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OPEN DIFFERENTIABLE MAPPINGS

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It is a well-known result that a $\mathcal{C}^1$-mapping defined on an open subset Ω of a Banach space E with values in a Banach space F whose differential is everywhere onto is an open mapping from Ω onto an open subset of F. We prove that this result still holds if E is finite-dimensional and both the critical set and the set of critical values have small topological dimensions. The restriction on the set of critical values can be removed if E and F have same dimension.

1. Introduction

Let Ω be an open subset of the Banach space E, F be a Banach space and $\mathcal{L}(E,F)$ denote the Banach space of continuous linear mappings from E to F. Let $f : \Omega \to F$ be a $\mathcal{C}^1$ function. A point $a \in \Omega$ is said to be *critical* if the differential $f'(a) \in \mathcal{L}(E,F)$ is not onto. We call *critical set* of f the set Γ of critical points of f, and a point $b \in F$ is said to be a *critical value* of f if there is some critical point $a \in \Omega$ such that $f(a) = b$. It is a well-known fact that the set $\{u \in \mathcal{L}(E,F) : u(E) = F\}$ of continuous onto linear mappings is open in $\mathcal{L}(E,F)$; so the set Γ is closed in Ω.

If V is a connected open set in $\mathbb{R}^n$ we say that a closed set $Z \subset \mathbb{R}^n$ *cuts* V if the open set $V \setminus Z$ is not connected.

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2. The main theorem

Theorem 2.1. *Let Ω be an open subset of $\mathbb{R}^n$ and $\varphi : \Omega \to \mathbb{R}^p$ ($2 \le p \le n$) a $\mathcal{C}^1$ function. Assume that the set Γ of critical points of φ is 0-dimensional and that the set $\varphi(\Gamma)$ of critical values of φ has dimension $\le p-2$. Then φ is an open mapping.*

Remark first that such a result cannot hold when $n = p = 1$ without the hypothesis on the set of critical values. The $\mathcal{C}^\infty$ function $q : x \mapsto x^2$ has a unique critical point 0, and a unique critical value 0, but is not open : indeed $q(\mathbb{R}) = \mathbb{R}^+$ is not a neighborhood of $q(0) = 0$ in $\mathbb{R}$. Nevertheless for $p = 1$ if the topological dimension of $\varphi(\Gamma)$ is -1, the set $\varphi(\Gamma)$ has to be empty, so φ has no critical point and the above statement reduces to the standard theorem.

Up to replacing Ω by an open subset of Ω, it is enough to prove that $\varphi(\Omega)$ is open in $\mathbb{R}^p$. The classical "Local Inversion Theorem" (or a variant of it) shows that the set $\varphi(\Omega \setminus \Gamma)$ of regular values is open in $\mathbb{R}^p$. So we have only to show that $\varphi(\Omega)$ is a neighborhood of every point $b \in \varphi(\Omega)$ such that $\Omega \cap \varphi^{-1}(b) \subset \Gamma$.

Lemma 2.2. *Let $a \in \Gamma \cap \Omega$, $b = \varphi(a)$ and $r > 0$. Assume that $\Omega \cap \varphi^{-1}(b) \subset \Gamma$. Then there exists a connected open neighborhood V of a, contained in the ball $B(a,r)$ and some $\rho > 0$ such that $\partial V \cap \Gamma = \emptyset$ and $\|\varphi(z) - b\| \ge 3\rho$ for all z in the boundary ∂V of V.*

Proof. Without loss of generality we can and do assume that r is small enough for the closed ball $\tilde{B}(a,r)$ being contained in Ω. Since Γ is a closed 0-dimensional subset of Ω there is a compact open neighborhood K of a in $\Gamma \cap B(a,r)$. Then K and $H = (\Gamma \setminus K) \cup (\mathbb{R}^n \setminus B(a,r))$ are two disjoint closed subsets of $\mathbb{R}^n$. The open set $V_0 = \{x \in \mathbb{R}^n : d(x,K) < d(x,H)\}$ is a neighborhood of K in $\mathbb{R}^n$ contained in $B(a,r)$ and for every $x \in \partial V_0$ we have $d(x,K) = d(x,H)$, hence $d(x,K) = d(x,H) > 0$ since $K \cap H = \emptyset$. Replacing V_0 by the connected component V of a in V_0, we get a connected open set V whose boundary is contained in ∂V_0. In particular $\partial V \cap \Gamma \subset \partial V_0 \cap \Gamma = \emptyset$, and $b \notin \varphi(\partial V)$. Since ∂V is compact in Ω, $\varphi(\partial V)$ is compact in $\mathbb{R}^p$ and because $b \notin \varphi(\partial V)$, we have $\rho = \frac{1}{3} \inf_{w \in \varphi(\partial V)} \|w - b\| > 0$. □

Lemma 2.3. *Let $\Gamma_0 = \{y \in \varphi(\Gamma \cap \overline{V}) : \varphi^{-1}(y) \cap V \subset \Gamma\}$. Then Γ_0 is a closed set which does not cut the open ball $B(b,\rho)$.*

Proof. Since $\overline{V}$ is compact in Ω and Γ is closed, $\varphi(\Gamma \cap \overline{V})$ is compact. Let (y_m) be a sequence in Γ_0 which converges to some $y^* \notin \Gamma_0$: there exists $x^* \in V \setminus \Gamma$ such that $\varphi(x^*) = y^*$, and $y^* \in \varphi(\Gamma \cap \overline{V})$ since this last set is compact.

Since $x^* \notin \Gamma$, φ is open on a neighborhood W of x^* and $\varphi(V \cap W \setminus \Gamma)$ is a neighborhood of y^*. Thus there exists at least one index k such that $y_k \in$

$\varphi(V \cap W \setminus \Gamma)$ and then some $x_k \in V \cap W \setminus \Gamma$ such that $\varphi(x_k) = y_k$, in contradiction with $y_k \in \Gamma_0$. This shows that Γ_0 is closed. And since $\Gamma \cap \partial V = \emptyset$ we also have $\Gamma_0 = \{y \in \varphi(\Gamma \cap \overline{V}) : \varphi^{-1}(y) \cap \overline{V} \subset \Gamma\}$.

Denote by U_0 the ball $B(b,\rho)$. By hypothesis $\varphi(\Gamma)$ has dimension $\leq p-2$, and $\Gamma_0 \subset \varphi(\Gamma)$. Then $\Gamma_0 \cap U_0$ is a closed subset of U_0 contained in $\varphi(\Gamma)$, hence a closed subset of U_0 of dimension $\leq p-2$.

A closed set X of dimension $\leq p-2$ cannot cut a ball B of $\mathbb{R}^p$: if X cuts such a ball, by homeomorphism between B and $\mathbb{R}^p$ there would be some closed set X' in $\mathbb{R}^p$ having dimension $\leq p-2$ and cutting $\mathbb{R}^p$, then by inversion with respect to some point in $\mathbb{R}^p \setminus X'$ we could find a bounded open set in $\mathbb{R}^p$ (hence also a basis of the topology) whose boundary has dimension $\leq p-2$. So $\mathbb{R}^p$ would have topological dimension $\leq p-1$. Thus the open set $U_0 \setminus \Gamma_0$ is connected. □

Lemma 2.4. *Let a be a critical point of φ, $b = \varphi(a)$ and $r > 0$ such that the closed ball $\tilde{B}(a,r)$ is contained in Ω. Assume that $b \notin \varphi(B(a,r) \setminus \Gamma)$. Then there exists $\rho > 0$ such that $\varphi(B(a,r)) \supset B(b,\rho)$.*

Proof. By Lemma 2.2 we can find some neighborhood V of a contained in $B(a,r)$, hence relatively compact in Ω, and $\rho > 0$ such that $\|\varphi(z) - b\| \geq 3\rho$ for all $z \in \partial V$. If w satisfies $\|w - b\| < \rho$ we then have $\|\varphi(z) - w\| > 2\rho > \|b - w\|$ for all $z \in \partial V$. The $\mathcal{C}^1$ function ψ_w defined by $\psi_w(z) = \|\varphi(z) - w\|^2$ attains its minimum over the compact set $\overline{V}$ at a point z_0 which cannot belong to ∂V since $\psi_w(z_0) \leq \psi_w(a) < \rho^2 \leq \inf_{z \in \partial V} \varphi_w(z)$. It follows that $z_0 \in V$ and that the differential of ψ_w vanishes at z_0. Then $\langle \psi'_w(z_0), h \rangle = 2\langle \varphi(z_0) - w, \varphi'(z_0).h \rangle$ for all $h \in \mathbb{R}^n$. And if moreover $\varphi'(z_0)$ is onto, there must exist $h^* \in \mathbb{R}^n$ such that $\varphi'(z_0).h^* = \varphi(z_0) - w$, hence $0 = \langle \psi'_w(z_0), h^* \rangle = 2\,\|\varphi(z_0) - w\|^2$ and $\varphi(z_0) = w$.

This implies that either $\varphi(z_0) = w$ or $z_0 \in \Gamma$. It follows from Lemma 2.3 that the open set $U = B(b,\rho) \setminus \Gamma_0$ is connected. By compactness of $\overline{V}$, the set $F = \varphi(\overline{V}) \cap U$ is closed in U. The open set $V_1 = V \cap \varphi^{-1}(B(b,\rho))$ is a neighborhood of a, and since Γ has empty interior there exists some $z^* \in V_1 \setminus \Gamma$, hence $\varphi(z^*) \in B(b,\rho) \cap \varphi(\overline{V}) \setminus \Gamma_0 = F$. This shows that F is a nonempty closed subset of the connected open set U.

We now prove that F is open in U and this will imply that $F = U$ hence that $U \subset \varphi(\overline{V})$, whence $B(b,\rho) \subset \overline{U} \subset \varphi(\overline{V}) \subset \varphi(B(a,r))$, since $\varphi(\overline{V})$ is compact. So let $c \in F \subset B(b,\rho) \setminus \Gamma_0$. Then $d(c, \varphi(\partial V)) > \rho$ and $c \notin \varphi(\partial V)$. Since $c \in U$ and U is open, there exists $\varepsilon \in]0,\rho[$ such that $d(c, \mathbb{R}^p \setminus U) > 2\varepsilon$. Then if $w \in B(c,\varepsilon)$, and if ψ_w attains its minimum over $\overline{V}$ at z_0, we have $d(w, \mathbb{R}^p \setminus U) \geq d(c, \mathbb{R}^p \setminus U) - \|c - w\| > \varepsilon$ and

$$\|w - \varphi(z_0)\|^2 = \psi_w(z_0) \leq \psi_w(c) < \varepsilon^2 < d(w, \mathbb{R}^p \setminus U)^2$$

which shows that $\varphi(z_0) \in U = B(b,\rho) \setminus \Gamma_0$. Thus there exists $z_1 \in V \setminus \Gamma$ such that $\varphi(z_1) = \varphi(z_0)$, hence $\psi_w(z_1) = \psi_w(z_0)$, and φ_w attains its minimum over

$\overline{V}$ also at z_1. As above $\varphi'_w(z_1)$ is onto and we deduce that $w = \varphi(z_1)$, that $w \in U \subset \varphi(\overline{V}) \setminus \varphi(\partial V)$, hence that $z_1 \in V$ and that $w \in F$. Thus $F \supset B(c,\varepsilon)$, and F is open in U. □

Remark 2.5. It appears quite clearly from the above proof that what is really used is not that the set $\varphi(\Gamma)$ of critical values has dimension $\leq p-2$ but that the (smaller) set of "hypercritical values" of φ has dimension $\leq p-2$, where hypercritical points of φ are defined as points $a \in \Omega$ for which exists a neighborhood V_a such that $\varphi^{-1}(\varphi(a)) \cap V_a \subset \Gamma$ and hypercritical values as images under φ of hypercritical points.

Proof of Theorem 2.1.

We have to prove that $\varphi(\Omega)$ is a neighborhood of each point $b \in \varphi(\Omega)$ such that $\Omega \cap \varphi^{-1}(b) \subset \Gamma$, what is the result of Lemma 2.4. □

Corollary 2.6. *Let Ω be an open subset of $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^p$ ($2 \leq p \leq n$) be a $\mathcal{C}^1$ function. If the critical set of f is countable, then f is open.*

Proof. If Γ is countable it has dimension 0. And the set $f(\Gamma)$ of critical values is countable hence 0-dimensional too. □

3. The case $n = p$

We now show that when $n = p$, the hypothesis on the dimension of the set of critical values can be removed. A bounded open set V will be said to *have a piecewise-$\mathcal{C}^1$ boundary* if its boundary is the union of finitely many piecewise-$\mathcal{C}^1$ hypersurfaces. For example a finite union of cubes in $\mathbb{R}^n$ has a piecewise-$\mathcal{C}^1$ boundary.

Lemma 3.1. *Let Ω be an open subset of $\mathbb{R}^n$, $\varphi : z \mapsto (X_1(z), X_2(z), \ldots, X_n(z))$ be a $\mathcal{C}^1$ mapping from Ω to $\mathbb{R}^n$, J_φ its jacobian determinant and V be a relatively compact open subset of Ω with piecewise-$\mathcal{C}^1$ boundary. Then*

$$\int_{\partial V} X_1\, dX_2 \wedge dX_3 \wedge \cdots \wedge dX_n(z) = \int_V J_\varphi(z)\, dz$$

Proof. Let $\rho : \mathbb{R}^n \to \mathbb{R}$ be a non-negative $\mathcal{C}^\infty$ function with compact support such that $\iint \rho(x,y)\, dx\, dy = 1$ and φ_k the "regularized by convolution" function $\varphi * \rho_k$ where $\rho_k(z) = k^n \rho(k.z)$ for $z \in \mathbb{R}^n$. Then the $\mathcal{C}^\infty$ functions (φ_k) are defined on a neighborhood of $\overline{V}$ for k large enough and this sequence converges in $\mathcal{C}^1$-norm to φ.

For proving the above equality for the function φ it is enough to prove it for the functions φ_k since both members depend continuously on φ in $\mathcal{C}^1$ topology.

So we can assume φ to be $\mathcal{C}^\infty$ and apply Stokes formula to the $(n-1)$-form $\omega = X_1\, dX_2 \wedge dX_3 \wedge \cdots \wedge dX_n$:

$$\int_V d\omega = \int_{\partial V} \omega$$

And since $d\omega = dX_1 \wedge dX_2 \wedge dX_3 \wedge \cdots \wedge dX_n = J_\varphi . dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n$, the result follows. □

Theorem 3.2. *Let Ω be an open subset of $\mathbb{R}^n$, φ be a $\mathcal{C}^1$ mapping from Ω to $\mathbb{R}^n$, θ a linear functional on $\mathbb{R}^n$ and V a connected relatively compact open subset of Ω with piecewise-$\mathcal{C}^1$ boundary. Assume that $\theta \circ \varphi(z) < 0$ for all $z \in \partial V$ and that $\theta \circ \varphi(a) = 0$ for some $a \in V$. Assume moreover that the critical set of φ is nowhere dense in V. Then the jacobian determinant J_φ of φ takes on V both positive and negative values.*

Proof. Since $\theta \neq 0$, we can find linear functionals $\eta_1, \eta_2, \ldots, \eta_n$ on $\mathbb{R}^n$ such that $\eta_1 = \theta$ and $(\eta_1, \eta_2, \ldots, \eta_n)$ is a basis of the dual space. Then define $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\alpha(z) = (\langle \eta_1, z - \varphi(a)\rangle, \langle \eta_2, z\rangle, \ldots, \langle \eta_n, z\rangle)$$

Then α is an affine isomorphism of $\mathbb{R}^n$ to itself and the function $\tilde{\varphi} = \alpha \circ \varphi$ satisfies $\sup_{x \in \partial V} X_1(x) < X_1(a) = 0$ where $X_1(x) = \langle \theta, \varphi(x) - \varphi(a)\rangle$ and $X_k(x) = \langle \eta_k, \varphi(x)\rangle$ $(2 \leq k \leq n)$. So J_α is constant, and we can replace φ by $\tilde{\varphi}$ and assume $\varphi(z) = \big(X_1(z), X_2(z), \ldots, X_n(z)\big)$.

Assume towards a contradiction that J_φ takes only non-negative or only non-positive values. Up to replacing X_n by $-X_n$ if necessary we can and do assume that $J_\varphi(z) \geq 0$ for all $z \in V$. Fix $\lambda > 0$. Since ∂V is compact and $X_1 < 0$ on ∂V there is some $\delta > 0$ such that $X_1(z) \leq -\delta$ for all $z \in \partial V$. Then the set $W = \{z \in V : X_1(z) > -\frac{\delta}{2}\}$ is an open set containing a. Applying lemma 3.1 to the function $\psi_\lambda : z \mapsto (\mathrm{e}^{\lambda X_1(z)}, X_2(z), \ldots, X_n(z))$ we get $J_{\psi_\lambda}(z) = \mathrm{e}^{\lambda X_1(z)} . J_\varphi(z)$ and

$$\int_{\partial V} \mathrm{e}^{\lambda X_1}\, dX_2 \wedge \ldots dX_n = \int_V \mathrm{e}^{\lambda X_1} . J_\varphi(z)\, dz \geq \mathrm{e}^{-\lambda\delta/2} \int_W J_\varphi(z)\, dz$$

Since X_k and ∇X_k ($2 \leq k \leq n$) are bounded on ∂V, there is some $M \in \mathbb{R}^+$ such that $\left| \int_{\partial V} u(z)\, dX_2 \wedge \ldots dX_n \right| \leq M \sup_{z \in \partial V} |u(z)|$ for any continuous function u on ∂V. So we get $\left| \int_{\partial V} \mathrm{e}^{\lambda X_1}\, dX_2 \wedge \ldots dX_n \right| \leq M . \mathrm{e}^{-\lambda\delta}$ and

$$0 \leq \int_W J_\varphi(z)\, dz \leq \inf_{\lambda > 0} \mathrm{e}^{\lambda\delta/2} . M \mathrm{e}^{-\lambda\delta} = \inf_{\lambda > 0} M . \mathrm{e}^{-\lambda\delta/2} = 0$$

This implies that the continuous non-negative function J_φ vanishes everywhere on W, so that every point of W is critical, in contradiction with the hypothesis since $W \neq \emptyset$. □

Theorem 3.3. *Let Ω be an open subset of $\mathbb{R}^n$, φ be a $\mathcal{C}^1$ mapping from Ω to $\mathbb{R}^n$ and V a connected relatively compact open subset of Ω with piecewise-$\mathcal{C}^1$ boundary ∂V. Assume that $d(b,\varphi(\partial V)) > d(b,\varphi(\overline{V})) > 0$ for some b. Assume moreover that the critical set of φ is nowhere dense in V. Then the jacobian determinant J_φ of φ takes on V both positive and negative values.*

Proof. The continuous function $x \mapsto \|b-\varphi(x)\|$ attains its minimum over $\overline{V}$ at a point a which cannot belong to ∂V since $\|b-\varphi(a)\| = d(b,\varphi(\overline{V})) < d(b,\varphi(\partial V))$. Morover, from the assumption $d(b,\varphi(\overline{V})) > 0$, one has $r_* = \|b-\varphi(a)\| > 0$. Consider the inversion h with pole b which transforms $\varphi(a)$ into itself : $h(y) = b + r_*^2 \cdot \dfrac{y-b}{\|y-b\|^2}$. Then h is a $\mathcal{C}^1$-diffeomorphism from $\mathbb{R}^n \setminus \{b\}$ onto itself and it is easily checked that $J_h(y) = -r_*^{2n}.\,\|y-b\|^{-2n} < 0$ for all $y \neq b$ in $\mathbb{R}^n$, and $\psi = h \circ \varphi$ satisfies $J_\psi(x) = J_h(\varphi(x)).J_\varphi(x)$, $\psi(\overline{V}) \subset \tilde{B}(b,r_*)$ and for $X : x \mapsto \langle \varphi(a)-b, \psi(x)-b\rangle - r_*^2$ we have $X(a) = 0 > \sup_{x\in\partial V} X(x)$ since for $x \in \partial V$ we have

$$\|b-\psi(x)\| = \frac{r_*^2}{\|b-\varphi(x)\|} \leq \frac{r_*^2}{d(b,\varphi(\partial V))} < \frac{r_*^2}{d(b,\varphi(\overline{V}))}$$

hence $X(x) \leq \|\varphi(a)-b\|\,.\,\|\psi(x)-b\| - r_*^2 \leq r_*^2 \dfrac{\|\varphi(a)-b\| - d(b,\varphi(\partial V))}{d(b,\varphi(\partial V))} < 0$ and $\psi(x) = h\circ\varphi(x) \neq h\circ\varphi(a) = \varphi(a)$. Then it follows from the previous theorem that J_ψ takes on V both positive and negative values. And so does J_φ since $J_h < 0$. □

Theorem 3.4. *Let Ω be an open subset of $\mathbb{R}^n$ and $\varphi : \Omega \to \mathbb{R}^n$ a $\mathcal{C}^1$ function. Assume that the set Γ of critical points of φ is 0-dimensional. Then φ is an open mapping.*

We can assume that Ω is connected. First remark that Γ cannot cut Ω : hence the jacobian determinant J_φ has constant sign, and up to composing φ with a symmetry we can assume that $J_\varphi \geq 0$ everywhere on Ω.

Lemma 3.5. *Let a be a critical point of φ, $b = \varphi(a)$ and $r > 0$ such that the closed ball $\tilde{B}(a,r)$ is contained in Ω. Assume that $b \notin \varphi(B(a,r)\setminus\Gamma)$. Then there exists $\rho > 0$ such that $\varphi(B(a,r)) \supset B(b,\rho)$.*

Proof. By Lemma 2.2 we can find some connected neighborhood V of a contained in $B(a,r)$, hence relatively compact in Ω, and $\rho > 0$ such that $\partial V \cap \Gamma = \emptyset$

and $\|\varphi(z)-b\| \geq 3\rho$ for all $z \in \partial V$. Since $\partial V \cap \Gamma = \emptyset$, there exists an integer $q > \frac{\sqrt{n}}{\rho}$ such that $\inf_{z\in\partial V} d(z,\Gamma) > \frac{\sqrt{n}}{q}$. For all $\ell = (\ell_1, \ell_2, \ldots, \ell_n) \in \mathbb{Z}^n$ define Q_ℓ as the cube $\prod_{j=1}^{n} \left[\frac{\ell_j}{q}, \frac{\ell_j+1}{q}\right] \subset \mathbb{R}^n$.

Then $V' = \bigcup\{Q_\ell : Q_\ell \cap V \neq \emptyset\}$ contains V and its boundary is disjoint from Γ : indeed if $z \in \partial V'$ there is $\ell \in \mathbb{Z}^n$ such that $z \in \partial Q_\ell$ and $Q_\ell \not\subset V$, hence $Q_\ell \cap \partial V \neq \emptyset$ and $d(z, \partial V) \leq \operatorname{diam}(Q_\ell) = \frac{\sqrt{n}}{q} < \inf_{z\in\partial V} d(z,\Gamma)$.

Moreover $\partial V'$ is the union of a subfamily of the faces of the Q_ℓ's hence a finite union of piecewise-$\mathcal{C}^1$ hypersurfaces. So replacing V by the interior of V' we can assume that V has a piecewise-$\mathcal{C}^1$ boundary and that $\|\varphi(z)-b\| \geq 3\rho - \frac{\sqrt{n}}{q} > 3\rho - \rho = 2\rho$ for all $z \in \partial V$.

If w satisfies $\|w-b\| < \rho$ and $w \notin \varphi(\overline{V})$ we then have $\|\varphi(z)-w\| > 2\rho - \rho > \|b-w\|$ for all $z \in \partial V$. The $\mathcal{C}^1$ function ψ_w defined by $\psi_w(z) = \|\varphi(z)-w\|^2$ attains its minimum over the compact set $\overline{V}$ at a point $z_0 \notin \partial V$ since $\psi_w(z_0) \leq \psi_w(a) < \rho^2 \leq \inf_{z\in\partial V} \varphi_w(z)$. Moreover $\varphi_w(z_0) > 0$. Since the closed set Γ has dimension 0, it cannot cut V and is nowhere dense in V. By Theorem 3.3 we conclude that J_φ has to take negative values in contradiction with the fact that $J_\varphi \geq 0$ on Ω. This contradiction implies that $w \in \varphi(\overline{V})$ and since $w \notin \varphi(\partial V)$ that $w \in \varphi(V)$. So $\varphi(B(a,r)) \supset \varphi(V) \supset B(b,\rho)$. □

Proof of Theorem 3.4. As for the proof of Theorem 2.1, notice that $\varphi(\Omega)$ is a neighborhood of $\varphi(\Omega \setminus \Gamma)$ by the Local Inversion Theorem and a neighborhood of every hypercritical value in virtue of lemma 3.5. □

4. When the critical set is no longer zero-dimensional

We now show by an example that the hypothesis on the critical set of φ cannot be weakened, even if $n = p$.

Example 4.1. *Let H be a Hilbert space of dimension ≥ 1. Then there exists a non-open $\mathcal{C}^1$ mapping Φ from $H \times \mathbb{R}$ to itself such that the critical set has dimension 1 and that there exists a unique critical value.*

Consider the "vertical" segment $J = \{0\} \times [-1,1]$ of the Hilbert space $E = H \oplus \mathbb{R}$. For $(x,z) \notin J$, there is a unique $\rho > 0$ such that

$$\frac{\|x\|^2}{\rho} + \frac{z^2}{1+\rho} = 1 .$$

Indeed the function $f : t \mapsto \dfrac{\|x\|^2}{t} + \dfrac{z^2}{1+t}$ is decreasing on $]0,+\infty[$, satisfies $\lim_{t\to+\infty} f(t) = 0 < 1$, $\lim_{t\to 0^+} f(t) = +\infty > 1$ if $x \neq 0$, and $\lim_{t\to 0^+} f(t) = z^2 > 1$ if $x = 0$.

More precisely we have $(1+\rho)\|x\|^2 + \rho z^2 = \rho(1+\rho)$, hence

$$\rho^2 - \rho(\|x\|^2 + z^2 - 1) - \|x\|^2 = 0 \ ,$$

and ρ is the largest root of this equation. It follows that if $(x,z) \notin J$, we have $\rho > 0$ and there are some $u \in H$ and $\psi \in [-\pi/2, \pi/2]$ such that $\|u\| = 1$ and

$$(x,z) = \big(\sqrt{\rho}.u.\cos\psi, \sqrt{1+\rho}.\sin\psi\big)$$

We then put $\omega = (0,0) \in E$,

$$\Phi(x,z) = \begin{cases} \omega & \text{if } (x,z) \in J \\ (\rho^2.u.\cos\psi, \rho^2 \sin\psi) & \text{if not} \end{cases}$$

so $\Phi(x,z) \in E$ and $\|\Phi(x,z)\| = \rho^2$. We will show that Φ is $\mathcal{C}^1$, that its critical set is J, which has dimension 1, and that the only critical value is ω, since $\Phi(J) = \{\omega\}$.

Looking at ρ as a function from E to $\mathbb{R}_+$, we then have $\Phi(x,z) = (X,Z)$ with

$$\begin{cases} X = x.\rho^{3/2}(x,z) \\ Z = z.\dfrac{\rho^2(x,z)}{\sqrt{1+\rho(x,z)}} \end{cases}$$

Since

$$\rho(x,z) = \frac{1}{2}\Big(\|x\|^2 + z^2 - 1 + \sqrt{(\|x\|^2 + z^2 - 1)^2 + 4\|x\|^2}\Big)$$

the function ρ is continuous on E and vanishes on J. We have to show that Φ is $\mathcal{C}^1$ on $E \setminus J$ and that $\|\nabla\Phi(x,z)\|$ tends to 0 as (x,z) tends to a point $(0,t)$ of J.

From the above formula it is clear that ρ is $\mathcal{C}^1$ at each $q = (x,z)$ where $\|x\|^2 > 0$ or $|z| > 1$, it is outside J. For the study of Φ around J, we can restrict ourself to the case where $q \in Q = \{(x,z) : \rho(x,z) \leq 1\}$, which is a neighborhood of J.

Lemma 4.2. *If $q \in Q$ and $\delta = d(q,J)$ we have $\rho \leq 6\delta$.*

Proof. Since $\dfrac{\|x\|^2}{\rho} + \dfrac{z^2}{1+\rho} = 1$, we have $\|x\|^2 \leq \rho$ and $z^2 \leq 1+\rho$, from what we get $\|x\| \leq 1$ and $z^2 \leq 2$, hence $|z| \leq \sqrt{2}$, for $q = (x,z) \in Q$. Moreover

$$\|x\| \leq \delta = d(q,J) \leq \|x\| + (|z|-1)_+ \leq 1 + (\sqrt{2}-1) = \sqrt{2} \ .$$

If $q \in Q$ and $\rho \geq 5\delta$, we have $\|x\|^2 \leq \delta^2$ and $|z| \leq 1+\delta$, hence

$$\frac{(1+\delta)^2}{1+\rho} \geq \frac{z^2}{1+\rho} = 1 - \frac{\|x\|^2}{\rho} \geq 1 - \frac{\delta^2}{5\delta} = 1 - \frac{\delta}{5}$$

whence

$$\begin{aligned} \rho &\leq \frac{(1+\delta)^2}{1-\delta/5} - 1 = \frac{1+2\delta+\delta^2-1+\delta/5}{1-\delta/5} \leq \frac{2\delta+\delta^2+\delta/5}{1-\delta/5} \\ &\leq \delta.\frac{2+\sqrt{2}+1/5}{1-\sqrt{2}/5} \leq 6\delta \end{aligned}$$

It follows that $\rho < 5\delta$ or $\rho \leq 6\delta$, hence that $\rho \leq 6\delta$ for each $q \in Q$. □

Lemma 4.3. *We have* $\dfrac{\|x\|^2}{\rho^2} + \dfrac{z^2}{(1+\rho)^2} = \dfrac{\|x\|^2+\rho^2}{\rho^2(1+\rho)}$.

Proof. Indeed

$$\begin{aligned} \frac{\|x\|^2}{\rho^2} + \frac{z^2}{(1+\rho)^2} &= \frac{\|x\|^2}{\rho^2} + \frac{1}{1+\rho} \cdot \Big(1 - \frac{\|x\|^2}{\rho}\Big) \\ &= \frac{(1+\rho)\,\|x\|^2 + \rho^2 - \rho\,\|x\|^2}{\rho^2(1+\rho)} = \frac{\|x\|^2+\rho^2}{\rho^2(1+\rho)}, \end{aligned}$$

which is the announced equality. □

Lemma 4.4. *We have*

$$\langle \frac{\partial\rho}{\partial x}, h\rangle = \frac{2\rho(1+\rho)}{\|x\|^2+\rho^2} \cdot \langle x, h\rangle \quad ; \quad \frac{\partial\rho}{\partial z} = \frac{2\rho^2 z}{\|x\|^2+\rho^2}$$

and $\|\nabla\rho(q)\| \leq 2\sqrt{6}.\rho(q)^{-1/2}$ *whenever* $q \in Q \setminus J$.

Proof. Since $\theta(x,z) := \dfrac{\|x\|^2}{\rho} + \dfrac{z^2}{1+\rho} = 1$, differentiating with respect to x, we get for $h \in H$:

$$\begin{aligned} 0 &= \frac{\partial\theta}{\partial x}.h = \frac{\langle 2x, h\rangle}{\rho} - \frac{\|x\|^2}{\rho^2} \cdot \langle \frac{\partial\rho}{\partial x}, h\rangle - \frac{z^2}{(1+\rho)^2} \cdot \langle \frac{\partial\rho}{\partial x}, h\rangle \\ &= \frac{\langle 2x, h\rangle}{\rho} - \langle \frac{\partial\rho}{\partial x}, h\rangle \cdot \frac{\|x\|^2+\rho^2}{\rho^2(1+\rho)} \end{aligned}$$

whence $\langle \dfrac{\partial\rho}{\partial x}, h\rangle = \dfrac{2\rho(1+\rho)}{\|x\|^2+\rho^2}.\langle x, h\rangle$. Analogously,

$$0 = \frac{\partial\theta}{\partial z} = -\frac{\|x\|^2}{\rho^2} \cdot \frac{\partial\rho}{\partial z} + \frac{2z}{1+\rho} - \frac{z^2}{(1+\rho)^2} \cdot \frac{\partial\rho}{\partial z} = \frac{2z}{1+\rho} - \frac{\partial\rho}{\partial z} \cdot \frac{\|x\|^2+\rho^2}{\rho^2(1+\rho)}$$

whence $\dfrac{\partial\rho}{\partial z} = \dfrac{2\rho^2 z}{\|x\|^2+\rho^2}$.

Since $\|x\|^2 \le \rho \le 1$ and $z^2 \le 1+\rho \le 2$, we deduce :

$$\|\nabla\rho(q)\|^2 = 4\rho^2\cdot\frac{(1+\rho)^2\|x\|^2+\rho^2z^2}{(\|x\|^2+\rho^2)^2} \le 4\rho^2\cdot\frac{\rho(1+\rho)^2+\rho^2(1+\rho)}{\rho^4}$$
$$= 4\cdot\frac{(1+\rho)(2+\rho)}{\rho} \le 4\cdot\frac{(2\times 3)}{\rho} = \frac{24}{\rho}$$

whence $\|\nabla\rho(q)\| \le 2\sqrt{6}.\rho(q)^{-1/2}$. □

Lemma 4.5. *For $q=(x,z)\in Q\setminus J$, we have $\|\nabla\Phi(q)\| \le 41.\delta^{1/2}(q)$.*

Proof. For $(h,s)\in E$, we have $X'(q).(h,s) = \rho^{3/2}.h + \frac{3}{2}x.\rho^{1/2}(q).\langle\nabla\rho(q),(h,s)\rangle$, hence

$$\|\nabla X(q)\| \le \rho^{3/2}(q) + \frac{3}{2}\|x\|.\rho^{1/2}(q).2\sqrt{6}.\rho^{-1/2}(q) \le \rho(q)+3\sqrt{6}.\|x\|$$
$$\le 6\delta(q)+3\sqrt{6}.\delta(q) \le 14.\delta(q) \le 14\sqrt[4]{2}.\delta^{1/2}(q) \le 17.\delta^{1/2}(q)$$

Finally $\langle Z'(u),(h,s)\rangle = s.\dfrac{\rho^2}{\sqrt{1+\rho}} + z.\dfrac{2\rho(1+\rho)-\rho^2/2}{(1+\rho)^{3/2}}.\langle\nabla\rho(q),(h,s)\rangle$,
whence since $|z|\le\sqrt{2}$, $\rho\le 1$ and $0 \le 2\rho(1+\rho)-\dfrac{\rho^2}{2} \le 2\rho(1+\rho)$:

$$\|\nabla Z(q)\| \le \frac{\rho^2}{\sqrt{1+\rho}} + \sqrt{2}.\big(2\sqrt{6}.\rho^{-1/2}(q)\big).\frac{2\rho}{\sqrt{1+\rho}}$$
$$\le \rho^2(q)+8\sqrt{3}.\rho^{1/2}(q) \le \rho^{1/2}(q).\Big(\rho^{3/2}(q)+8\sqrt{3}\Big)$$
$$\le \sqrt{6}.\delta^{1/2}(q).\big(1+8\sqrt{3}\big) \le 37.\delta^{1/2}(q)\ .$$

Hence
$\|\nabla\Phi(q)\| \le \big(\|\nabla X(q)\|^2+\|\nabla Z(q)\|^2\big)^{1/2} \le \sqrt{(17^2+37^2).\delta(q)} \le 41.\delta^{1/2}(q)$. □

Lemma 4.6. *The function Φ is $\mathcal{C}^1$ on E, and its critical points are the points of J.*

Proof. We have previously seen that Φ is continuous on E, $\mathcal{C}^1$ on $E\setminus J$ and that $\|\nabla\Phi(q)\| \to 0$ as $\delta(q)=d(q,J)\to 0$. It follows that Φ is differentiable at each point q of J with $\Phi'(q)=0$ and that Φ' is continuous on E. Since $\Phi'(q)=0$ for every $q\in J$, J is contained in the critical set of Φ, and since Φ vanishes on J, there is a unique critical value $\omega = (0,0)$ corresponding to these critical points. It remains only to prove that Φ is regular at each point of $E \setminus J$. It is clear that $\Phi(E \setminus J) \subset E \setminus \{\omega\}$.

We will show that $\Psi = \Phi^{-1}$ is differentiable on $E \setminus \{\omega\}$, and it will follow that, for $q \in E \setminus J$ and $v = \Phi(q)$, $\Phi'(q) \circ \Psi'(v) = \Psi'(v) \circ \Phi'(q) = \mathrm{Id}_E$, hence that $\Phi'(q)$ is invertible. Indeed for $v = \Phi(q) = (X,Z) \neq \omega$ in E, we must have $\rho(q) = \|v\|^{1/2}$. Hence for $r = \|v\|^{1/2} = (\|X\|^2 + Z^2)^{1/4}$ the point q has to belong to the ellipsoid $\mathcal{E} = \{(x,z) \in E : \dfrac{\|x\|^2}{r} + \dfrac{z^2}{1+r} = 1\}$ and we must have for $q = (x,z) = \Psi(X,Z)$,

$$\begin{cases} x = X.r^{-3/2} \\ z = Z.\dfrac{\sqrt{1+r}}{r^2} \end{cases}$$

Since on $E \setminus \{\omega\}$ the mapping : $v = (X,Z) \mapsto r = (\|X\|^2 + Z^2)^{1/4}$ is $\mathcal{C}^1$ and does not vanish, this shows that Ψ is $\mathcal{C}^1$ on $E \setminus \{\omega\}$, hence that Φ is regular at every point of $E \setminus J$. □

Lemma 4.7. *The mapping Φ is not open.*

Proof. It is enough to notice that the image C under Φ of the open unit ball B of E is not a neighborhood of $\Phi(\omega) = \omega$, and even that $C \cap (\{0\} \times \mathbb{R}) = \{\omega\}$. Indeed, if $\Phi(q) = (0,t)$ for some $t \neq 0$ we must have $r = |t|^{1/2}$, hence $q = \left(0, t.\dfrac{\sqrt{1+|t|^{1/2}}}{|t|}\right)$, and $\|q\| = \sqrt{1+|t|^{1/2}} > 1$, it is $q \notin B$. □

It follows from what precedes that if H has dimension ≥ 2 the critical set of Φ has a topological dimension $1 \leq \dim(E) - 2$ and that Φ is not open.

5. Questions

Let us complete this note by asking two questions.

In the statement of Theorem 2.1 an hypothesis is done on the topological dimension of the set of critical values (or more exactly on the dimension of the set of hypercritical values). It follows from Theorem 3.4 that if $n = p$ this hypothesis is useless. It is then quite natural to ask whether this hypothesis could be removed also when $n > p$, since Example 4.1 shows that in turn the hypothesis on the dimension of critical points cannot be weakened.

- *Does there exist a $\mathcal{C}^1$ non-open function from $\Omega \subset \mathbb{R}^3$ to $\mathbb{R}^2$ whose critical set has dimension 0?*

The methods used in the above proofs rely heavily on the local compactness of Ω. So it can be guessed that these results hold only for functions defined on open subsets of a finite-dimensional space. For example the following question can be raised.

- *Let H be a infinite-dimensional Hilbert space. If $f : H \to H$ is $\mathcal{C}^1$ with a unique critical point, must f be open ?*

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