r)$ the r-*aureola* around $A \subset X$, and by $\overline{A}$ the closure of A. Further, 2^X, $\mathcal{F}(X)$ and $\mathcal{K}(X)$ denote respectively the family of all nonempty subsets of X, the family of nonempty closed sets and the family of nonempty

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compact sets. We endow $\mathcal{F}(X)$ (consequently $\mathcal{K}(X)$) with the *Hausdorff metric* given for $A_1,\ A_2 \in \mathcal{F}(X)$ by

$$h(A_1, A_2) = \inf\{r > 0 \ : \ A_1 \subset \mathcal{O}_r(A_2),\ A_2 \subset \mathcal{O}_r(A_1)\}.$$

So topologized family is called a *hyperspace*. For the topological notions in $\mathcal{F}(X)$, like closure or compactness, we use prefix „h-", e.g., h-closure or h-compactness. We shall also need the double exponentiate $\mathcal{F}(\mathcal{F}(X))$ furnished with the respective Hausdorff metric denoted in this case by H. For basic properties of h we refer to [15], [13] or [4].

Lemma 1. *Let $\{A_t\}_t \subset \mathcal{F}(X)$ be h-compact. Then the union $\bigcup_t A_t$ is a closed set. If additionally every A_t is bounded, then $\bigcup_t A_t$ is also bounded.*

Proof. We start with closedness. Denote $\mathcal{A} = \{A_t\}_t$ and take any convergent sequence $\{x_n\}_{n=1}^{\infty} \subset \bigcup \mathcal{A}$, $x_n \to x$. We will show that $x \in \bigcup \mathcal{A}$. One can associate with each x_n a set $A_{t_n} \in \mathcal{A}$ s.t. $A_{t_n} \ni x_n$. Since the family $\mathcal{A}$ is h-compact, there exists a subsequence $(A_{t_{k_n}})_{n=1}^{\infty}$ of $(A_{t_n})_{n=1}^{\infty}$ s.t. $A_{t_{k_n}} \to A$ for some $A \in \mathcal{A}$. Correspondingly, for the subsequence $(x_{k_n})_{n=1}^{\infty}$ we have $x_{k_n} \to x$. Now fix $\epsilon > 0$ and observe that $x_{k_n} \in A_{t_{k_n}} \subset \mathcal{O}_{\epsilon/2}A$, $x \in \mathcal{O}_{\epsilon/2}\{x_{k_n}\}$ for large enough n. Hence

$$x \in \mathcal{O}_{\epsilon/2}\{x_{k_n}\} \subset \mathcal{O}_{\epsilon/2}A_{t_{k_n}} \subset \mathcal{O}_{\epsilon}A,$$

By closedness of A we obtain $x \in A \in \mathcal{A}$ i.e. $x \in \bigcup \mathcal{A}$.

Now we show boundedness. The family $\mathcal{A}$ admits (by its h-compactness) a finite (1/2)-net $\{A_{t_1}, \dots, A_{t_s}\} \subset \mathcal{A}$, i.e.,

$$\forall t \ \ \exists i = 1, \dots, s \ : \ h(A_t, A_{t_i}) < \frac{1}{2}.$$

Hence $\bigcup \mathcal{A} \subset \bigcup_{i=1}^{s} \mathcal{O}_1 A_{t_i}$ is bounded. The proof is completed. $\square$

Remark 1. We cannot simply invoke in the above proof the preservation of compactness under continuous map. The reason is that the mapping $\cup$: $\mathcal{K}(\mathcal{F}(X)) \to \mathcal{F}(X)$ given for all $\mathcal{A} \in \mathcal{K}(\mathcal{F}(X)$ by $\cup(\mathcal{A}) = \bigcup \mathcal{A}$ is *not* continuous; otherwise $\bigcup \mathcal{A}$ would be always compact (comp. [15], [26], [18]).

Lemma 2. (On closing unions I). *For any family $\{A_t\}_t$ of sets $A_t \subset X$ we have*

$$\overline{\bigcup_t \overline{A_t}} = \overline{\bigcup_t A_t}.$$

Lemma 3. (On closing unions II). *Let* $\mathcal{A} \subset \mathcal{F}(X)$ *and* $\mathcal{B} = \overline{\mathcal{A}}^h \subset \mathcal{F}(X)$ *be the* h*-closure of* $\mathcal{A}$. *Then* $\overline{\bigcup \mathcal{A}} = \overline{\bigcup \mathcal{B}}$.

Proof. We only have to check that $\overline{\bigcup \mathcal{B}} \subset \overline{\bigcup \mathcal{A}}$. Fix $x \in \overline{\bigcup \mathcal{B}}$. This means that $x_n \to x$ for some $\{x_n\}_{n=1}^{\infty} \subset \bigcup \mathcal{B}$. Next, we can associate with each x_n some $B_n \in \mathcal{B}$ s.t. $B_n \ni x_n$. Recalling now that every B_n is an h-limit of sets $A_n^m \in \mathcal{A}$ we obtain

$$\begin{array}{ccccc} A_n^m & \xrightarrow[m \to \infty]{} & B_n & \ni & x_n \\ & \cdots\cdots & \vdots & & \downarrow \\ & & & & x. \end{array}$$

Standard diagonal argument shows that there exists $\{x'_n\}_{n=1}^{\infty} \subset \bigcup \mathcal{A}$ s.t. $d(x'_n, x_n) < \frac{1}{n}$ and so $d(x'_n, x) \to 0$. Therefore $x \in \overline{\bigcup \mathcal{A}}$. $\square$

2. The Hausdorff measure of noncompactness on the hyperspace.

Let C be a nonempty subset of (X, d). The *Hausdorff measure of noncompactness* relative to C is the functional $\beta_C : 2^X \to [0, \infty]$ given by

$$\beta_C(A) = \inf\{\, r > 0 \;:\; \exists_{c_1,\ldots,c_k \in C} \; \bigcup_{i=1}^{k} B(c_i, r) \supset A \,\}.$$

In case $C = X$ we will omit the subscript X and simply write β. Replacing (X, d) with the hyperspace $(\mathcal{F}(X), h)$ we get in that way the Hausdorff measure of noncompactness relative to $\mathcal{C} \subset \mathcal{F}(X)$ denoted by $\beta^{\#}_{\mathcal{C}}$. The measures $\beta^{\#}$ and $\beta^{\#}_{\mathcal{K}(X)}$ will be of our primal interest. One could also consider $(2^X, h)$ furnished with an infinite semimetric h and the respective Hausdorff measure of noncompactness, but it is easily seen that such a measure relativized to $\mathcal{F}(X)$ is just $\beta^{\#}$. For basic facts concerning measures of noncompactness we refer to [1] and [3].

Lemma 4. *For* $\mathcal{A} \subset \mathcal{F}(X)$ *we have*

$$\beta\Big(\bigcup \mathcal{A}\Big) = \beta^{\#}_{\mathcal{K}(X)}\big(\mathcal{A}\big).$$

Proof. Label $\mathcal{A} = \{A_t\}_t$. To prove „$\leq$" choose ν and $\overline{\nu}$ so that $\overline{\nu} > \nu > \beta^{\#}_{\mathcal{O}(X)}(\mathcal{A})$. Hence there exists a finite ν-net $\{K_1, \ldots, K_k\} \subset \mathcal{O}(X)$ for $\mathcal{A}$, i.e.,

$$\forall_t \;\; \exists_i \;\; h(A_t, K_i) < \nu.$$

Thus $\bigcup \mathcal{A} \subset \bigcup_{i=1}^{k} \mathcal{O}_\nu K_i$. Being each K_i compact we can find a finite $(\overline{\nu} - \nu)$-net $K_i \subset \bigcup_{j \in J(i)} B(x_j{}^i, \overline{\nu} - \nu)$. Altogether $\bigcup \mathcal{A} \subset \bigcup_i \bigcup_j B(x_j{}^i, \overline{\nu})$.

Now we prove „$\geq$”. Fix $\nu > \beta(\bigcup \mathcal{A})$ and pick up a suitable ν-net $\bigcup \mathcal{A} \subset \bigcup_{i=1}^{k} B(x_i, \nu)$. Further, for every t denote $P_t = \{x_i \ : \ x_i \in \mathcal{O}_\nu A_t\}$. Finally observe that $\{P_t\}_t \subset \mathcal{O}(X)$ is a finite ν-net for $\mathcal{A}$. Indeed, $A_t \subset \bigcup_{i=1}^{k} B(x_i, \nu) = \mathcal{O}_\nu P_t$, and $P_t \subset \mathcal{O}_\nu A_t$, because

$$B(x_i, \nu) \cap A_t \neq \emptyset \Leftrightarrow x_i \in \mathcal{O}_\nu A_t \Leftrightarrow x_i \in P_t.$$

Therefore $\beta^{\#}_{\mathcal{O}(X)}(\mathcal{A}) < \nu$. $\square$

Lemma 5. (Estimate for infinite unions). *If* $\{A_t\}_{t\in T} \subset \mathcal{F}(X)$*, then*

$$\sup_{t\in T} \beta(A_t) + \beta^{\#}(\{A_t\}_{t\in T}) \leq \beta\Big(\bigcup_{t\in T} A_t\Big) \leq \sup_{t\in T} \beta(A_t) + 2\,\beta^{\#}(\{A_t\}_{t\in T}).$$

In particular, for h-compact $\{A_t\}_{t\in T}$, $\beta\big(\bigcup_{t\in T} A_t\big) = \sup_{t\in T} \beta(A_t)$.

Proof. For the proof of the upper estimation of $\beta\big(\bigcup_{t\in T} A_t\big)$ we refer to [24]. Thus it is enough to verify that

$$\beta^{\#}(\{A_t\}_{t\in T}) \leq \beta\Big(\bigcup_{t\in T} A_t\Big) - \sup_{t\in T} \beta(A_t) = \inf_{t\in T}\Big[\beta\Big(\bigcup_{t\in T} A_t\Big) - \beta(A_t)\Big].$$

Fix $t_0 \in T$ and $r > \beta\big(\bigcup_{t\in T} A_t\big) - \beta(A_{t_0}) \geq 0$. Putting $\nu = r + \beta(A_{t_0})$ we can find a finite ν-net $\{p_1, \ldots, p_k\}$ for the sum $\bigcup_{t\in T} A_t$. Next we can associate with each t the following set $P_t = \{p_i \ : \ p_i \in \mathcal{O}_\nu A_t\}$. Similarly as in the proof of Lemma 4 we see that $\{P_t\}_{t\in T}$ is a finite ν-net for $\{A_t\}_{t\in T}$. Being $r = \nu - \beta(A_{t_0})$ and $t_0 \in T$ arbitrary, the desired inequality follows. $\square$

Proposition 1. ([24] Prop. 2) *If* $\{\mathcal{A}_t\}_{t\in T} \subset \mathcal{F}(\mathcal{F}(X))$ *is* H*-compact, then*

$$\beta^{\#}\Big(\bigcup_{t\in T} \mathcal{A}_t\Big) = \sup_{t\in T} \beta^{\#}(\mathcal{A}_t).$$

For any family $\mathcal{A} \subset \mathcal{F}(X)$ we define a new family $\Sigma(\mathcal{A}) = \{\overline{\bigcup \mathcal{B}} \ : \ \emptyset \neq \mathcal{B} \subset \mathcal{A}\}$.

Lemma 6. *Suppose* $\mathcal{C} \subset \mathcal{F}(X)$ *is finitely additive, i.e.,* $C_1, C_2 \in \mathcal{C} \Rightarrow C_1 \cup C_2 \in \mathcal{C}$*). Then for* $\mathcal{A} \subset \mathcal{F}(X)$ *we have*

$$\beta^{\#}_{\mathcal{C}}[\Sigma(\mathcal{A})] = \beta^{\#}_{\mathcal{C}}(\mathcal{A}).$$

Proof. The inequality „$\geq$" is immediate, due to monotonicity of $\beta^{\#}_{\mathcal{C}}$. To prove the reverse inequality label $\mathcal{A} = \{A_t\}_{t\in T}$ and fix $\overline{r} > r > \beta^{\#}_{\mathcal{C}}(\mathcal{A})$. Thus there exists a finite r-net $\mathcal{E} \subset \mathcal{C}$ for $\mathcal{A}$, i.e.,

$$\forall_{t\in T}\ \exists_{E_t\in\mathcal{E}}\ h(A_t, E_t) < r.$$

We shall check that $\Sigma(\mathcal{E})$ is a finite $\overline{r}$-net for $\Sigma(\mathcal{A})$. Of course $\Sigma(\mathcal{E}) \subset \mathcal{C}$ is finite. Moreover, each member of $\Sigma(\mathcal{A})$ has the form $\overline{\bigcup_{t\in T_0} A_t}$, where $\emptyset \neq T_0 \subset T$. Therefore we obtain

$$h\Big(\overline{\bigcup_{t\in T_0} A_t}, \overline{\bigcup_{t\in T_0} E_t}\Big) \leq \sup_{t\in T_0} h(A_t, E_t) \leq r < \overline{r},$$

and $\overline{\bigcup_{t\in T_0} E_t} \in \Sigma(\mathcal{E})$. $\square$

As a direct consequence we get

Proposition 2. *For $\mathcal{A} \subset \mathcal{F}(X)$ we have:*

$$\beta^{\#}\big[\Sigma(\mathcal{A})\big] = \beta^{\#}(\mathcal{A}),\quad \beta^{\#}_{\mathcal{K}(X)}\big[\Sigma(\mathcal{A})\big] = \beta^{\#}_{\mathcal{K}(X)}(\mathcal{A}).$$

In particular, if $\mathcal{A}$ is h-compact, then so is $\Sigma(\mathcal{A})$.

3. The Barnsley-Hutchinson operator. The space of multifunctions.

Let $\varphi : X \multimap X$ be a multifunction with nonempty closed values. The *image* of $(A \subset X$ is the set $\varphi(A) = \bigcup_{a\in A}\varphi(a)$. We associate with φ the *Barnsley-Hutchinson operator* $F : \mathcal{F}(X) \to \mathcal{F}(X)$ given by $F(A) = \overline{\varphi(A)}$. Further, it induces the operator $F_{\#} : \mathcal{F}(\mathcal{F}(X)) \to \mathcal{F}(\mathcal{F}(X))$ taking any $\mathcal{A} \in \mathcal{F}(\mathcal{F}(X))$ to the h-closure of its image under F, namely $F_{\#}(\mathcal{A}) = \overline{\{F(A) : A \in \mathcal{A}\}}$. The operator $F_{\#}$ is alike a Barnsley-Hutchinson operator associated with F. It plays rather technical role in Sections 4 and 5.

To justify our assumption on values of φ to be closed sets we point out that given any multifunction $\varphi : X \multimap X$ and its closure $\overline{\varphi} : X \multimap X$, $\overline{\varphi}(x) = \overline{\varphi(x)}\ \forall_{x\in X}$, both of them generate the same Barnsley-Hutchinson operator (just apply Lemma 2).

Similar definitions hold for a system of multifunctions $\{\varphi_t : X \multimap X\}_{t\in T}$, called also a *multivalued iterated function system.* The study of such a system can be reduced to study of a single multifunction, namely its union $\bigcup_{t\in T}\varphi_t : X \multimap X$ given by $\big(\bigcup_{t\in T}\varphi_t\big)(x) = \bigcup_{t\in T}\varphi_t(x)$ for $x \in X$. This union in the case of finite or even compactly infinite system preserves most of typical assumptions put on maps φ_t, like contractivity, compactness, condensity, Hausdorff continuity etc. (see e.g. [26], [18], [24]). The concepts of continuity and compactness for multivalued maps are discussed in Section 4. For basic facts from set-valued analysis see e.g. [13] and [4].

We shall need the space $\mathcal{M}(X, X)$ of all multifunctions with nonempty closed values furnished with the *Chebyshev uniform metric*

$$h_{\sup}(\varphi_1, \varphi_2) = \sup_{x \in X} h[\varphi_1(x), \varphi_2(x)]$$

for $\varphi_1, \varphi_2 : X \multimap X$ ([24]). The symbol $\mathcal{F}(X)^{\mathcal{F}(X)}$ shall stand for the space of operators acting on $\mathcal{F}(X)$ endowed with the Chebyshev metric $h_{\sup}$, analogously $\mathcal{F}(\mathcal{F}(X))^{\mathcal{F}(\mathcal{F}(X))}$ shall stand for the space of operators acting on $\mathcal{F}(\mathcal{F}(X))$ endowed with respective Chebyshev metric $H_{\sup}$.

Let us associate with $\varphi_1, \varphi_2 \in \mathcal{M}(X, X)$ the induced Barnsley-Hutchinson operators $F_1, F_2 \in \mathcal{F}(X)^{\mathcal{F}(X)}$, $F_{1\,\#}, F_{2\,\#} \in \mathcal{F}(\mathcal{F}(X))^{\mathcal{F}(\mathcal{F}(X))}$. Noting that

$$h_{\sup}(\varphi_1, \varphi_2) = \sup_{A \in \mathcal{F}(X)} h[\varphi_1(A), \varphi_2(A)]$$

and $F_\#(\{A\}) = \{F(A)\}$ we get the isometry

Proposition 3. *We have*

$$h_{\sup}(\varphi_1, \varphi_2) = h_{\sup}(F_1, F_2) = H_{\sup}(F_{1\,\#}, F_{2\,\#}).$$

The evaluation map $\mathrm{ev}_{\mathcal{A}} : \mathcal{F}(\mathcal{F}(X))^{\mathcal{F}(\mathcal{F}(X))} \to \mathcal{F}(\mathcal{F}(X))$ is defined for $\mathcal{A} \in \mathcal{F}(\mathcal{F}(X))$ and $\Gamma : \mathcal{F}(\mathcal{F}(X)) \to \mathcal{F}(\mathcal{F}(X))$ as $\mathrm{ev}_{\mathcal{A}}(\Gamma) = \Gamma(\mathcal{A})$. Its important feature describes (comp. [24] Lemma 4)

Lemma 7. *The evaluation map $\mathrm{ev}_{\mathcal{A}}$ is nonexpansive, i.e.,*

$$H[\mathrm{ev}_{\mathcal{A}}(\Gamma_1), \mathrm{ev}_{\mathcal{A}}(\Gamma_2)] \leq H_{\sup}[\Gamma_1, \Gamma_2].$$

4. Hyper-condensing maps.

In this section we introduce the notion of a hyper-condensing multifunction. We shall say that a multifunction $\varphi : X \multimap X$ is *hyper-condensing*, iff

$$\forall_{\mathcal{A}\subset\mathcal{F}(X),\ \beta^{\#}(\mathcal{A})<\infty}\ \beta^{\#}\{\overline{\varphi(A)}\ :\ A\in\mathcal{A}\}\begin{cases} <\beta^{\#}(\mathcal{A}), & \text{when } \beta^{\#}(\mathcal{A})>0 \\ =0, & \text{when } \beta^{\#}(\mathcal{A})=0 \end{cases}.$$

This seemingly new concept – suggested by L. Górniewicz – is very similar to the one of condensing map. Condensing maps allow us to treat both contractions with compact values and compact maps together. What really matters, for condensing maps we can formulate (due to G. Darbo and B. N. Sadovskiĭ; see e.g. [1], [13]) a joint generalization of two fundamental fixed point theorems: the Banach Principle and the Schauder Principle. Similarly, the idea underlying the notion of hyper-condensing map is to build a bridge between multivalued contractions (with noncompact values) and h-compact multifunctions (developed in [6]; see also [5]).

We proceed to compare hyper-condensing maps with other classes of multifunctions. Recall that a *comparison function* is the function $\eta : [0,\infty) \to [0,\infty)$ fulfilling

(i) $\eta(0) = 0$, $\eta(r) < r$ for $r > 0$,
(ii) $r_1 \le r_2 \Rightarrow \eta(r_1) \le \eta(r_2)$ (monotonicity),
(iii) $\lim_{s\to r^+} \eta(s) = \eta(r)$ (right continuity).

We refer to [17] for a useful discussion of various conditions usually put on comparison functions.

We say that $\varphi : X \multimap X$ is

- *bounded*, iff $\overline{\varphi(X)}$ is bounded,
- *compact*, iff $\overline{\varphi(X)}$ is compact,
- *contractive*, iff there exists a constant $L \ge 0$ s.t.

$$\forall_{x_1,x_2\in X}\ h\big(\varphi(x_1),\varphi(x_2)\big) \le L\, d(x_1,x_2),$$

- *weakly contractive*, iff there exists a comparison function η s.t.

$$\forall_{x_1,x_2\in X}\ h\big(\varphi(x_1),\varphi(x_2)\big) \le \eta\big(d(x_1,x_2)\big),$$

- *condensing*, iff

$$\forall_{A\subset X,\ \beta(A)<\infty}\ \beta(\varphi(A))\begin{cases} <\beta(A), & \text{when } \beta(A)>0 \\ =0, & \text{when } \beta(A)=0 \end{cases},$$

- *h-bounded*, iff $\overline{\{\varphi(x)\ :\ x \in X\}}^{h} \subset \mathcal{F}(X)$ is h-bounded,
- *h-compact*, iff $\overline{\{\varphi(x)\ :\ x \in X\}}^{h} \subset \mathcal{F}(X)$ is h-compact,
- *Hausdorff upper semicontinuous* (shortly h-u.s.c.), iff

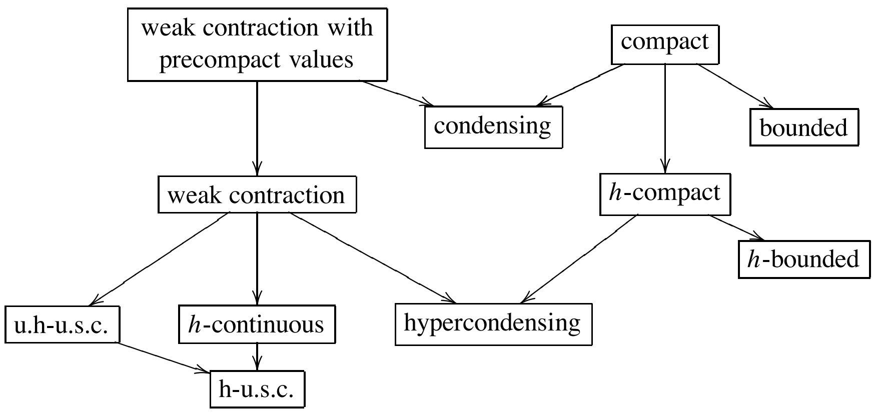
$$\forall_{x\in X}\ \forall_{\epsilon>0}\ \exists_{\delta>0}\ \varphi(\ B(x,\delta)\ ,\subset \mathcal{O}_{\epsilon}\varphi(x),$$

- *uniformly Hausdorff upper semicontinuous* (shortly u.h-u.s.c.), iff

$$\forall_{A\in\mathcal{F}(X)}\ \forall_{\epsilon>0}\ \exists_{\delta>0}\ \ \varphi(\mathcal{O}_{\delta}A) \subset \mathcal{O}_{\epsilon}\varphi(A),$$

- *h-continuous*, iff the mapping $\varphi : (X, d) \to (\mathcal{F}(X), h)$ is continuous.

For the notion of u.h-u.s.c. map see [23]. Relations between all introduced classes of maps are established below.



We would like to point out that the "h-modification" of usual notions have some limitations. It is illustrated by the following three propositions

Proposition 4. (a) *If φ is contraction with constant L, then*

$$\forall_{A_1,A_2\subset X}\ \ h\big(\varphi(A_1), \varphi(A_2)\big) \leq L\, h(A_1, A_2).$$

(b) *If φ is weakly contractive with comparison function η, then*

$$\forall_{A_1,A_2\subset X}\ \ h\big(\varphi(A_1), \varphi(A_2)\big) \leq \eta\big(h(A_1, A_2)\big).$$

Proposition 5. ([6], [5])

(a) *φ is bounded if and only if φ is h-bounded with bounded values;*

(b) *φ) is compact if and only if φ is h-compact with compact values;*

(c) *φ is h-compact if and only if $\overline{\{\overline{\varphi(A)} \,:\, A \in \mathcal{F}(X)\}}^h$ is h-compact.*

Proposition 6. *A multifunction φ is condensing if and only if*

$$\begin{array}{l}\forall_{\mathcal{A}\subset\mathcal{F}(X),\ \beta^{\#}_{\mathcal{K}(X)}(\mathcal{A})<\infty} \\ \beta^{\#}_{\mathcal{K}(X)}\{\overline{\varphi(A)} \,:\, A \in \mathcal{A}\}\end{array} \begin{cases} < \beta^{\#}_{k(X)}(\mathcal{A}), & \text{when } \beta^{\#}_{\mathcal{K}(X)}(\mathcal{A}) > 0 \\ = 0, & \text{when } \beta^{\#}_{\mathcal{K}(X)}(\mathcal{A}) = 0 \end{cases}.$$

We skip the easy proofs noticing that the verification of Proposition 5 (c) involves Proposition 2 and Proposition 6 involves Lemmata 2 and 4.

Remark 2. One can easily see that $\varphi : X \circ X$ is hyper-condensing (h-compact) whenever $F : \mathcal{F}(X) \to \mathcal{F}(X)$ is $\beta^{\#}$-condensing (resp. compact).

Next we show typical hyper-condensing maps which are not condensing.

Example 1. Put $\varphi : X \multimap X$, $\varphi(x) = D(0,1)\ \ \forall_{x\in X}$, where X is an infinite dimensional Banach space and $D(0,1)$ is the closed unit ball at 0. This multifunction has noncompact values, it is bounded, h-compact and contractive; so it is hyper-condensing.

Example 2. Put $\varphi : X \multimap X$, $\varphi(x) = X\ \ \forall_{x\in X}$, where X is an infinite dimensional Banach space. Although unbounded, it is h-compact. In particular it is hyper-condensing and h-bounded.

Finally we prove (comp. [24] Th. 2)

Theorem 1. (On compact unions of hyper-condensing maps). *Let $\{\varphi_t : X \multimap X\}_{t\in T}$ be an $h_{\sup}$-compact family of hyper-condensing multifunctions. Then $\bigcup_{t\in T} \varphi_t : X \multimap X$ is also a hyper-condensing map.*

Proof. Let F_t and $F_{t\,\#}$ be the Barnsley-Hutchinson operators corresponding to φ_t and $\mathcal{A} \in \mathcal{F}(\mathcal{F}(X))$, $0 < \beta^{\#}(\mathcal{A}) < \infty$. We calculate

$$\beta^{\#}\left\{\overline{\bigcup_{t\in T}\varphi_t(A)} \,:\, A \in \mathcal{A}\right\} \overset{(1)}{=} \beta^{\#}\left\{\overline{\bigcup_{t\in T} F_t(A)} \,:\, A \in \mathcal{A}\right\}$$

$$\overset{(2)}{\leq} \beta^{\#}\Sigma\left[\bigcup_{t\in T}\overline{\{F_t(A) \,:\, A \in \mathcal{A}\}}^h\right] \overset{(3)}{=} \beta^{\#}\bigcup_{t\in T}\overline{\{F_t(A) \,:\, A \in \mathcal{A}\}}^h$$

$$\overset{(4)}{=} \sup_{t\in T} \beta^{\#}\overline{\{F_t(A) : A \in \mathcal{A}\}}^{h} \overset{(5)}{=} \beta^{\#}\overline{\{F_{t_0}(A) : A \in \mathcal{A}\}}^{h} \overset{(6)}{<} \beta^{\#}(\mathcal{A}),$$

where $t_0 \in T$. (1) is due to Lemma 2, (2) holds by the definition of Σ and monotonicity of $\beta^{\#}$, (3) is due to Proposition 2, and (6) is hyper-condensity of φ_{t_0}. For (4) and (5) firstly observe that the family $\overline{\{F_t(A) : A \in \mathcal{A}\}}^{h}_{t\in T} \subset \mathcal{F}(\mathcal{F}(X))$ is H-compact. Indeed, from Proposition 3 $\{F_{t\ \#}\}_{t\in T} \subset \mathcal{F}(\mathcal{F}(X))^{\mathcal{F}(\mathcal{F}(X))}$ is $H_{\sup}$-compact, so we can apply Lemma 7. To the end, Proposition 1 yields (4) and continuity of $\beta^{\#}$ yields (5). □

5. Fixed points and attractors.

We would like to know whether the Barnsley-Hutchinson operator F admits a fixed point? We know that F generated by a system of (weak) contractions possess a unique fixed point among all bounded nonempty closed sets; it is called a fractal ([14], [9], [26], [2]). On the other hand, F generated by a system of compact or, more generally, condensing maps often has many fixed points. Therefore, to get uniqueness we must look up for maximal fixed points (see e.g. [10], [7], [17], [18], [7]). But it turns out that there is no difference between searching for fixed points and maximal fixed points ([25], see also [12] Th. 11 p. 198, [11] Th. 2.2, [19] Lemma (L), [22] Th. 2).

Theorem 2. *For the Barnsley-Hutchinson operator $F : \mathcal{F}(X) \to \mathcal{F}(X)$ the following are equivalent:*

(o) *There exists $A \in \mathcal{F}(X)$ such that $A \subset F(A)$,*
(i) *F possess a fixed point,*
(ii) *F possess a maximal fixed point (no fixed points „above"),*
(iii) *F possess the greatest fixed point (all fixed points „under").*

The following technical statement relates fixed points on the double exponentiate $\mathcal{F}(\mathcal{F}(X))$ to fixed points on $\mathcal{F}(X)$.

Proposition 7. *If $F_{\#}(\mathcal{A}) = \mathcal{A}$ for a nonempty h-compact $\mathcal{A} \subset \mathcal{F}(X)$, then the set $\bigcup \mathcal{A} \in \mathcal{F}(X)$ is a fixed point of the operator F. If additionally $\mathcal{A}$ is the maximal fixed point of $F_{\#}$, then $\bigcup \mathcal{A}$ is the maximal fixed point of F.*

Proof. Firstly observe that $\bigcup \mathcal{A}$ is indeed closed (Lemma 1). Further we have

$$F\Big(\bigcup \mathcal{A}\Big) = \overline{\bigcup_{A\in\mathcal{A}} \varphi(A)} = \overline{\bigcup_{A\in\mathcal{A}} F(A)} = \overline{\bigcup F_{\#}(\mathcal{A})} = \overline{\bigcup \mathcal{A}} = \bigcup \mathcal{A}.$$

The second equality above follows from Lemma 2, the third one holds by Lemma 3, and the fourth one is nothing but $F_{\#}(\mathcal{A}) = \mathcal{A}$.

Let $\mathcal{A} = F_\#(\mathcal{A})$ be maximal. Any fixed point $B = F(B)$ satisfies $F_\#(\mathcal{A} \cup \{B\}) = \mathcal{A} \cup \{B\}$, so $B \in \mathcal{A}$. Thus $B \subset \bigcup \mathcal{A} = F\big(\bigcup \mathcal{A}\big)$, i.e., $\bigcup \mathcal{A}$ is the maximal fixed point of F. $\square$

Now we are ready to state the main theorem which improves and generalizes several earlier results ([10], [9], [2], [22], [23], [24]).

Theorem 3. (On attractors of hyper-condensing maps). *Let $\varphi : X \multimap X$ be an h-bounded hyper-condensing map and let $F : \mathcal{F}(X) \to \mathcal{F}(X)$ be the corresponding Barnsley-Hutchinson operator. Then*

(i) *There exists the maximal fixed point $A_* \in \mathcal{F}(X)$ of F.*
(ii) *The set $M = \bigcap_{n=1}^{\infty} F^n(X)$ is the global attractor of F, i.e., $F^n(X) \to M$ w.r.t. h (F^n denotes the n-fold composition of F).*
(iii) *We have that $A_* \subset M$. If φ is u.h-u.s.c., then $A_* = M$.*
(iv) *If φ is bounded, then both A_* and M are bounded.*

Proof. Let us consider the transfinite iteration $\{F_\#^\alpha(\mathcal{F}(X))\}_{\alpha<\chi}$,

$$F_\#^\alpha(\mathcal{F}(X)) = \begin{cases} \mathcal{F}(X), & \text{for } \alpha = 0, \\ F_\#\big[{F_\#}^{\alpha-1}(\mathcal{F}(X))\big], & \text{for isolated } \alpha, \\ \bigcap_{\gamma<\alpha} F_\#^\gamma(\mathcal{F}(X)), & \text{for limit } \alpha, \end{cases}$$

where χ is the first ordinal of cardinality greater than $\mathcal{F}(X)$. This transfinite sequence is decreasing. By Proposition 2 we have $\beta^\#\big[F_\#(\mathcal{F}(X))\big] = \beta^\#\{\varphi(x) : x \in X\} < \infty$, since φ is h-bounded. From Lemma 1.6.11 in [1] we obtain

$$\lim_{n\to\infty} \beta^\#\big[F_\#^n(\mathcal{F}(X))\big] = 0,$$

because φ is hyper-condensing. Therefore by the Kuratowski Intersection Theorem (e.g. [20], chap. III 30. I)

$$\mathcal{A}_\infty = \bigcap_{n<\omega} F_\#^n(\mathcal{F}(X))$$

is a nonempty h-compact family such that

$$(*) \qquad F_\#^n(\mathcal{F}(X)) \overrightarrow{n \to \infty} \mathcal{A}_\infty \text{ w.r.t. } H.$$

(Symbol ω denotes the first infinite ordinal). Moreover $F_\#(\mathcal{A}_\infty) \subset \mathcal{A}_\infty$. This way we have secured that starting from $\alpha = \omega$ our transfinite sequence consists from h-compact families and it is well-defined.

Ad (i). The iteration process must stabilize on some nonempty h-compact family $F_{\#}^{\delta}(\mathcal{F}(X)) = F_{\#}^{\delta+1}(\mathcal{F}(X))$, $\delta < \chi$ (see [22] Th.2). Applying now Proposition 7 to $\mathcal{A} = F_{\#}^{\delta}(\mathcal{F}(X))$ we get that $A_* = \bigcup \mathcal{A}$ is the maximal fixed point of F.

Ad (ii). Observe that $F^n(X) \in F_{\#}^n(\mathcal{F}(X))$ for $n < \omega$. Due to $(*)$ the h-compact family $\mathcal{A}_\infty$ attracts sequence $\{F^n(X)\}_{n=1}^\infty$, which in turn admits an h-convergent subsequence $F^{k_n}(X) \to M \in \mathcal{A}_\infty$. Since $\{F^n(X)\}_{n=1}^\infty$ is monotone, it must be convergent itself. This yields that $M = \bigcap_{n=1}^\infty F^n(X)$.

Ad (iii). Since $A_* = F^n(A_*) \subset F^n(X)$, for $n < \omega$, we have $A_* \subset M$. If additionally φ is u.h-u.s.c., then M is the fixed point of F as was proved in [23].

Ad (iv). From (iii) we know that $A_* \subset M \subset F(X)$. The latter set is bounded provided φ is bounded. □

Remark 3. One could try to assume in (ii) the continuity of $F_{\#} : (\mathcal{K}(\mathcal{F}(X)), H) \to (\mathcal{K}(\mathcal{F}(X)), H)$. But this would result in continuity of $F : (\mathcal{F}(X), h) \to (\mathcal{F}(X), h)$ implying for $\varphi : X \multimap X$ both h-continuity and u.h-u.s.c. So our hypothesis is weaker (u.h-u.s.c. map need not be continuous).

An accompanying theorem (comp. [26], [18], [24]) is

Theorem 4. (On attractors of compact families of hyper-condensing maps). *Let $\{\varphi_t : X \multimap X\}_{t\in T}$ be an $h_{\sup}$-compact family of h-bounded hyper-condensing multifunctions and let $F : \mathcal{F}(X) \to \mathcal{F}(X)$ be the corresponding Barnsley-Hutchinson operator. Then there hold (i), (ii) as above and*

(iii) *$A_* \subset M$; if all φ_t are u.h-u.s.c., then $A_* = M$.*
(iv) *If all φ_t are bounded, then both A_* and M are bounded.*

Proof. The thesis combines Theorems 3 and 1. □

Our theorems are applicable to multifunctions like those in Examples 1 and 2 which are usually excluded from general statements. Nevertheless we quote an example of a very simple situation which cannot be handled in the framework of hyper-condensing maps.

Example 3. Let $\varphi : X \multimap X$, $\varphi(x) = \{x\}$ for $x \in X$. This „multivalued identity" generates the operators $F : \mathcal{F}(X) \to \mathcal{F}(X)$ and $F_{\#} : \mathcal{F}(\mathcal{F}(X)) \to \mathcal{F}(\mathcal{F}(X))$, both being identities. One easily sees that all closed sets are fixed points of F with the whole space X as the maximal fixed point. Nevertheless this fact cannot be deduced from Theorem 3, because $F_{\#}$ is not condensing under any measure of noncompactness on $\mathcal{F}(\mathcal{F}(X))$.

REFERENCES

[1] R.R. Akhmerov - M.I. Kamenskiĭ- A.S. Potapov - A.E. Rodkina - B.N. Sadovskiĭ, *Measures of Noncompactness and Condensing Operators,* Operator Theory – Advances and Applications, Birkhäuser Verlag, Basel, 1992.

[2] J. Andres - J. Fišer, *Metric and topological multivalued fractals,* Int. J. Bifurc. Chaos, 14 n. 4 (2004), pp. 1277–1289.

[3] J.M. Ayerbe Toledano - T. Domínguez Benavides - G. López Acedo, *Measures of Noncompactness in Metric Fixed Point Theory,* Birkhäuser, Basel, 1997.

[4] G. Beer, *Topologies on Closed and Closed Convex Sets,* Mathematics and Its Applications, Kluwer Academic Publishers, Dordrecht, 1993.

[5] F.S. De Blasi - P.Gr. Georgiev, *Hukuhara's topological degree for non compact valued multifunctions,* Publ. Res. Inst. Math. Sci., 39 (2003), pp. 183–203.

[6] F.S. De Blasi - J. Myjak, *A remark on the definition of topological degree for set-valued mappings,* J. Math. Anal. Appl., 92 (1983), pp. 445–551.

[7] A. Edalat, *Dynamical systems, measures and fractals via domain theory,* Inform. and Comput., 120 n.1 (1995), pp. 32–48.

[8] J.K. Hale, *Asymptotic Behavior of Dissipative Systems,* Mathematical Surveys and Monographs, AMS, Providence, 1988.

[9] M. Hata, *On some properties of set-dynamical systems,* Proc. Japan Acad., Ser. A, 61 n. 4 (1985), pp. 99–102.

[10] S. Hayashi, *Self-similar sets as Tarski's fixed points,* Publ. RIMS Kyoto Univ., 21 (1985), pp. 1059–1066.

[11] S. Heikkilä, *On fixed points through a generalized iteration method with applications to differential and integral equations involving discontinuities,* Nonlinear Anal., 14 n. 5 (1990), pp. 413–426.

[12] P. Hitzler - A.K. Seda, *Generalized metrics and uniquely determined logic programs,* Theoret. Comput. Sci., 305 (2003), pp. 187–219.

[13] S. Hu - N.S. Papageorgiou, *Handbook of Multivalued Analysis. Vol. I: Theory,* Mathematics and its Applications, Kluwer Academic Publishers, Dordrecht, 1997.

[14] J.E. Hutchinson, *Fractals and self similarity,* Indian. Univ. Math. J., 30 n.5 (1981), pp. 713–747.

[15] A. Illanes - S.B. Nadler Jr., *Hyperspaces. Fundamentals and Recent Advances,* Pure and Applied Mathematics, Marcel Dekker Inc., New York - Basel, 1999.

[16] J.R. Jachymski, *Equivalence of some contractivity properties over metrical structures,* Proc. Amer. Math. Soc., 125 n. 8 (1997), pp. 2327–2335.

[17] J. Jachymski - L. Gajek - P. Pokarowski, *The Tarski-Kantorovitch Principle and the Theory of Iterated Function Systems,* Bull. Austral. Math. Soc., 61 (2000), pp. 247–261.

[18] B. Kieninger, *Iterated Function Systems on Compact Hausdorff Spaces,* Berichte aus der Mathematik, Shaker-Verlag, Aachen, 2002.

[19] B. Knaster, *Un théorème sur les fonctions d'ensembles,* Ann. Soc. Polon. Math., 6 (1928), pp. 133-134.

[20] C. Kuratowski, *Topologie, vol.I,* Monografie Matematyczne, PWN, Warszawa 1958 (French).

[21] A. Lasota - J. Myjak, *Attractors of Multifunctions,* Bull. Pol. Ac. Math., 48 (2000) pp. 319–334.

[22] K. Leśniak, *Extremal sets as fractals,* Nonlinear Anal. Forum, 7 n. 2 (2002), pp. 199–208.

[23] K. Leśniak, *Stability and invariance of multivalued iterated function systems,* Math. Slovaca, 53 n. 4 (2003), pp. 393–405.

[24] K. Leśniak, *Infinite iterated function systems: a multivalued approach,* Bull. Pol. Ac. Sci., Ser. A, 52 n.1 (2004), pp. 1–8.

[25] E.A. Ok, *Fixed set theory for closed correspondences with applications to self-similarity and games,* Nonlinear Anal., 56 (2004), pp. 309–330.

[26] K.R. Wicks, *Fractals and Hyperspaces,* Lecture Notes in Mathematics 1492, Springer-Verlag, Berlin, 1991.

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