LE MATEMATICHE
Vol. LXXII (2017) – Fasc. I, pp. 177–184
doi: 10.4418/2017.72.1.14

EXISTENCE AND MULTIPLICITY OF NON-ZERO SOLUTIONS FOR THE NEUMANN PROBLEM VIA SPHERICAL MAXIMA

MEHDI KHODABAKHSHI

In this paper, we are interested in the existence and multiplicity of non-zero solutions for a two-point boundary value problems subject to Neumann conditions. Our approach is based on a result on spherical maxima sharing the same Lagrange multiplier that was established recently by Ricceri.

1. Introduction and preliminaries

In this article, we consider the Neumann problem

$$\begin{cases} -u'' + u = \lambda \alpha(t) f(u) & t \in [0,1] \\ u'(0) = u'(1) = 0 \end{cases} \tag{1}$$

where $\alpha : [0,1] \to \mathbb{R}$ is a non-zero, non-negative continuous function, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with $f(0) = 0$ and $\lambda > 0$ is a real parameter. In particular, A. Iannizzotto, in [2], proved the the following result:

Entrato in redazione: 12 agosto 2016

AMS 2010 Subject Classification: 34B15
Keywords: Neumann problem; Spherical maxima; Palais-Smale condition; Variational methods; Critical point

Theorem 1.1. *Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function such that, for some $s>0$, one has*

$$\sup_{|\xi|\le s}\int_0^{\xi} f(t)dt = \sup_{|\xi|\le\sqrt{2}s}\int_0^{\xi} f(t)dt.$$

Then, for every continuous, non-zero and non-negative function $\alpha:[0,1]\to\mathbb{R}$, the problem

$$\begin{cases} -u''+u=\alpha(t)f(u) & t\in[0,1] \\ u'(0)=u'(1)=0 \end{cases}$$

has at least a solution whose norm in $H^1(]0,1[)$ is less than or equal to s.

Notice that when $f(0)=0$ the quoted result does not guarantee the existence of a non-zero solution.
Our purpose in this work is to show that, under the additional condition

$$\sup_{|\xi|\le s}\int_0^{\xi} f(t)dt > 0,$$

there is a *non-zero* solution for some suitable $\lambda>0$, and also in Theorem 2.3, we obtain at least two non-zero solutions for the problem (1), by using the Ambrosetti-Rabinowitz condition (see [1]).
Here, we denote by X the Sobolev space $H^1(]0,1[)$, and consider the following scalar product on X :

$$<u,v>=\int_0^1 u'(t)v'(t)dt+\int_0^1 u(t)v(t)dt$$

with the induced norm

$$\|u\| = \left(\int_0^1 |u(t)|^2dt+\int_0^1 \left|u'(t)\right|^2dt\right)^{\frac{1}{2}}.$$

Let us define $F(\xi):=\int_0^{\xi} f(t)dt$, for every $\xi\in\mathbb{R}$. By the compact embedding $X\hookrightarrow C([0,1])$, there exists a positive constant $c>1$ such that

$$\|u\|_\infty \le c\,\|u\|\,, \qquad (\forall u\in X)$$

where c is the best constant of the embedding and thanks to [2], $c=\sqrt{2}$. Moreover we introduce the functional $J:X\to\mathbb{R}$ associated with (1),

$$J(u)=\int_0^1 \alpha(t)F(u(t))dt.$$

Standard arguments show that J is a well defined and continuously Gâteaux differentiable functional whose Gâteaux derivative is given by

$$J'(u)(v) = \int_0^1 \alpha(t) f(u(t)) v(t) dt$$

for every $u, v \in X$.

By classical results, the critical points of the functional $\frac{1}{2}\|.\|^2 - \lambda J(.)$ in $H^1(]0,1[)$ are exactly the classical solutions of (1).

Definition 1.2. A Gâteaux differentiable function I satisfies the Palais-Smale condition if any sequence $\{u_n\}$ such that

(a) $\{I(u_n)\}$ is bounded,

(b) $\lim_{n\to+\infty} \|I'(u_n)\|_{X^*} = 0$,

has a convergent subsequence.

Our main tool to investigate the existence and multiplicity of solutions for the problem (1) is a result on spherical maxima sharing the same Lagrange multiplier due to B. Ricceri [3, Theorem 1], which we now recall in a convenient form. First, we begin by introducing some notations. If X is a real Hilbert space, for each $\rho > 0$, set

$$S_\rho = \{x \in X : \|u\| = \rho\},$$
$$B_\rho = \{x \in X : \|u\| \le \rho\},$$

and

$$\beta_\rho = \sup_{B_\rho} J, \qquad \delta_\rho = \sup_{x \in B_\rho \setminus \{0\}} \frac{J(x)}{\|x\|^2}.$$

Theorem 1.3. *For some $\rho > 0$, assume that J is Gâteaux differentiable in $int(B_\rho)\setminus\{0\}$ and*

$$\frac{\beta_\rho}{\rho} < \delta_\rho. \tag{2}$$

For each $r \in]\beta_\rho, +\infty[$, put

$$\eta(r) = \sup_{y \in B_\rho} \frac{\rho - \|y\|^2}{r - J(y)}$$

and

$$\Gamma(r) = \left\{ x \in B_\rho : \frac{\rho - \|x\|^2}{r - J(x)} = \eta(r) \right\}.$$

Then the following assertions hold:

i) *the function* η *is convex and decreasing in* $]\beta_\rho, +\infty[$, *with* $\lim_{r\to+\infty}\eta(r) = 0$;

ii) *for each* $r \in]\beta_\rho, \rho\delta_\rho[$, *the set* $\Gamma(r)$ *is non-empty and, for every* $\hat{x} \in \Gamma(r)$, *one has*

$$0 < \|\hat{x}\|^2 < \rho$$

and

$$\hat{x} \in \left\{ x \in S_{\|\hat{x}\|^2} : J(x) = \sup_{S_{\|\hat{x}\|^2}} J \right\}$$

$$\subseteq \left\{ x \in \mathrm{int}(B_\rho) : \|x\|^2 - \eta(r)J(x) = \inf_{y\in B_\rho} \left(\|y\|^2 - \eta(r)J(y) \right) \right\}$$

$$\subseteq \left\{ x \in X : x = \frac{\eta(r)}{2} J'(x) \right\}.$$

In this work, the valuable and main way for providing condition (2), that was proved in ([3], Proposition 2), is the following

Proposition 1.4. *For some* $s > 0$, *assume that* J *is Gâteaux differentiable in* $B_s \backslash \{0\}$ *and that there exists a global maximum* $\hat{x}$ *of* $J_{|B_s}$ *such that*

$$\langle J'(\hat{x}), \hat{x} \rangle < 2J(\hat{x}).$$

Then (2) *holds with* $\rho = \|\hat{x}\|^2$.

2. Main results

Our main result is the following.

Theorem 2.1. *Assume that* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function such that*

i) *there exist a* $s > 0$ *and* $\xi_0 \in \mathbb{R}$ *with* $|\xi_0| \leq \sqrt{s}$ *such that*

$$F(\xi_0) = \sup_{|\xi|\leq\sqrt{2s}} F(\xi)$$

ii) $F(\xi_0) > 0$.

Then, for each non-zero, non-negative, continuous function $\alpha : [0,1] \to \mathbb{R}$ *there exists a non-degenerate interval*

$I_\alpha \subset]0, +\infty[$, *such that, for every* $\lambda \in I_\alpha$, *the problem* (1) *has at least a non-zero solution whose norm in* $H^1(]0,1[)$ *is less than* $|\xi_0|$.

Proof. Our aim is to apply Theorem 1.3 with $X = H^1(]0,1[)$ and
$J(u) = \int_0^1 \alpha(t)F(u(t))dt$.

Set $\rho = |\xi_0|^2$. By the compact embedding $X \hookrightarrow C([0,1])$, it is easy to show that the functional J is a sequentially weakly continuous. Then, for every $a > 0$, the functional $\|.\|^2 - aJ(.)$ is sequentially weakly lower semicontinuous. Fix a non-zero, non-negative continuous function $\alpha : [0,1] \to \mathbb{R}$. Let us prove that $\frac{\beta_\rho}{\rho} < \delta_\rho$. Denote by u_0 the constant function $u_0(t) = \xi_0$ for every $t \in [0,1]$. Clearly $u_0 \in X$. By *i*) and since $\|u\|_\infty \leq \sqrt{2}\|u\|$ for every $u \in H^1(]0,1[)$, then we have

$$\sup_{\|u\|^2 \leq s} J(u) \leq \sup_{|\xi| \leq \sqrt{2s}} F(\xi) \left(\int_0^1 \alpha(t)dt \right) = F(\xi_0) \left(\int_0^1 \alpha(t)dt \right)$$

then $\sup_{\|u\|^2 \leq s} J(u) = J(u_0)$. On the other hand, since

$$\langle J'(u_0), u_0 \rangle = \int_0^1 \alpha(t) f(\xi_0) u_0(t) dt \ ,$$

by *ii*) and $f(\xi_0) = 0$, we get

$$\langle J'(u_0), u_0 \rangle = 0 < 2F(\xi_0) \left(\int_0^1 \alpha(t)dt \right) = 2J(u_0).$$

So, thanks to Proposition 1.4, the condition $\frac{\beta_\rho}{\rho} < \delta_\rho$ is verified. Therefore, all conclusions of Theorem 1.3 hold. Put

$$I_\alpha := \left\{ \frac{\eta(r)}{2} : r \in]\beta_\rho, \rho\delta_\rho[\right\}. \tag{3}$$

Then, since the function $\eta :]\beta_\rho, +\infty[\to \mathbb{R}$ is continuous and decreasing, I_α is a non-degenerate interval, Now, fix $\lambda \in I_\alpha$. So, there is a $r \in]\beta_\rho, \rho\delta_\rho[$ such that $\lambda = \frac{\eta(r)}{2}$. Then, by *ii*) of Theorem 1.3, there exists $\hat{x} \in \Gamma(r)$ such that

$$0 < \|\hat{x}\| < |\xi_0|$$

and

$$\hat{x} = \frac{\eta(r)}{2} J'(\hat{x}) = \lambda J'(\hat{x}),$$

which means that the problem (1) has at least a non-zero solution whose norm in $H^1(]0,1[)$ is less than $|\xi_0|$. □

Remark 2.2. In view of *ii*) of Theorem 1.3, it is clear that the non-zero solution of the conclusion of Theorem 1.3 is a local minimum of the functional $\frac{1}{2}\|.\|^2 - \lambda J(.)$.

Theorem 2.3. *Let the assumptions of Theorem 1.3 be satisfied. Moreover, assume that*

i) $\limsup_{\xi \to 0} \frac{F(\xi)}{\xi^2} \le 0$;

ii) *there are constants* $\mu > 2$ *and* $r > 0$ *such that for* $|\xi| \ge r$,

$$0 < \mu F(\xi) \le \xi f(\xi).$$

Then, for each non-zero, non-negative, continuous function $\alpha : [0,1] \to \mathbb{R}$ *there exists a non-degenerate interval* $I_\alpha \subset]0,+\infty[$ *such that, for every* $\lambda \in I_\alpha$, *the problem* (1) *has at least two non-zero solutions, one of which has norm in* $H^1(]0,1[)$ *less than* $|\xi_0|$.

Proof. Let $\alpha : [0,1] \to \mathbb{R}$ be a non-zero, non-negative continuous function. By Theorem 1.3 and Remark 2.2, there exists a non-degenerate interval $\lambda \in I_\alpha$, such that, for every $\lambda \in I_\alpha$, the functional $\frac{1}{2}\|.\|^2 - \lambda J(.)$ has a non-zero local minimum $\hat{x}$ whose norm in $H^1(]0,1[)$ is less than $|\xi_0|$. By *i*), there exists a $\sigma > 0$ such that

$$F(\xi) \le \frac{1}{8\lambda\left(\sup\limits_{[0,1]} \alpha\right)} \xi^2,$$

for every $\xi \in]-\sigma,\sigma[$. Set $V = \left\{u \in H^1(]0,1[) : \|u\| < \frac{\sigma}{\sqrt{2}}\right\}$. Then we have

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 - \lambda J(u) \ge \frac{1}{2}\|u\|^2 - \lambda \frac{1}{8\lambda\left(\sup\limits_{[0,1]} \alpha\right)} \left(\int_0^1 \alpha(t)|u(t)|^2 dt\right)$$

$$\ge \frac{1}{2}\|u\|^2 - \lambda \frac{1}{8\lambda\left(\sup\limits_{[0,1]} \alpha\right)} \left(\sup\limits_{[0,1]} \alpha\right) \|u\|^2_\infty = \frac{1}{4}\|u\|^2 \ge 0$$

for every $u \in V$. So, 0 is another local minimum of the functional $\frac{1}{2}\|.\|^2 - \lambda J(.)$. On the other hand, a very classical reasoning shows that, by *ii*), the same functional satisfies the Palais-Smale condition. So, by Corollary 1 of [4], this functional has a third critical point, and the proof is compelete. □

Example 2.4. In connection with Theorem 2.1, consider the following problem

$$\begin{cases} -u'' + u = \lambda \alpha(t) f(u) & in\ [0,1] \\ u'(0) = u'(1) = 0, \end{cases}$$

where $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(t) = \begin{cases} -t^3 - t^2 & t \le 0, \\ \frac{1}{1000}(5t^4 + 3t^2) & t > 0. \end{cases}$$

Then, for each non-zero, non-negative, continuous function $\alpha : [0,1] \to \mathbb{R}$ there exists a non-degenerate interval $I_\alpha \subseteq]0,+\infty[$ such that, for every $\lambda \in I_\alpha$, the above problem has at least a non-zero solutions whose norm in $H^1(]0,1[)$ is less than 1.
To prove this, we can apply Theorem 2.1 by taking $\xi_0 = -1$ and $s = 1$. A simple computation shows that

$$\sup_{|\xi| \le \sqrt{2}} \int_0^{\xi} f(t)dt = \int_0^{\xi_0} f(t)dt.$$

All the assumptions of Theorem 2.1 are so verified and the conclusion follows.

Example 2.5. In connection with Theorem 2.3, consider the following problem

$$\begin{cases} -u'' + u = \lambda \alpha(t) f(u) & in\ [0,1] \\ u'(0) = u'(1) = 0, \end{cases}$$

where $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(t) = \begin{cases} -(t+1)^2 & t \le -1 \\ -8\pi \sin(16\pi t) & -\frac{1}{8} < t \le \frac{1}{8} \\ 16\pi \sin\left(16\pi\left(t - \frac{1}{4}\right)\right) & \frac{1}{8} < t \le \frac{3}{8} \\ (t-1)^2 & t \ge 1 \\ 0 & o.w \end{cases}$$

Then, for each non-zero, non-negative, continuous function $\alpha : [0,1] \to \mathbb{R}$ there exists a non-degenerate interval $I_\alpha \subseteq]0,+\infty[$ such that, for every $\lambda \in I_\alpha$, the above problem has at least two non-zero solutions, one of which has norm in $H^1(]0,1[)$ less than $\frac{5}{16}$.
To prove this, we can apply Theorem 2.3, by taking $\xi_0 = \frac{5}{16}$, $s = \frac{1}{2}$, $r = 1$ and $\mu = 3$. A simple computation shows that

$$\sup_{|\xi| \le 1} \int_0^{\xi} f(t)dt = \int_0^{\xi_0} f(t)dt,$$

and also

$$0 < \mu F(x) \leq xf(x),$$

for every $|x| \geq 1$. Further, since the function F is negative in some neighborhood of origin so, we have

$$\limsup_{\xi \to 0} \frac{F(\xi)}{\xi^2} \leq 0.$$

Finally, all the assumptions of Theorem 2.3 are so verified and the conclusion follows.

REFERENCES

[1] A. Ambrosetti, P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. 14 (1973), 349–381.

[2] A. Iannizzotto, *A sharp existence and localization theorem for a Neumann problem*, Arch. Math. 82 (2004), 352–360.

[3] B. Ricceri, *A note on spherical maxima sharing the same Lagrange multiplier* , Fixed Point Theory and Applications. 2014 (2014): 25.

[4] P. Pucci, J. Serrin, *A mountain pass theorem*, J. Differential Equations. 60 (1985), 142–149.

MEHDI KHODABAKHSHI
Department of Mathematics,Faculty of Mathematical and Computer Sciences
University of Shahid Chamran ,Ahwaz,Iran
e-mail: m.khodabakhshi11@gmail.com