THREE SOLUTIONS FOR A NEUMANN BOUNDARY VALUE PROBLEM INVOLVING THE $p$-LAPLACIAN

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In this note we prove the existence of an open interval $]\lambda', \lambda''[\,$ for each $\lambda$ of which a Neumann boundary value problem involving the $p$-Laplacian and depending on $\lambda$ admits at least three solutions. The result is based on a recent three critical points theorem.

1. Introduction.

Let $\Omega$ be a nonempty bounded open set of the real Euclidean space $\mathbb{R}^n$, with a boundary of class $C^1$, $a \in L^\infty(\Omega)$, with $\text{essinf}_\Omega a > 0$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ a function, and $p \geq 2$.

Let us consider the following problem

\[
\begin{aligned}
-\Delta_p u + a(x)|u|^{p-2}u &= \lambda f(x, u) & \text{in } \Omega \\
\frac{\partial u}{\partial \nu} &= 0 & \text{on } \partial \Omega,
\end{aligned}
\]

where $\Delta_p = \text{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $\lambda \in ]0, +\infty[$, and $\nu$ is the outer unit normal to $\partial \Omega$.

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A weak solution to problem (P) is a function \( u \in W^{1,p}(\Omega) \) such that
\[
\int_{\Omega} |\nabla u(x)|^{p-2}\nabla u(x) \nabla v(x) \, dx + \int_{\Omega} a(x)|u(x)|^{p-2}u(x)v(x) \, dx \\
- \lambda \int_{\Omega} f(x, u(x))v(x) \, dx = 0, \quad \forall \ v \in W^{1,p}(\Omega).
\]

Problems of the above type were widely studied in these latest years and we refer to [1], [2], [5], [7], [8], (see also [6] and [9], for the case \( n = 1 \) and \( p = 2 \)) and the references therein for more details. In particular, in [7] the authors obtained the existence of an open interval \( \Lambda \subseteq [0, \infty[ \) such that for each \( \lambda \in \Lambda \) problem (P) admits at least three weak solutions which are uniformly bounded with respect to \( \lambda \), without, however, establishing where \( \Lambda \) is located; while in [8], under a different set of assumptions, the existence of three weak solutions to (P) for \( \lambda = 1 \) was proved.

The aim of this note is to establish the existence of a precise open interval \( ]\lambda', \lambda''[ \), \( 0 < \lambda' < \lambda'' \leq +\infty \), for each \( \lambda \) of which problem (P) admits at least three weak solutions. Our main result is Theorem 1 and, as a way of example, we present, here, a particular case.

**Theorem A.** Let \( h : \mathbb{R} \to \mathbb{R} \) be a bounded continuous function such that
\[
\lim_{u \to 0^+} \frac{h(u)}{u} = 0 \quad \text{and} \quad \int_0^\delta h(u) \, du > 0 \quad \text{for some} \quad \delta > 0.
\]

Then, for every non-negative function \( g \in C^0([0, 1]) \) such that
\[
\|g\|_1 > \frac{\delta^2}{2\int_0^\delta h(u) \, du}
\]
the problem
\[
\begin{cases}
-u'' + u = g(t)h(u) \\
u'(0) = u'(1) = 0,
\end{cases}
\]
(P1)

admits at least two non-negative and non-trivial classical solutions.

Example 2 at the end of the paper shows a Neumann problem that, owing to our results, admits three solutions, but to which Theorem 2.1 of [8] cannot be applied.

Our results are based on the following recent three critical points theorem obtained in [3].
Theorem B. (Theorem B of [3]) Let $X$ be a real reflexive Banach space, $\Phi : X \to \mathbb{R}$ a continuously Gateaux differentiable and sequentially weakly lower semicontinuous functional whose Gateaux derivative admits a continuous inverse on $X^*$, $\Psi : X \to \mathbb{R}$ a continuously Gateaux differentiable functional whose Gateaux derivative is compact. Assume that:

(i) $\lim_{\|u\| \to +\infty} (\Phi(u) + \lambda \Psi(u)) = +\infty$ for all $\lambda \in [0, +\infty[$;

(ii) there is $r \in \mathbb{R}$ such that:

$$\inf_X \Phi < r,$$

and

$$\varphi_1(r) < \varphi_2(r),$$

where

$$\varphi_1(r) := \inf_{u \in \Phi^{-1}(]-\infty,r[)} \frac{\Psi(u) - \inf_{\Phi^{-1}(]-\infty,r[)} \Psi}{r - \Phi(u)},$$

$$\varphi_2(r) := \inf_{u \in \Phi^{-1}(]-\infty,r[)} \sup_{v \in \Phi^{-1}(]r,\infty[)} \frac{\Psi(u) - \Psi(v)}{\Phi(v) - \Phi(u)},$$

and $\Phi^{-1}(]-\infty,r[)^w$ is the closure of $\Phi^{-1}(]-\infty,r[)$ in the weak topology.

Then, for each $\lambda \in \left[\frac{1}{\varphi_2(r)}, \frac{1}{\varphi_1(r)}\right]$ the functional $\Phi + \lambda \Psi$ has at least three critical points in $X$.

Other applications of Theorem B can be found in [3] and [4].

In order to apply Theorem B to our problem, let $X$ be the space $W^{1,p}(\Omega)$ equipped with the norm

$$\|u\| := \left( \int_\Omega |\nabla u(x)|^p \, dx + \int_\Omega a(x)|u(x)|^p \, dx \right)^{\frac{1}{p}},$$

which is equivalent to the usual one, while on the space $C^0(\overline{\Omega})$ we consider the norm $\|u\|_\infty := \sup_{u \in \overline{\Omega}} |u(x)|$.

If $p > n$, $X$ is compactly embedded in $C^0(\overline{\Omega})$, so that

$$c := \sup_{u \in X \setminus \{0\}} \frac{\|u\|_\infty}{\|u\|} < +\infty.$$  

Clearly, $c^p \|a\|_1 \geq 1$, where $\|a\|_1 := \int_\Omega |a(x)| \, dx$.

For other basic notations and definitions we refer to [11].
2. Results.

Our main result is the following

**Theorem 1.** Let $p > n$ and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that, for every $\rho > 0$, $\sup_{|u| \leq \rho} |f(\cdot, u)| \in L^1(\Omega)$. Put

$$F(x, \xi) := \int_0^\xi f(x, u) \, du \quad \text{for every } (x, \xi) \in \Omega \times \mathbb{R},$$

and assume that there exist three positive constants $\gamma$, $\delta$, and $s$, with $\gamma < \delta$ and $s < p$, and a function $\mu \in L^1(\Omega)$ such that:

1. \[ \frac{\int_{|\xi| \leq \gamma} F(x, \xi) \, dx}{\gamma^p} < \frac{1}{1 + c\rho \|a\|_1} \frac{\int_{\Omega} F(x, \delta) \, dx}{\delta^p} ; \]
2. \[ F(x, \xi) \leq \mu(x)(1 + |\xi|^s) \quad \text{for all } (x, \xi) \in \Omega \times \mathbb{R} . \]

Then, setting

$$\lambda' := \frac{\|a\|_1 \delta^p}{p \left( \int_{\Omega} F(x, \delta) \, dx - \int_{\Omega} \sup_{|\xi| \leq \gamma} F(x, \xi) \, dx \right)}$$

and

$$\lambda'' := \frac{\gamma^p}{c^p \rho \int_{|\xi| \leq \gamma} F(x, \xi) \, dx},$$

for each $\lambda \in [\lambda', \lambda'']$ problem (P) admits at least three weak solutions.

**Proof.** For each $u \in X$, put

$$\Phi(u) := \frac{1}{p} \|u\|^p$$

and

$$\Psi(u) := - \int_{\Omega} F(x, u(x)) \, dx .$$

Since $p > n$, $X$ is compactly embedded in $C^0(\overline{\Omega})$ and it is well known that $\Phi$ and $\Psi$ are (well defined and) continuously Gâteaux differentiable functionals with

$$\Phi'(u)(v) = \int_{\Omega} \left( |\nabla u(x)|^{p-2} \nabla u(x) \nabla v(x) + a(x)|u(x)|^{p-2} u(x) v(x) \right) \, dx$$
and
\[ \Psi'(u)(v) = - \int_{\Omega} f(x, u(x))v(x) \, dt \]
for every \( u, v \in X \), as well as \( \Psi' \) is compact.

Furthermore, by Proposition 25.20 (i) of [11], \( \Phi \) is sequentially weakly lower semicontinuous, while Proposition 1 of [7] ensures that \( \Phi' \) admits a continuous inverse on \( X^* \).

Hypothesis (i) of Theorem B follows in a simple way thanks to (jj).

In order to prove (ii) of Theorem B, put \( r := \frac{1}{p} \left( \frac{\gamma}{\alpha} \right)^p \).

From hypothesis (j), we get
\[
\frac{\int_{\Omega} \sup_{|\xi| \leq \gamma} F(x, \xi) \, dx}{c^p \|a\|_1 \gamma^p} < \frac{1}{c^p \|a\|_1 (1 + c^p \|a\|_1)} \frac{\int_{\Omega} F(x, \delta) \, dx}{\delta^p} = \left( \frac{1}{c^p \|a\|_1} - \frac{1}{1 + c^p \|a\|_1} \right) \frac{\int_{\Omega} F(x, \delta) \, dx}{\delta^p},
\]
then
\[
\frac{1}{1 + c^p \|a\|_1} \frac{\int_{\Omega} F(x, \delta) \, dx}{\delta^p} + \frac{\int_{\Omega} \sup_{|\xi| \leq \gamma} F(x, \xi) \, dx}{c^p \|a\|_1 \gamma^p} < \frac{1}{c^p \|a\|_1} \frac{\int_{\Omega} F(x, \delta) \, dx}{\delta^p},
\]
thus, being \( \gamma < \delta \), we have
\[
\frac{1}{1 + c^p \|a\|_1} \frac{\int_{\Omega} F(x, \delta) \, dx}{\delta^p} < \frac{\int_{\Omega} F(x, \delta) \, dx - \int_{\Omega} \sup_{|\xi| \leq \gamma} F(x, \xi) \, dx}{c^p \|a\|_1 \delta^p},
\]
hence, using again (j), we get
\[
\frac{\int_{\Omega} \sup_{|\xi| \leq \gamma} F(x, \xi) \, dx}{\gamma^p} < \frac{\int_{\Omega} F(x, \delta) \, dx - \int_{\Omega} \sup_{|\xi| \leq \gamma} F(x, \xi) \, dx}{c^p \|a\|_1 \delta^p},
\]
from which, multiplying by \( c^p \gamma^p \), we obtain
\[
\frac{c^p p \int_{\Omega} \sup_{|\xi| \leq \gamma} F(x, \xi) \, dx}{\gamma^p} < \frac{\int_{\Omega} F(x, \delta) \, dx - \int_{\Omega} \sup_{|\xi| \leq \gamma} F(x, \xi) \, dx}{\|a\|_1 \delta^p}. \tag{2}
\]
We claim that:

\[(C1) \quad \varphi_1(r) \leq \sup_{|\xi| \leq \gamma} \int_{\Omega} F(x, \xi) \, dx \]

and

\[(C2) \quad \varphi_2(r) \geq p \frac{\int_{\Omega} F(x, \delta) \, dx - \int_{\Omega} \sup_{|\xi| \leq \gamma} F(x, \xi) \, dx}{\|a\|_1 \delta^p},\]

from which (ii) of Theorem B follows.

In fact, taking into account that the function identically 0 obviously belongs to \(\Phi_1(\mathbb{R}, r]\), and that \(\Psi(0) = 0\), we get

\[\varphi_1(r) \leq \sup_{\Phi_1(\mathbb{R}, r]} \int_{\Omega} F(\xi, u(x)) \, dx / r ,\]

and, since \(\Phi_1(\mathbb{R}, r]\) = \(\Phi_1(\mathbb{R}, r]\), we have

\[\frac{\sup_{\Phi_1(\mathbb{R}, r]} \int_{\Omega} F(\xi, u(x)) \, dx}{r} = \frac{\sup_{\Phi_1(\mathbb{R}, r]} \int_{\Omega} F(x, u(x)) \, dx}{r} ,\]

thus, taking into account that \(|u(x)| \leq c(pr)^{1/p} = \gamma\), for every \(u \in X\) such that \(\Phi(u) \leq r\) and for each \(x \in \Omega\), we obtain

\[\frac{\sup_{\Phi_1(\mathbb{R}, r]} \int_{\Omega} F(\xi, u(x)) \, dx}{r} \leq \frac{\int_{\Omega} \sup_{|\xi| \leq \gamma} F(x, \xi) \, dx}{r} .\]

So, (C1) follows at once by the definition of \(r\).

Moreover, for each \(v \in X\) such that \(\Phi(v) \geq r\), we have

\[\varphi_2(r) \geq \inf_{u \in \Phi_1(\mathbb{R}, r]} \frac{\int_{\Omega} F(\xi, u(x)) \, dx - \int_{\Omega} F(x, u(x)) \, dx}{\Phi(v) - \Phi(u)} ,\]

thus, from \(|u(x)| \leq c(pr)^{1/p} = \gamma\), for every \(u \in X\) such that \(\Phi(u) < r\) and for each \(x \in \Omega\), we obtain

\[\inf_{u \in \Phi_1(\mathbb{R}, r]} \frac{\int_{\Omega} F(\xi, u(x)) \, dx - \int_{\Omega} F(x, u(x)) \, dx}{\Phi(v) - \Phi(u)} .\]
\[
\inf_{u \in \Phi^{-1}(l-\infty, r]} \frac{\int_{\Omega} F(x, v(x)) \, dx - \int_{\Omega} \sup_{|\xi| \leq \gamma} F(x, \xi) \, dx}{\Phi(v) - \Phi(u)} \geq \frac{\int_{\Omega} F(x, v(x)) \, dx - \int_{\Omega} \sup_{|\xi| \leq \gamma} F(x, \xi) \, dx}{\|v\|^p},
\]
from which, being \(0 < \Phi(v) - \Phi(u) \leq \Phi(v)\) for every \(u \in \Phi^{-1}(l-\infty, r]\), and under further condition
\[(3) \int_{\Omega} F(x, v(x)) \, dx \geq \int_{\Omega} \sup_{|\xi| \leq \gamma} F(x, \xi) \, dx,
\]
we can write
\[
\inf_{u \in \Phi^{-1}(l-\infty, r]} \frac{\int_{\Omega} F(x, v(x)) \, dx - \int_{\Omega} \sup_{|\xi| \leq \gamma} F(x, \xi) \, dx}{\Phi(v) - \Phi(u)} \geq \frac{\int_{\Omega} F(x, v(x)) \, dx - \int_{\Omega} \sup_{|\xi| \leq \gamma} F(x, \xi) \, dx}{\|v\|^p}.
\]
If we put \(v(x) := \delta\), for each \(x \in \Omega\), we have \(\|v\| = \|a\|^{1/p} \delta\), hence, by (1) and \(\gamma < \delta\), we get \(\Phi(v) > r\). Moreover, with this choice of \(v\), (2) ensures (3), thus (C2) is also proved.

Hence the conclusion follows by Theorem B, by observing that
\[
\frac{1}{\varphi_2(r)} \leq \frac{\|a\|_1 \delta^p}{p \left( \int_{\Omega} F(x, \delta) \, dx - \int_{\Omega} \sup_{|\xi| \leq \gamma} F(x, \xi) \, dx \right)}
\]
and
\[
\frac{1}{\varphi_1(r)} \geq \frac{\gamma^p}{c^p \int_{\Omega} \sup_{|\xi| \leq \gamma} F(x, \xi) \, dx}.
\]

\[\square\]

**Remark 1.** In applying Theorem 1, it is enough to know an explicit upper bound for the constant \(c\) defined in (1). In this connection, when \(\Omega\) is a convex set, we have the following estimate (see [1] for more details):

\[(4) \quad c \leq 2^{\frac{n-1}{p}} \max \left\{ \left( \frac{1}{\|a\|_1} \right)^{\frac{1}{p}} \frac{\text{diam}(\Omega)}{n^{\frac{1}{p}}}, \frac{n^{\frac{1}{p}} \left( \frac{p-1}{p} \right) \text{meas}(\Omega)}{n^{\frac{1}{p}} \|a\|_1} \right\}.
\]
Example 1. The problem
\begin{equation*}
\begin{cases}
-\Delta u + \frac{2u}{|x|^2}u = \lambda(x^2 + y^2)
\left[2e^{-u^2}u^{17}(9 - u^2) + 1\right] & \text{in } \Omega \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega,
\end{cases}
\end{equation*}
where \( \Omega \) is the open unit ball in \( \mathbb{R}^2 \), admits at least three weak solutions for each \( \lambda \in ]15 \cdot 10^{-4}, 15 \cdot 10^{-2} [ \).

In fact, choosing \( \gamma := 1, \delta := 2, s := 2 \), and
\begin{equation*}
a(x, y) := \frac{2 + x}{\pi}, f(x, y, u) := (x^2 + y^2)
\left[2e^{-u^2}u^{17}(9 - u^2) + 1\right]
\end{equation*}
for every \( (x, y) \in \Omega \) and every \( u \in \mathbb{R} \), and taking into account (4), it is simple to verify all the hypotheses of Theorem 1, and that \( ]15 \cdot 10^{-4}, 15 \cdot 10^{-2} [ \subset \lambda' , \lambda'' [ \).

Remark 2. In the ordinary case, whenever \( \Omega := ]\alpha, \beta [ \) is a bounded open interval in \( \mathbb{R} \), a function \( u : ]\alpha, \beta [ \to \mathbb{R} \) is said a generalized solution to problem
\begin{equation*}
(PO) \quad \begin{cases}
- ( |u'|^{p-2}u')' + a(t)|u|^{p-2}u = \lambda f(t, u) & \text{in } ]\alpha, \beta [ \\
u'(\alpha) = u'(\beta) = 0,
\end{cases}
\end{equation*}
if \( u \in C^1(]\alpha, \beta [) \), \( |u'|^{p-2}u' \in AC(]\alpha, \beta [) \), \( u'(\alpha) = u'(\beta) = 0 \) and
\begin{equation*}
- ( |u'(t)|^{p-2}u'(t))' + a(t)|u(t)|^{p-2}u(t) = \lambda f(t, u(t))
\end{equation*}
for almost every \( t \in ]\alpha, \beta [ \).

Using standard arguments, and taking into account that the correspondence \( u \mapsto |u|^{p-2}u \) is an homeomorphism in \( \mathbb{R} \), one can show that weak solutions coincides with generalized ones, provided \( a \in L^\infty(]\alpha, \beta [) \) and \( f \) is as in Theorem 1.

We point out the following simple consequence of Theorem 1, then we give the proof of Theorem A and finally we present an easy example of application.

Theorem 2. Let \( p \geq 2 \), and let \( g \in L^1(]\alpha, \beta [) \), \( h \in C^0(\mathbb{R}) \) be two non-negative functions, with \( \|g\|_1 > 0 \). Assume that there exists four positive constants \( \gamma, \delta, \eta \) and \( s \), with \( \gamma < \delta \) and \( s < p \), such that:
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\[
\begin{aligned}
\gamma \int_0^\delta h(u) \, du &< \frac{1}{1 + 2\rho \max \{1, (\beta - \alpha)^2 \rho - 1\} \|a\|_\infty \|a\|_1 \gamma^p} \int_0^\gamma h(u) \, du \quad \delta \rho ; \\
\gamma \int_0^\delta h(u) \, du &< \frac{1}{1 + 2\rho \max \{1, (\beta - \alpha)^2 \rho - 1\} \|a\|_\infty \|a\|_1 \gamma^p} \int_0^\gamma h(u) \, du \\
\end{aligned}
\]

(k) \( \int_0^{\xi} h(u) \, du \leq \eta (1 + |\xi|^{\rho}) \) for all \( \xi \in \mathbb{R} \).

Then, for each \( \lambda \) in

\[
\left[ \frac{\delta^p \|a\|_1}{p \|g\|_1} \int_\gamma^\delta h(u) \, du \right] < 2\rho \max \{1, (\beta - \alpha)^2 \rho - 1\} \|a\|_\infty \|a\|_1 \gamma^p \int_0^{\xi} h(u) \, du \]

the problem

\[
\left\{ -\left(|u'|^{\rho - 2} u'\right)' + a(t)|u|^{\rho - 2} u = \lambda g(t) h(u) \text{ in } ]a, \beta[ \\
u'(a) = u'(\beta) = 0, \right. \]

admits at least three generalized solutions.

\textbf{Proof.} Taking into account Remarks 1 and 2, the conclusion follows immediately from Theorem 1, by using \( f := gh \) and \( \mu := \eta g \). \( \square \)

\textbf{Proof of Theorem A.} Put

\[
\overline{h}(u) := \begin{cases} 
0 & \text{if } u < 0 \\
h(u) & \text{if } u \geq 0.
\end{cases}
\]

Clearly, \( \overline{h} \) is a continuous function on \( \mathbb{R} \). Moreover, taking into account that

\[
\lim_{\gamma \to 0^+} \frac{\max_{|\xi| \leq Y} \int_0^\xi \overline{h}(u) \, du}{\gamma^2} = 0,
\]

it is enough to pick \( \gamma > 0 \) such that

\[
\frac{\max_{|\xi| \leq Y} \int_0^\xi \overline{h}(u) \, du}{\gamma^2} < \frac{1}{3} \frac{\int_0^\delta \overline{h}(u) \, du}{\delta^2},
\]

\[
\frac{\gamma^2}{4 \|g\|_1 \max_{|\xi| \leq Y} \int_0^\xi \overline{h}(u) \, du} > 1
\]

and

\[
\max_{|\xi| \leq Y} \int_0^\xi \overline{h}(u) \, du < \int_0^\delta \overline{h}(u) \, du - \frac{\delta^2}{2 \|g\|_1}
\]
so that, taking into account Remark 1, using $f := gh$, and observing that $1 \in [\lambda', \lambda'']$, Theorem 1 ensures that the problem

\begin{equation}
(P2) \quad \begin{cases}
-u'' + u = g(t)h(u) \\
u'(0) = u'(1) = 0,
\end{cases}
\end{equation}

admits at least two non-null classical solutions. We claim that these solutions are non-negative. In fact, arguing by contradiction, if one of them, say $u_1$, is negative at one point of $[0, 1]$, there exists an interval $[a, b] \subseteq [0, 1]$ such that $-u_1''(t) + u_1(t) = 0$ for every $t \in [a, b]$ and, further, it must be true one of the following conditions:

- $u_1(a) = u_1(b) = 0$ (if $0 < a < b < 1$),
- $u_1'(0) = u_1'(1) = 0$ (if $0 = a < b = 1$),
- $u_1'(0) = u_1'(1) = 0$ (if $0 = a < b = 1$).

Therefore, $u_1(t) = 0$ for every $t \in [a, b]$ and this is a contradiction, so our claim is proved. Finally, the conclusion follows taking into account that the nonnegative solutions of (P2) are also solutions of (P1).

**Remark 3.** If in Theorem A we also assume that

$$\lim_{u \to 0^+} \frac{h(u)}{u} = 0 \quad \text{and} \quad \int_{-\delta}^0 h(u) \, du < 0 \quad \text{for some} \quad \delta > 0,$$

then, for every non-negative function $g \in C^0([0, 1])$ such that

$$\|g\|_1 > \max \left\{ \frac{\delta^2}{2 \int_0^\delta h(u) \, du}, \frac{\delta^2}{2 \int_{-\delta}^0 h(u) \, du} \right\},$$

problem (P1) admits also two non-positive solutions. It is enough to apply the same Theorem A to the function $h^*(u) := -h(-u)$.

**Example 2.** Let $h : \mathbb{R} \to \mathbb{R}$ the function defined as follows

$$h(u) := \begin{cases} 
0 & \text{if} \quad u \leq 0 \\
3u^2 & \text{if} \quad 0 < u < 1 \\
3 & \text{if} \quad u \geq 1.
\end{cases}$$

Therefore, owing to Theorem A, for every non-negative function $g \in C^0([0, 1])$ such that $\|g\|_1 > \frac{1}{2}$, the problem

\begin{equation}
\begin{cases}
-u'' + u = g(t)h(u) \\
u'(0) = u'(1) = 0,
\end{cases}
\end{equation}

admits at least two non-negative and non-trivial classical solutions.

On the other hand, the function $H(\xi) := \int_{-\xi}^\xi h(u) \, du$ does not satisfy the assumption 1. of Theorem 2.1 of [8].
REFERENCES


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