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THREE SOLUTIONS FOR A NEUMANN BOUNDARY VALUE PROBLEM INVOLVING THE p-LAPLACIAN

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In this note we prove the existence of an open interval $]\lambda', \lambda''[$ for each λ of which a Neumann boundary value problem involving the p-Laplacian and depending on λ admits at least three solutions. The result is based on a recent three critical points theorem.

1. Introduction.

Let Ω be a nonempty bounded open set of the real Euclidean space $\mathbb{R}^n$, with a boundary of class C^1, $a \in L^\infty(\Omega)$, with $\operatorname{essinf}_\Omega a > 0$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ a function, and $p \geq 2$.

Let us consider the following problem

$$\text{(P)} \qquad \begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = \lambda f(x,u) & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_p = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $\lambda \in]0, +\infty[$, and ν is the outer unit normal to $\partial\Omega$.

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A weak solution to problem (P) is a function $u \in W^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)\,dx + \int_\Omega a(x)|u(x)|^{p-2}u(x)v(x)\,dx$$
$$- \lambda \int_\Omega f(x, u(x))v(x)\,dx = 0, \quad \forall\ v \in W^{1,p}(\Omega).$$

Problems of the above type were widely studied in these latest years and we refer to [1], [2], [5], [7], [8], (see also [6] and [9], for the case $n = 1$ and $p = 2$) and the references therein for more details. In particular, in [7] the authors obtained the existence of an open interval $\Lambda \subseteq [0, \infty[$ such that for each $\lambda \in \Lambda$ problem (P) admits at least three weak solutions which are uniformly bounded with respect to λ, without, however, establishing where Λ is located; while in [8], under a different set of assumptions, the existence of three weak solutions to (P) for $\lambda = 1$ was proved.

The aim of this note is to establish the existence of a precise open interval $]\lambda', \lambda''[$, $0 < \lambda' < \lambda'' \leq +\infty$, for each λ of which problem (P) admits at least three weak solutions. Our main result is Theorem 1 and, as a way of example, we present, here, a particular case.

Theorem A. *Let $h : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function such that*

$$\lim_{u \to 0^+} \frac{h(u)}{u} = 0 \quad \textit{and} \quad \int_0^\delta h(u)\,du > 0 \ \textit{for some}\ \delta > 0.$$

Then, for every non-negative function $g \in C^0([0, 1])$ such that

$$\|g\|_1 > \frac{\delta^2}{2\int_0^\delta h(u)\,du}$$

the problem

$$\text{(P1)} \qquad \begin{cases} -u'' + u = g(t)h(u) \\ u'(0) = u'(1) = 0, \end{cases}$$

admits at least two non-negative and non-trivial classical solutions.

Example 2 at the end of the paper shows a Neumann problem that, owing to our results, admits three solutions, but to which Theorem 2.1 of [8] cannot be applied.

Our results are based on the following recent three critical points theorem obtained in [3].

Theorem B. (Theorem B of [3]) *Let X be a real reflexive Banach space, $\Phi : X \to \mathbb{R}$ a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that:*

(i) $\lim_{\|u\|\to+\infty} (\Phi(u) + \lambda\Psi(u)) = +\infty$ *for all* $\lambda \in [0, +\infty[$;

(ii) *there is $r \in \mathbb{R}$ such that:*

$$\inf_X \Phi < r,$$

and

$$\varphi_1(r) < \varphi_2(r),$$

where

$$\varphi_1(r) := \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\Psi(u) - \inf_{\overline{\Phi^{-1}(]-\infty,r[)}^w} \Psi}{r - \Phi(u)},$$

$$\varphi_2(r) := \inf_{u\in\Phi^{-1}(]-\infty,r[)} \sup_{v\in\Phi^{-1}([r,+\infty[)} \frac{\Psi(u) - \Psi(v)}{\Phi(v) - \Phi(u)},$$

and $\overline{\Phi^{-1}(] - \infty, r[)}^w$ is the closure of $\Phi^{-1}(] - \infty, r[)$ in the weak topology.

Then, for each $\lambda \in]\frac{1}{\varphi_2(r)}, \frac{1}{\varphi_1(r)}[$ the functional $\Phi + \lambda\Psi$ has at least three critical points in X.

Other applications of Theorem B can be found in [3] and [4].

In order to apply Theorem B to our problem, let X be the space $W^{1,p}(\Omega)$ equipped with the norm

$$\|u\| := \Big(\int_\Omega |\nabla u(x)|^p\, dx + \int_\Omega a(x)|u(x)|^p\, dx\Big)^{\frac{1}{p}},$$

which is equivalent to the usual one, while on the space $C^0(\overline{\Omega})$ we consider the norm $\|u\|_\infty := \sup_{u\in\overline{\Omega}} |u(x)|$.

If $p > n$, X is compactly embedded in $C^0(\overline{\Omega})$, so that

$$c := \sup_{u\in X\setminus\{0\}} \frac{\|u\|_\infty}{\|u\|} < +\infty. \tag{1}$$

Clearly, $c^p\|a\|_1 \geq 1$, where $\|a\|_1 := \int_\Omega |a(x)|\, dx$.
For other basic notations and definitions we refer to [11].

2. Results.

Our main result is the following

Theorem 1. *Let $p > n$ and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that, for every $\rho > 0$, $\sup_{|u|\le\rho} |f(.,u)| \in L^1(\Omega)$. Put*

$$F(x,\xi) := \int_0^\xi f(x,u)\,du \quad \text{for every} \quad (x,\xi) \in \Omega \times \mathbb{R},$$

and assume that there exist three positive constants γ, δ, and s, with $\gamma < \delta$ and $s < p$, and a function $\mu \in L^1(\Omega)$ such that:

(j) $$\frac{\int_\Omega \sup_{|\xi|\le\gamma} F(x,\xi)\,dx}{\gamma^p} < \frac{1}{1+c^p\|a\|_1} \frac{\int_\Omega F(x,\delta)\,dx}{\delta^p};$$

(jj) $F(x,\xi) \le \mu(x)(1+|\xi|^s)$ *for all* $(x,\xi) \in \Omega \times \mathbb{R}$.

Then, setting

$$\lambda' := \frac{\|a\|_1 \delta^p}{p\Big(\int_\Omega F(x,\delta)\,dx - \int_\Omega \sup_{|\xi|\le\gamma} F(x,\xi)\,dx\Big)}$$

and

$$\lambda'' := \frac{\gamma^p}{c^p p \int_\Omega \sup_{|\xi|\le\gamma} F(x,\xi)\,dx},$$

for each $\lambda \in]\lambda', \lambda''[$ problem (P) *admits at least three weak solutions.*

Proof. For each $u \in X$, put

$$\Phi(u) := \frac{1}{p}\|u\|^p$$

and

$$\Psi(u) := -\int_\Omega F(x,u(x))\,dx.$$

Since $p > n$, X is compactly embedded in $C^0(\overline{\Omega})$ and it is well known that Φ and Ψ are (well defined and) continuously Gâteaux differentiable functionals with

$$\Phi'(u)(v) = \int_\Omega \big(|\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x) + a(x)|u(x)|^{p-2}u(x)v(x)\big)\,dx$$

and

$$\Psi'(u)(v) = -\int_{\Omega} f(x, u(x))v(x)\,dt$$

for every $u, v \in X$, as well as Ψ' is compact.

Furthermore, by Proposition 25.20 (i) of [11], Φ is sequentially weakly lower semicontinuous, while Proposition 1 of [7] ensures that Φ' admits a continuous inverse on X^*.

Hypothesis (i) of Theorem B follows in a simple way thanks to (jj).

In order to prove (ii) of Theorem B, put $r := \frac{1}{p}\left(\frac{\gamma}{c}\right)^p$.

From hypothesis (j), we get

$$\frac{\int_\Omega \sup_{|\xi|\le\gamma} F(x,\xi)\,dx}{c^p\|a\|_1\gamma^p} < \frac{1}{c^p\|a\|_1(1+c^p\|a\|_1)} \frac{\int_\Omega F(x,\delta)\,dx}{\delta^p}$$

$$= \left(\frac{1}{c^p\|a\|_1} - \frac{1}{1+c^p\|a\|_1}\right)\frac{\int_\Omega F(x,\delta)\,dx}{\delta^p},$$

then

$$\frac{1}{1+c^p\|a\|_1}\frac{\int_\Omega F(x,\delta)\,dx}{\delta^p} + \frac{\int_\Omega \sup_{|\xi|\le\gamma} F(x,\xi)\,dx}{c^p\|a\|_1\gamma^p} < \frac{1}{c^p\|a\|_1}\frac{\int_\Omega F(x,\delta)\,dx}{\delta^p},$$

thus, being $\gamma < \delta$, we have

$$\frac{1}{1+c^p\|a\|_1}\frac{\int_\Omega F(x,\delta)\,dx}{\delta^p} < \frac{\int_\Omega F(x,\delta)\,dx - \int_\Omega \sup_{|\xi|\le\gamma} F(x,\xi)\,dx}{c^p\|a\|_1\delta^p},$$

hence, using again (j), we get

$$\frac{\int_\Omega \sup_{|\xi|\le\gamma} F(x,\xi)\,dx}{\gamma^p} < \frac{\int_\Omega F(x,\delta)\,dx - \int_\Omega \sup_{|\xi|\le\gamma} F(x,\xi)\,dx}{c^p\|a\|_1\delta^p},$$

from which, multiplying by $c^p p$, we obtain

$$\frac{c^p\, p\int_\Omega \sup_{|\xi|\le\gamma} F(x,\xi)\,dx}{\gamma^p} < p\,\frac{\int_\Omega F(x,\delta)\,dx - \int_\Omega \sup_{|\xi|\le\gamma} F(x,\xi)\,dx}{\|a\|_1\delta^p}. \tag{2}$$

We claim that:

$$\text{(C1)} \qquad \varphi_1(r) \leq \frac{c^p \; p \; \int_\Omega \sup\limits_{|\xi|\leq\gamma} F(x,\xi)\,dx}{\gamma^p}$$

and

$$\text{(C2)} \qquad \varphi_2(r) \geq p \, \frac{\int_\Omega F(x,\delta)\,dx - \int_\Omega \sup\limits_{|\xi|\leq\gamma} F(x,\xi)\,dx}{\|a\|_1 \delta^p},$$

from which (ii) of Theorem B follows.

In fact, taking into account that the function identically 0 obviously belongs to $\Phi^{-1}(]-\infty, r[)$, and that $\Psi(0) = 0$, we get

$$\varphi_1(r) \leq \frac{\sup\limits_{\overline{\Phi^{-1}(]-\infty,r[)}^w} \int_\Omega F(x,u(x))\,dx}{r},$$

and, since $\overline{\Phi^{-1}(]-\infty, r[)}^w = \Phi^{-1}(]-\infty, r])$, we have

$$\frac{\sup\limits_{\overline{\Phi^{-1}(]-\infty,r[)}^w} \int_\Omega F(x,u(x))dx}{r} = \frac{\sup\limits_{\Phi^{-1}(]-\infty,r])} \int_\Omega F(x,u(x))\,dx}{r},$$

thus, taking into account that $|u(x)| \leq c(pr)^{\frac{1}{p}} = \gamma$, for every $u \in X$ such that $\Phi(u) \leq r$ and for each $x \in \Omega$, we obtain

$$\frac{\sup\limits_{\Phi^{-1}(]-\infty,r])} \int_\Omega F(x,u(x))\,dx}{r} \leq \frac{\int_\Omega \sup\limits_{|\xi|\leq\gamma} F(x,\xi)\,dx}{r}.$$

So, (C1) follows at once by the definition of r.

Moreover, for each $v \in X$ such that $\Phi(v) \geq r$, we have

$$\varphi_2(r) \geq \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\int_\Omega F(x,v(x))\,dx - \int_\Omega F(x,u(x))\,dx}{\Phi(v) - \Phi(u)},$$

thus, from $|u(x)| \leq c(pr)^{\frac{1}{p}} = \gamma$, for every $u \in X$ such that $\Phi(u) < r$ and for each $x \in \Omega$, we obtain

$$\inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\int_\Omega F(x,v(x))\,dx - \int_\Omega F(x,u(x))\,dx}{\Phi(v) - \Phi(u)}$$

$$\geq \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\int_\Omega F(x,v(x))\,dx - \int_\Omega \sup_{|\xi|\leq\gamma} F(x,\xi)\,dx}{\Phi(v)-\Phi(u)},$$

from which, being $0 < \Phi(v) - \Phi(u) \leq \Phi(v)$ for every $u \in \Phi^{-1}(]-\infty, r[)$, and under further condition

$$\int_\Omega F(x,v(x))\,dx \geq \int_\Omega \sup_{|\xi|\leq\gamma} F(x,\xi)\,dx, \tag{3}$$

we can write

$$\inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\int_\Omega F(x,v(x))\,dx - \int_\Omega \sup_{|\xi|\leq\gamma} F(x,\xi)\,dx}{\Phi(v)-\Phi(u)}$$

$$\geq p\,\frac{\int_\Omega F(x,v(x))\,dx - \int_\Omega \sup_{|\xi|\leq\gamma} F(x,\xi)\,dx}{\|v\|^p}.$$

If we put $v(x) := \delta$, for each $x \in \Omega$, we have $\|v\| = \|a\|_1^{\frac{1}{p}}\delta$, hence, by (1) and $\gamma < \delta$, we get $\Phi(v) > r$. Moreover, with this choice of v, (2) ensures (3), thus (C2) is also proved.

Hence the conclusion follows by Theorem B, by observing that

$$\frac{1}{\varphi_2(r)} \leq \frac{\|a\|_1\delta^p}{p\Big(\int_\Omega F(x,\delta)\,dx - \int_\Omega \sup_{|\xi|\leq\gamma} F(x,\xi)\,dx\Big)}$$

and

$$\frac{1}{\varphi_1(r)} \geq \frac{\gamma^p}{c^p p \int_\Omega \sup_{|\xi|\leq\gamma} F(x,\xi)\,dx}. \qquad \square$$

Remark 1. In applying Theorem 1, it is enough to know an explicit upper bound for the constant c defined in (1). In this connection, when Ω is a convex set, we have the following estimate (see [1] for more details):

$$c \leq 2^{\frac{p-1}{p}} \max\left\{\Big(\frac{1}{\|a\|_1}\Big)^{\frac{1}{p}}, \frac{\operatorname{diam}(\Omega)}{n^{\frac{1}{p}}}\Big(\frac{p-1}{p-n}\operatorname{meas}(\Omega)\Big)^{\frac{p-1}{p}} \frac{\|a\|_\infty}{\|a\|_1}\right\}. \tag{4}$$

Example 1. The problem

$$\begin{cases} -\Delta_3 u + \frac{2+x}{\pi}|u|u = \lambda(x^2+y^2)\Big[2e^{-u^2}u^{17}(9-u^2)+1\Big] & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is the open unit ball in $\mathbb{R}^2$, admits at least three weak solutions for each $\lambda \in]15 \cdot 10^{-4}, 15 \cdot 10^{-2}[$.

In fact, choosing

$$\gamma := 1, \delta := 2, s := 2,$$

and

$$a(x, y) := \frac{2+x}{\pi}, \; f(x, y, u) := (x^2+y^2)\Big[2e^{-u^2}u^{17}(9-u^2)+1\Big]$$

for every $(x, y) \in \Omega$ and every $u \in \mathbb{R}$, and taking into account (4), it is simple to verify all the hypotheses of Theorem 1, and that $]15 \cdot 10^{-4}, 15 \cdot 10^{-2}[\subset]\lambda', \lambda''[$.

Remark 2. In the ordinary case, whenever $\Omega :=]\alpha, \beta[$ is a bounded open interval in $\mathbb{R}$, a function $u : [\alpha, \beta] \to \mathbb{R}$ is said a generalized solution to problem

$$\text{(PO)} \qquad \begin{cases} -\big(|u'|^{p-2}u'\big)' + a(t)|u|^{p-2}u = \lambda f(t, u) & \text{in }]\alpha, \beta[\\ u'(\alpha) = u'(\beta) = 0, \end{cases}$$

if $u \in C^1([\alpha, \beta])$, $|u'|^{p-2}u' \in AC([\alpha, \beta])$, $u'(\alpha) = u'(\beta) = 0$ and

$$-\big(|u'(t)|^{p-2}u'(t)\big)' + a(t)|u(t)|^{p-2}u(t) = \lambda f(t, u(t))$$

for almost every $t \in [\alpha, \beta]$.

Using standard arguments, and taking into account that the correspondence $u \mapsto |u|^{p-2}u$ is an homeomorphism in $\mathbb{R}$, one can show that weak solutions coincides with generalized ones, provided $a \in L^\infty(]\alpha, \beta[)$ and f is as in Theorem 1.

We point out the following simple consequence of Theorem 1, then we give the proof of Theorem A and finally we present an easy example of application.

Theorem 2. *Let $p \geq 2$, and let $g \in L^1(]\alpha, \beta[)$, $h \in C^0(\mathbb{R})$ be two non-negative functions, with $\|g\|_1 > 0$. Assume that there exists four positive constants γ, δ, η and s, with $\gamma < \delta$ and $s < p$, such that:*

(k) $$\frac{\int_0^{\gamma} h(u)\,du}{\gamma^p} < \frac{1}{1+2^{p-1}\max\{1,(\beta-\alpha)^{2p-1}\|a\|_\infty^p\|a\|_1^{1-p}\}}\,\frac{\int_0^{\delta} h(u)\,du}{\delta^p};$$

(kk) $$\int_0^{\xi} h(u)\,du \le \eta(1+|\xi|^s) \text{ for all } \xi \in \mathbb{R}.$$

Then, for each λ *in*

$$\left]\frac{\delta^p\|a\|_1}{p\|g\|_1 \int_\gamma^\delta h(u)\,du}, \frac{\gamma^p\|a\|_1}{2^{p-1}\max\{1,(\beta-\alpha)^{2p-1}\|a\|_\infty^p\|a\|_1^{1-p}\}p\|g\|_1\int_0^\gamma h(u)\,du}\right[$$

the problem

$$\begin{cases} -\big(|u'|^{p-2}u'\big)' + a(t)|u|^{p-2}u = \lambda g(t)h(u) \text{ in }]\alpha,\beta[\\ u'(\alpha)=u'(\beta)=0, \end{cases}$$

admits at least three generalized solutions.

Proof. Taking into account Remarks 1 and 2, the conclusion follows immediately from Theorem 1, by using $f := gh$ and $\mu := \eta g$. □

Proof of Theorem A. Put

$$\overline{h}(u) := \begin{cases} 0 & \text{if} \quad u < 0 \\ h(u) & \text{if} \quad u \ge 0. \end{cases}$$

Clearly, $\overline{h}$ is a continuous function on $\mathbb{R}$. Moreover, taking into account that

$$\lim_{\gamma\to 0^+} \frac{\max_{|\xi|\le\gamma}\int_0^\xi \overline{h}(u)\,du}{\gamma^2} = 0,$$

it is enough to pick $\gamma > 0$ such that

$$\frac{\max_{|\xi|\le\gamma}\int_0^\xi \overline{h}(u)\,du}{\gamma^2} < \frac{1}{3}\frac{\int_0^\delta \overline{h}(u)\,du}{\delta^2},$$

$$\frac{\gamma^2}{4\|g\|_1\max_{|\xi|\le\gamma}\int_0^\xi \overline{h}(u)\,du} > 1$$

and

$$\max_{|\xi|\le\gamma}\int_0^\xi \overline{h}(u)\,du < \int_0^\delta \overline{h}(u)\,du - \frac{\delta^2}{2\|g\|_1}$$

so that, taking into account Remark 1, using $f := g\overline{h}$, and observing that $1 \in]\lambda', \lambda''[$, Theorem 1 ensures that the problem

$$\text{(P2)} \qquad \begin{cases} -u'' + u = g(t)\overline{h}(u) \\ u'(0) = u'(1) = 0, \end{cases}$$

admits at least two non-null classical solutions. We claim that these solutions are non-negative. In fact, arguing by a contradiction, if one of them, say u_1, is negative at one point of $[0, 1]$, there exists an interval $]a, b[\subseteq [0, 1]$ such that $-u_1''(t) + u_1(t) = 0$ for every $t \in]a, b[$ and, further, it must be true one of the following conditions: $u_1(a) = u_1(b) = 0$ (if $0 < a < b < 1$), $u_1'(0) = u_1(b) = 0$ (if $0 = a < b < 1$), $u_1(a) = u_1'(1) = 0$ (if $0 < a < b = 1$), $u_1'(0) = u_1'(1) = 0$ (if $0 = a < b = 1$). Therefore, $u_1(t) = 0$ for every $t \in [a, b]$ and this is a contradiction, so our claim is proved. Finally, the conclusion follows taking into account that the nonnegative solutions of (P2) are also solutions of (P1). □

Remark 3. If in Theorem A we also assume that

$$\lim_{u\to 0^-} \frac{h(u)}{u} = 0 \quad \text{and} \quad \int_{-\delta'}^{0} h(u)\, du < 0 \quad \text{for some} \quad \delta' > 0$$

then, for every non-negative function $g \in C^0([0, 1])$ such that

$$\|g\|_1 > \max\left\{ \frac{\delta^2}{2\int_0^{\delta} h(u)\, du}, \frac{\delta'^2}{2\int_0^{-\delta'} h(u)\, du} \right\}$$

problem (P1) admits also two non-positive solutions. It is enough to apply the same Theorem A to the function $h^*(u) := -h(-u)$.

Example 2. Let $h : \mathbb{R} \to \mathbb{R}$ the function defined as follows

$$h(u) := \begin{cases} 0 & \text{if} \quad u \le 0 \\ 3u^2 & \text{if} \quad 0 < u < 1 \\ 3 & \text{if} \quad u \ge 1. \end{cases}$$

Therefore, owing to Theorem A, for every non-negative function $g \in C^0([0, 1])$ such that $\|g\|_1 > \frac{1}{2}$, the problem

$$\begin{cases} -u'' + u = g(t)h(u) \\ u'(0) = u'(1) = 0, \end{cases}$$

admits at least two non-negative and non-trivial classical solutions.
On the other hand, the function $H(\xi) := \int_0^{\xi} h(u)\, du$ does not satisfy the assumption 1. of Theorem 2.1 of [8].

REFERENCES

[1] G. Anello - G. Cordaro, *An existence theorem for the Neumann problem involving the p-Laplacian,* J. Convex Anal., 10 (2003), pp. 185–198.

[2] G. Anello - G. Cordaro, *Existence of solutions of the Neumann problem for a class of equations involving the p-Laplacian via a variational principle of Ricceri,* Arch. Math. (Basel), 79 (2002), pp. 274–287.

[3] D. Averna - G. Bonanno, *A three critical points theorem and its applications to the ordinary Dirichlet problem,* Topol. Methods Nonlinear Anal., 22 (2003), pp. 93–104.

[4] D. Averna - G. Bonanno, *Three solutions for a quasilinear two point boundary value problem involving the one-dimensional p-Laplacian,* Proc. Edinb. Math. Soc., 47 (2004), pp. 257–270.

[5] P.A. Binding - P. Drábek - Y.X. Huang, *On Neumann boundary value problems for some quasilinear elliptic equations,* Electron. J. Differential Equations, 1997 n. 5 (1997), pp. 1–11.

[6] G. Bonanno, *Multiple solutions for a Neumann boundary value problem,* J. Nonlinear Convex Anal., 4 (2003), pp. 287–290.

[7] G. Bonanno - P. Candito, *Three solutions to a Neumann problem for elliptic equations involving the p-Laplacian,* Arch. Math. (Basel), 80 (2003), pp. 424–429.

[8] F. Faraci, *Multiplicity results for a Neumann problem involving the p-Laplacian,* J. Math. Anal. Appl., 277 (2003), pp. 180–189.

[9] A.R. Miciano - R. Shivaji, *Multiple positive solutions for a class of semipositone Neumann two point boundary value problems,* J. Math. Anal. Appl., 178 (1993), pp. 102–115.

[10] B. Ricceri, *A general variational principle and some of its applications,* J. Comput. Appl. Math., 113 (2000), pp. 401–410.

[11] E. Zeidler, *Nonlinear functional analysis and its applications,* Vol. II/B. Berlin-Heidelberg-New York, 1990.

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