LE MATEMATICHE
Vol. LXXII (2017) – Fasc. I, pp. 185–200
doi: 10.4418/2017.72.1.15

ON THE MINIMAL SUBMODULES OF A MODULE

H. ANSARI-TOROGHY - S. S. POURMORTAZAVI

For any module M over a commutative ring R, $Spec^s_R(M)$ (resp., $Min_R(M)$) is the collection of all second (resp., minimal) submodules of M. In this article we investigate the interplay between the topological properties of $Min_R(M)$ and module theoretic properties of M. Also, for various types of modules M, we obtain some conditions under which $Min_R(M)$ is homeomorphic with the maximal ideal space of some ring.

1. Introduction

Throughout this article, R denotes a commutative ring with identity and all modules are unitary. Also $\mathbb{P}$ and $\mathbb{Z}$ denote the set of prime integers and the ring of integers, respectively. If N is a subset of an R-module M, then $N \leq M$ denotes N is an R-submodule of M. For any ideal I of R containing $Ann_R(M)$, $\bar{R}$ and $\bar{I}$ denote $R/Ann_R(M)$ and $I/Ann_R(M)$, respectively. The *colon ideal of M into N* is defined to be $(N : M) = \{r \in R : rM \subseteq N\} = Ann_R(M/N)$. Also we use the notation $(0 :_M I)$ to denote the set $\{m \in M \mid rm = 0 \text{ for every } r \in I\}$.

Let M be an R-module. A non-zero submodule N of M is said to be *second* if for each $a \in R$ the homomorphism $N \stackrel{a}{\to} N$ is either surjective or zero. This implies that $Ann_R(N) = p$ is a prime ideal of R and S is said to be *p-second* (see [10]).

Entrato in redazione: 13 ottobre 2016

AMS 2010 Subject Classification: 13C13; 13C99
Keywords: Second submodule, minimal submodule, Zariski topology, spectral space.

M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$ (see [3]).

For a submodule N of M, the *second socle* (or *second radical*) of N is defined as the sum of all second submodules of M contained in N and denoted by $soc(N)$ (or $sec(N)$). In case N does not contain any second submodule, the socle of N is defined to be (0) . Also, $N \neq (0)$ is said to be a *socle submodule* of M if $soc(N) = N$ (see [2, 6]).

The second spectrum of M is defined as the set of all second submodules of M and denoted by $Spec^s_R(M)$ or X^s. We call the map $\psi : X^s \to Spec(\overline{R})$ given by $S \mapsto \overline{Ann_R(S)}$ as the *natural map* of X^s.

M is said to be X^s*-injective* (resp. *secondful*) if the natural map of X^s is injective (resp. surjective). Equivalently, M is X^s-injective if and only if $Ann_R(S_1) = Ann_R(S_2)$, $S_1, S_2 \in X^s$, implies that $S_1 = S_2$ if and only if for every $p \in Spec(R)$, $|Spec^s_p(M)| \leq 1$ (see [4, 7]).

The *Zariski topology* on $Spec^s_R(M)$ is the τ^s described by taking the set $\zeta^s(M) := \{V^s(N) : N \leq M\}$ as the set of closed sets of $Spec^s_R(M)$, where

$$V^s(N) = \{S \in Spec^s_R(M) : Ann_R(N) \subseteq Ann_R(S)\}$$

(see [1]).

There exists a topology on $Min_R(M)$ (we recall that $Min_R(M)$ is the collection of all minimal submodules of M. Of course, each element of $Min_R(M)$ is a non-zero submodule). having $\zeta^{sm}(M) := \{V^{sm}(N) : N \leq M\}$ as the set of closed sets of $Min_R(M)$, where

$$V^{sm}(N) = \{S \in Min_R(M) : Ann_R(N) \subseteq Ann_R(S)\}.$$

We denote this topology by τ^{sm}. In fact τ^{sm} is the same as the subspace topology induced by τ^s on $Min_R(M)$. In the rest of this article $Spec^s_R(M)$ (resp. $Min_R(M)$) is always equipped with the Zariski topology τ^s (resp. τ^{sm}).

In this article, we investigate the interplay between the topological properties of $Min_R(M)$ and module theoretic properties of M (see Proposition 2.4, Theorem 2.9, Theorem 2.16, Corollary 2.18, Proposition 2.21, and Theorem 2.26). Also we consider the conditions under which $Min_R(M)$ is a Noetherian topological space (see Proposition 2.4, Theorem 2.9, Theorem 2.17, and Corollary 2.18). Moreover, we study the topological space $Min_R(M)$ from the point of view of *Max*-spectral spaces (see Theorem 2.26). It is shown that if M is a Min-injective module over a PID, then $Min_R(M)$ is a Max-spectral topological space (see Theorem 2.26 (g)). These results enable us to provide a large family of modules such that their minimal submodules are Max-spectral.

2. Main results

As it was mentioned before, $Spec_R^s(M)$ (resp. $Min_R(M)$) is always equipped with Zariski topology τ^s (resp. τ^{sm}).

Definition 2.1. Let M be an R-module.

(a) The map $\phi : Min_R(M) \to Max(\overline{R})$ defined by $\phi(N) = \overline{Ann_R(N)}$ for every minimal submodule N of M is called the *natural map* of $Min_R(M)$. (Note that since N is minimal, $Ann_R(N)$ is a maximal ideal of R and hence $\overline{Ann_R(N)} \in Max(\overline{R})$.)

(b) We say that M is a *Min-surjective* module if either $M = 0$, or $M \neq 0$ and the natural map of $Min_R(M)$ is surjective.

(c) We say that M is *Min-injective* module if $M = 0$, or $M \neq 0$ and the natural map of $Min_R(M)$ is injective.

Example 2.2. (a) Every finite length R-module is Min-surjective by [1, Example 3.10]. However, the converse is not true in general. To see this, let $M = \bigoplus_{p\in\mathbb{P}} \mathbb{Z}_p$. Then we have

$$Spec_{\mathbb{Z}}^s(M) = Min_{\mathbb{Z}}(M) = \left\{ \mathbb{Z}_p \oplus (\bigoplus_{p\neq q\in\mathbb{P}} (0)) \mid p \in \mathbb{P} \right\}.$$

Clearly, M is a Min-surjective $\mathbb{Z}$-module while it is not a finite length $\mathbb{Z}$-module.

(b) Every X^s-injective module is Min-injective . However, the converse is not true in general. To see this, let $M = \mathbb{Z}(p^\infty) \oplus \mathbb{Z}(q^\infty)$, where p and q are prime number. Then we have

$$\begin{aligned} Spec_{\mathbb{Z}}^s(M) =&\{\mathbb{Z}(p^\infty) \oplus (0), (0) \oplus \mathbb{Z}(q^\infty), \mathbb{Z}(p^\infty) \oplus \mathbb{Z}(q^\infty), \\ &< 1/p + \mathbb{Z} > \oplus(0), (0)\oplus < 1/q + \mathbb{Z} >\} \end{aligned}$$

and

$$Min_{\mathbb{Z}}(M) = \{< 1/p + \mathbb{Z} > \oplus(0), (0)\oplus < 1/q + \mathbb{Z} >\}.$$

Clearly, M is a Min-injective $\mathbb{Z}$-module while it is not an X^s-injective $\mathbb{Z}$-module.

For an ideal I of R, we will denote $V(I)$ by the set $\{p \in Spec(R) \mid I \subseteq p\}$. Also we define $V^m(I)$ as $V^m(I) = V(I) \cap Max(R)$.

Lemma 2.3. *Let M be an R-module and let $\phi : (Min_R(M), \tau^{sm}) \to (Max(\overline{R}), \tau)$ be the natural map of $Min_R(M)$. Then the following hold. (We recall that $(Max(\overline{R}), \tau)$ is the subspace topology induced by Zariski topology on $Spec(\overline{R})$.)*

(a) ϕ is a continuous map.

(b) If M is Min-surjective, then ϕ is a closed and open mapping.

Proof. (a) This follows from the fact that $\phi^{-1}(V^m(\bar{I})) = V^{sm}((0 :_M I))$ for every ideal I of R containing $Ann_R(M)$.

(b) Let N be a submodule of M and let $V^{sm}(N)$ be a closed subset of $Min_R(M)$. Then as in the proof part (a), we have

$$\phi^{-1}(V^m(\overline{Ann_R(N)})) = V^{sm}(N).$$

Hence $\phi(V^{sm}(N)) = V^m(\overline{Ann_R(N)})$ because ϕ is surjective. Similarly, ϕ is open and the proof is completed. □

Proposition 2.4. *Let R be a ring such that the intersection of every infinite collection of maximal ideals of R is zero (for example, when R is PID or one dimensional Noetherian domain) and let M be an R-module. Then $Min_R(M)$ is a Noetherian topological space.*

Proof. Assume that $(Min_R(M), \tau^{sm})$ is not a Noetherian topological space. It turns out that there exists a descending chain of closed subsets of $Min_R(M)$.

$$V^{sm}(N_1) \supsetneqq V^{sm}(N_2) \supsetneqq \cdots \supsetneqq V^{sm}(N_k) \supsetneqq \cdots .$$

For each $i \in \mathbb{N}$, we set

$$I_i := \{m \in Max(R) \mid \exists S \in Min_R(M)\, s.t.\, Ann_R(S) = m, Ann_R(N_i) \subseteq Ann_R(S)\}.$$

Clearly, we have $I_1 \supsetneqq I_2 \supsetneqq \cdots \supsetneqq I_k \supsetneqq \cdots$. If for $i \in \mathbb{N}$, I_i is a finite set, then the above mentioned chain of closed subsets becomes eventually constant, a contradiction. Otherwise, for each $i \in \mathbb{N}$, we have $Ann_R(N_i) = 0$ and hence $V^{sm}(N_i) = Min_R(M)$, a contradiction. So the proof is completed. □

Note 2.5. Let M be an R-module and let W be a subset of $Min_R(M)$. We will denote the sum of all elements in W by $T(W)$ and the closure of W in $Min_R(M)$ (resp. $Spec^s_R(M)$) by $cl^m(W)$ (resp. $cl(W)$).

Lemma 2.6. *Let M be an R-module and W be a subset of $Min_R(M)$. Then $cl^m(W) = V^{sm}(T(W))$. Hence, W is closed if and only if $V^{sm}(T(W)) = W$.*

Proof. By [1, Proposition 5.1], we have $cl(W) = V^s(\sum_{S \in W} S)$. Hence $cl^m(W) = V^{sm}(T(W))$. □

Remark 2.7. For a proper ideal I of R, we recall that the *J-radical* of I, denoted by $J^m_R(I)$, is the intersection of all maximal ideals containing I. An ideal I of R is a J-radical ideal if $I = J^m_R(I)$ (see [9]).

Definition 2.8. Let M be an R-module. The *socle* of a submodule N of M, denoted by $Soc(N)$, is the summation of all members of $V^{sm}(N)$. In case that $V^{sm}(N) = \emptyset$, we define $Soc(N) = 0$. Asubmodule N of M is said to be a *socle submodule* if $N = Soc(N)$.

Theorem 2.9. *Let M be an R-module. Then the following statements are equivalent:*

(a) $Min_R(M)$ is a Noetherian topological space.

(b) The descending chain for socle submodules of M holds.

Proof. (a)⇒(b). Straightforward.
(b)⇒(a). Let

$$V^{sm}(N_1) \supseteq V^{sm}(N_2) \supseteq \cdots \supseteq V^{sm}(N_i) \supseteq \cdots$$

be a descending chain of closed subsets of $Min_R(M)$, where N_i is a submodule of M. Hence

$$Soc(N_1) \supseteq Soc(N_2) \supseteq \cdots \supseteq Soc(N_i) \supseteq \cdots$$

is an descending chain of socle submodules of M. So by hypothesis, there exists a $k \in \mathbb{N}$ suth that for all $n \geqslant 1$, we have $Soc(N_{k+n}) = Soc(N_k)$ and the proof is completed. □

Corollary 2.10. *Let M be a Noetherian R-module. Then $Min_R(M)$ is a Noetherian topological space.*

We recall that if I is an ideal of R, then the J-components of I are the minimal members of the family of J-radical prime ideals containing I (see [9, p. 631]).

Definition 2.11. Let M be an R-module and L a submodule of M. A submodule K of M is a *S-component* of L, if $Ann_R(K)$ is a J-component of $Ann_R(L)$.

Definition 2.12. An R-module M is said to have property (SFC) if every closed subset of $Min_R(M)$ has a finite number of irreducible components.

Example 2.13. Let M be an R-module. Then M has property (SFC) in each the following cases.

(a) $Min_R(M)$ is a Noetherian topological space

(b) R is a semi-local or PID (see Proposition 2.4 and part (a)).

(c) M is Noetherian (see Corollary 2.10 and part (a)).

When $M = R$, then R has property (SFC) if and only if every ideal of R has a finite number of J-components (see [9, p. 632]).

Lemma 2.14. *Let M be a Min-surjective R-module. Then the following hold.*

(a) If N is a submodule of M, then

$$J_R^m(Ann_R(N)) = Ann_R(Soc(N)).$$

(b) If q is a J-radical ideal of R containing $Ann_R(M)$, then there exists a submodule K of M such that $Ann_R(K) = q$.

Proof. (a) We have

$$J_R^m(Ann_R(N)) = \bigcap_{m \in V^m(Ann_R(N))} m$$

and

$$Ann_R(Soc(N)) = Ann_R(\sum_{S \in V^{sm}(N)} S) = \bigcap_{S \in V^{sm}(N)} Ann_R(S).$$

Since M is Min-surjective, for every $m \in V^m(Ann_R(N))$, there exists $S_m \in Min_R(M)$ such that $Ann_R(S_m) = m$. So we have

$$\bigcap_{m \in V^m(Ann_R(N))} m = \bigcap_{S \in V^{sm}(N)} Ann_R(S).$$

(b) Since M is Min-surjective, for every $m \in V^m(Ann_R(N))$, there exists $S_m \in Min_R(M)$ such that $Ann_R(S_m) = m$. So we have

$$q = J_R^m(q) = \bigcap_{m \in V^m(q)} m = \bigcap_{m \in V^m(q)} Ann_R(S_m) = Ann_R(\sum_{m \in V^m(q)} S_m). \quad \square$$

Remark 2.15. If S is a commutative ring with a non-zero identity, then there exists a one-to-one correspondence between the J-radical prime ideals of ring S and irreducible closed subsets of $Max(S)$ (see [9, p. 632]).

Theorem 2.16. *Let M be a Min-surjective R-module. Then the following hold.*

(a) If $Y \subseteq Min_R(M)$, then Y is an irreducible closed subset of $Min_R(M)$ if and only if $Y = V^{sm}(N)$ for some submodule N of M such that $Ann_R(N)$ is a J-radical prime ideal of R.

(b) If $W \subseteq Min_R(M)$ and L is a submodule of M, then W is an irreducible component of $V^{sm}(L)$ if and only if $W = V^{sm}(N')$ for some S-component N' of L.

(c) If $Z \subseteq Min_R(M)$, then Z is an irreducible component of $Min_R(M)$ if and only if $Z = V^{sm}((0 :_M p))$ for some J-component ideal p of $Ann_R(M)$.

(d) *M has property (SFC) if and only if every submodule of M has only finitely many of S-components.*

Proof. (a) Let Y be an irreducible closed subset of $Min_R(M)$. Since Y is closed, $Y = V^{sm}(N)$ for some submodule N of M. It turns out that $\phi(V^{sm}(N)) = V^m(\overline{Ann_R(N)})$ is an irreducible closed subset of $Max(\overline{R})$ by Lemma 2.3. Now by Remark 2.15 and Lemma 2.14,

$$J^m_{\overline{R}}(\overline{Ann_R(N)}) = \overline{Ann_R(Soc(N))}$$

is a J-radical prime ideal of $\overline{R}$ so that $Ann_R(Soc(N))$ is a J-radical prime ideal of R. Conversely, let $V^{sm}(K)$ be a closed subset of $Min_R(M)$, where K is a submodule of M such that $Ann_R(K)$ is a J-radical prime ideal of R. We show that $V^{sm}(K)$ is irreducible. To see this, let E and E' be submodules of M with

$$V^{sm}(K) \subseteq V^{sm}(E) \cup V^{sm}(E').$$

Hence as in the proof of Lemma 2.3 (b), we have

$$V^m(\overline{Ann_R(K)}) \subseteq V^m(\overline{Ann_R(E)}) \cup V^m(\overline{Ann_R(E')}).$$

Since $Ann_R(K)$ is a J-radical prime ideal of R, it is easy to check that $\overline{Ann_R(K)}$ is a J-radical prime ideal of $\overline{R}$. Therefore $V^m(\overline{Ann_R(K)})$ is an irreducible closed subset of $Max(\overline{R})$ by Remark 2.15. Hence

$$V^m(\overline{Ann_R(K)}) \subseteq V^m(\overline{Ann_R(E)}) \vee V^m(\overline{Ann_R(K)}) \subseteq V^m(\overline{Ann_R(E')}).$$

Suppose that $V^m(\overline{Ann_R(K)}) \subseteq V^m(\overline{Ann_R(E)})$. This implies that $V^{sm}(K) \subseteq V^{sm}(E)$. By similar arguments, $V^{sm}(K) \subseteq V^{sm}(E')$ when $V^m(\overline{Ann_R(K)}) \subseteq V^m(\overline{Ann_R(E')})$.

(b) $(\Rightarrow)$. Let W be an irreducible component of $V^{sm}(L)$. Then W is an irreducible closed subset of $Min_R(M)$. So by part (a), $W = V^{sm}(N')$ for some submodule N' of M such that $Ann_R(N')$ is a J-radical prime ideal of R. We claim that N' is an S-component of L or equivalently, $Ann_R(N')$ is a J-component of $Ann_R(L)$. Clearly $Ann_R(L) \subseteq Ann_R(N')$ by using Lemma 2.14 (a). So by the above arguments, it is enough to show that $Ann_R(N')$ is a minimal member of the family of J-radical prime ideals containing $Ann_R(L)$. To see this, let q be a J-radical prime ideal of R with

$$Ann_R(L) \subseteq q \subseteq Ann_R(N').$$

Since M is Min-surjective, there exists a submodule Q of M such that $q = Ann_R(Q)$ by Lemma 2.14 (b). Hence

$$V^{sm}(N') \subseteq V^{sm}(Q) \subseteq V^{sm}(L).$$

Also $V^{sm}(Q)$ is an irreducible closed subset of $V^{sm}(L)$ by part (a). Since $W = V^{sm}(N')$ is an irreducible component of $V^{sm}(L)$, by the above arguments, we have $V^{sm}(Q) = V^{sm}(N')$. Now by using Lemma 2.14 (a), $q = Ann_R(N')$ as desired.

($\Leftarrow$). Let N' be an S-component of L. Then $V^{sm}(N')$ is an irreducible closed subset of $V^{sm}(L)$ by part (a). Let L' be a submodule of M such that $Ann_R(L')$ is a J-radical prime ideal of R and $V^{sm}(N') \subseteq V^{sm}(L') \subseteq V^{sm}(L)$. Hence

$$Ann_R(Soc(L)) \subseteq Ann_R(Soc(L')) \subseteq Ann_R(Soc(N')).$$

By using Lemma 2.14 (a), we have

$$Ann_R(L) \subseteq J^m_R(Ann_R(L)) \subseteq J^m_R(Ann_R(L')) \subseteq J^m_R(Ann_R(N')).$$

Since $Ann_R(L')$ and $Ann_R(N')$ are J-radical prime ideals,

$$Ann_R(L) \subseteq Ann_R(L') \subseteq Ann_R(N').$$

Since N' be an S-component of L, we have $Ann_R(L') = Ann_R(N')$. Hence $V^{sm}(N') = V^{sm}(L')$.

(c) This follows from part (b) and Lemma 2.14 (b) and the fact that if N is a submodule of M, then

$$V^{sm}((0 :_M Ann_R(N))) = V^{sm}(N).$$

(d) This follows from part (b). □

Let X be a topological space. We consider strictly decreasing chain $Z_0 \subsetneq Z_1 \subsetneq, \cdots \subsetneq Z_r$ of length r of irreducible closed subsets Z_i of X. The supremum of the lengths, taken over all such chains, is called the combinatorial dimension of X and denoted by $dim(X)$. For the empty set $\emptyset$, the combinatorial dimension of $\emptyset$ is defined to be -1.

Theorem 2.17. *Let M be a Min-surjective R-module. Then the following hold*

(a) $Min_R(M)$ is a Noetherian topological space if and only if $Max(\overline{R})$ is a Noetherian topological space.

(b) *$Min_R(M)$ is a connected topological space if and only if $Max(\overline{R})$ is a connected topological space.*

(c) *$Min_R(M)$ is an irreducible topological space if and only if $Max(\overline{R})$ is an irreducible topological space.*

(d) *$Min_R(M)$ is a quasi-compact topological space.*

(e) *$dim((Min_R(M),\tau^{sm})) = dim((Max(\overline{R}),\tau))$, where $(Max(\overline{R}),\tau)$ is the subspace topology induced by Zariski topology on $Spec(\overline{R})$.*

Proof. (a) The necessity is clear. To show the converse, by Theorem 2.9, it is enough to show that the descending chain condition for socle submodules of M holds. To see this, let $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_i \cdots$ be an descending chain of socle submodules of M. Then by Lemma 2.14 (a),

$$\overline{Ann_R(N_1)} \subseteq \overline{Ann_R(N_2)} \subseteq \cdots \subseteq \overline{Ann_R(N_i)} \subseteq \cdots$$

be an ascending chain of J-radical ideals of $\overline{R}$. Now since $Max(\overline{R})$ is a Noetherian topological space, there exists a $k \in \mathbb{N}$ such that for all $n \in \mathbb{N}$,

$$\overline{Ann_R(N_{k+n})} = \overline{Ann_R(N_k)}.$$

Hence for all $n \in \mathbb{N}$,

$$\begin{aligned} V^{sm}(N_{k+n}) &= V^{sm}((0:_M Ann_R(N_{k+n}))) \\ &= V^{sm}((0:_M Ann_R(N_k))) = V^{sm}(N_k). \end{aligned}$$

So for all $n \in \mathbb{N}$, we have

$$N_{k+n} = Soc(N_{k+n}) = Soc(N_k) = N_k,$$

as desired.

(b) First assume that $Min_R(M)$ is a connected topological space. Then $Max(\overline{R}) = \phi(Min_R(M))$ is connected by Lemma 2.3. To see the reverse implication, we assume that $Max(\overline{R})$ is a connected topological space. If $Min_R(M)$ is a disconnected topological space, then there exist submodules N and K of M such that

$$Min_R(M) = V^{sm}(N) \cup V^{sm}(K)$$

and

$$V^{sm}(N) \cap V^{sm}(K) = \emptyset,$$

where $V^{sm}(N) \neq \emptyset$, and $V^{sm}(K) \neq \emptyset$. Hence as in the proof of Lemma 2.3 we have

$$Max(\overline{R}) = V^m(\overline{Ann_R(N)}) \cup V^m(\overline{Ann_R(K)}).$$

On the other hand we have

$$V^m(\overline{Ann_R(N)}) \cap V^m(\overline{Ann_R(K)}) = \emptyset,$$

$V^m(\overline{Ann_R(N)}) \neq \emptyset$, and $V^m(\overline{Ann_R(K)}) \neq \emptyset$ (Note that if $m \in V^m(\overline{Ann_R(N)}) \cap V^m(\overline{Ann_R(K)})$, then $Ann_R(N) \subseteq m$ and $Ann_R(K) \subseteq m$. Since M is Min-surjective, there exists $S \in Min_R(M)$ such that $Ann_R(S) = m$. It follows that $S \in V^m(\overline{Ann_R(N)}) \cap V^m(\overline{Ann_R(K)})$, a contradiction). Therefore $Max(\overline{R})$ is a disconnected topological space, a contradiction. Hence $Min_R(M)$ is a connected topological space.

(c) We have similar argument as in part (b).

(d) Let $\{V^{sm}(N_\alpha) \mid \alpha \in \Lambda\}$ be a family of closed subset of $Min_R(M)$ such that $\bigcap_{\alpha\in\Lambda} V^{sm}(N_\alpha) = \emptyset$, where N_α is a submodule of M for every $\alpha \in \Lambda$. Then $\{\phi(V^{sm}(N_\alpha)) \mid \alpha \in \Lambda\}$ is a family of closed subset of $Max(\overline{R})$ because ϕ is closed by Lemma 2.3 (b). Since ϕ is surjective, it is easy to see that $\bigcap_{\alpha\in\Lambda} \phi(V^{sm}(N_\alpha)) = \emptyset$ (Note that if $\overline{m} \in \bigcap_{\alpha\in\Lambda} \phi(V^{sm}(N_\alpha))$, then since ϕ is surjective, there exists $Q \in Min_R(M)$ such that $\overline{Ann_R(Q)} = \overline{m}$. Hence $Q \in \bigcap_{\alpha\in\Lambda} V^{sm}(N_\alpha)$, a contradiction). As $Max(\overline{R})$ is quasi-compact by [5, Exercise 7, p. 64], there exists a finite subset Γ of Λ such that $\bigcap_{\alpha\in\Gamma} \phi(V^{sm}(N_\alpha)) = \emptyset$. This implies that $\bigcap_{\alpha\in\Gamma} V^{sm}(N_\alpha) = \emptyset$ and hence $Min_R(M)$ is quasi-compact.

(e) Let $Z_0 \supsetneq Z_1 \supsetneq \cdots \supsetneq Z_n$ be a descending chain of irreducible closed subset of $Min_R(M)$. Then by Theorem 2.16 (a), for $i(1 \leq i \leq n)$, there exists submodule L_i of M such that $Ann_R(L_i)$ is a J-radical prime ideal of R and $Z_i = V^{sm}(L_i)$. It follows that

$$V^m(\overline{Ann_R(L_0)}) \supsetneq V^m(\overline{Ann_R(L_1)}) \supsetneq \cdots \supsetneq V^m(\overline{Ann_R(L_n)})$$

is a descending chain of irreducible closed subset of $Max(\overline{R})$ by Remark 2.15. Hence $dim(Min_R(M), \tau^{sm}) \leq dim(Max(\overline{R}), \tau)$. Now let $A_0 \supsetneq A_1 \supsetneq \cdots \supsetneq A_t$ be a descending chain of irreducible closed subset of $Max(\overline{R})$. By Remark 2.15, for each $i(1 \leq i \leq t)$, there exists a J-radical prime ideal $\overline{p_i}$ of $\overline{R}$ such that $A_i = V(\overline{p_i})$. This yields that $p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_t$ is an ascending chain of J-radical prime ideal of R. Since M is Min-surjective, by Lemma 2.14 (b), for every p_i $(1 \leq i \leq t)$, there exists a submodule Q_i of M such that $p_i = Ann_R(Q_i)$. Hence by Theorem 2.16 (a),

$$V^{sm}(Q_0) \supsetneq V^{sm}(Q_1) \supsetneq \cdots \supsetneq V^{sm}(Q_t)$$

is a descending chain of irreducible closed subset of $Min_R(M)$. It follows that $dim(Min_R(M), \tau^{sm}) \geq dim(Max(\overline{R}))$ and the proof is completed. □

Corollary 2.18. *Let M be a Min-surjective R-module. Then the following hold.*

(a) If R is Noetherian, then $Min_R(M)$ is a Noetherian topological space.

(b) If D is the family of all J-radical prime ideal of R, then we have

$$dim(Min_R(M), \tau^{sm}) = sup\{n \mid p_0 \subsetneq \cdots \subsetneq p_n \text{ is an ascending chain of } D\}.$$

Proof. (a) This follows from Theorem 2.17 (a).

(b) Apply the technique of Theorem 2.17 (e). □

We recall that a topological space X is spectral if it is homeomorphic to the prime spectrum $Spec(S)$ of some ring S, endowed with the Zariski topology (see [8]).

Definition 2.19. We say that a topological space W is a *Max-spectral space* if W is homeomorphic with the maximal ideal space of some ring S (with the topology inherited from $Spec(S)$).

Remark 2.20. Max-spectral spaces have been characterized by Hochster [8, p. 57, Proposition 11] as the topological spaces W which satisfy the following conditions:

(a) W is a T_1-space.

(b) W is quasi-compact.

Proposition 2.21. *Let M be an R-module. Then the following are equivalent.*

(a) M is Min-injective.

(b) $Min_R(M)$ is a T_0-space.

(c) $Min_R(M)$ is a T_1-space.

Proof. (c) $\Rightarrow$ (b). This is clear.
(b) $\Rightarrow$ (a). We assume that M is not Min-injective. Hence there exist $S_1, S_2 \in Min_R(M)$ such that $\overline{Ann_R(S_1)} = \overline{Ann_R(S_2)}$ and $S_1 \neq S_2$. Since $(Min_R(M), \tau^{sm})$ is T_0, there exists a submodule N of M such that $S_1 \in Min_R(M) \backslash V^{sm}(N)$ and $S_2 \notin Min_R(M) \backslash V^{sm}(N)$. But this is a contradiction because $Ann_R(S_1) = Ann_R(S_2)$

implies that $S_1, S_2 \in Min_R(M) \backslash V^{sm}(N)$.
(a) $\Rightarrow$ (c). Let $S_1, S_2 \in Min_R(M)$, where $S_1 \neq S_2$. Clearly,

$$S_2 \notin Min_R(M) \backslash V^{sm}(S_2).$$

We show that $S_1 \in Min_R(M) \backslash V^{sm}(S_2)$. To see this, let $S_1 \notin Min_R(M) \backslash V^{sm}(S_2)$. Since $Ann_R(S_1), Ann_R(S_2) \in Max(R)$ and $Ann_R(S_2) \subseteq Ann_R(S_1)$, we have $Ann_R(S_2) = Ann_R(S_1)$. Since M is Min-injective, $S_1 = S_2$ which is a contradiction. Similarly, we have

$$S_1 \notin Min_R(M) \backslash V^{sm}(S\cdot) \text{ and } S_2 \notin Min_R(M) \backslash V^{sm}(S_1). \quad \square$$

Corollary 2.22. *Let M be an R-module.*

(a) If $Min_R(M)$ is a Max-spectral topological space, then M is Min-injective.

(b) If M is secondful and $Spec^s_R(M)$ is a Max-spectral topological space, then $Spec^s_R(M) = Min_R(M)$.

Proof. This follows from Remark 2.20, Proposition 2.21, and from [1, Theorem 2.10]. $\square$

Remark 2.23. (a) Let $(M_i)_{i \in I}$ be a family of R-modules and let $M = \bigoplus_{i \in I} M_i$. If M is an X^s-injective module, then

$$Spec^s(M) = \left\{ S \oplus \left(\bigoplus_{j \neq i \in I} (0) \right) \mid j \in I, S \in Spec^s(M_j) \right\}$$

[4, Proposition 3.13 (ii)].

(b) Let $M = \mathbb{Z}(p^\infty) \bigoplus \left(\bigoplus_{p \neq q \in \mathbb{P}} \mathbb{Z}_q \right)$, where $\mathbb{P}$ is the set of all prime integers. Then M is a secondful X^s-injective $\mathbb{Z}$-module and so it is a spectral space by [1, Theorem 6.5]. But

$$\begin{aligned} Spec^s_{\mathbb{Z}}(M) = & \left\{ \mathbb{Z}(p^\infty) \oplus \left(\oplus_{p \neq q \in \mathbb{P}} (0) \right), (1/p + \mathbb{Z}) \oplus \left(\oplus_{q \neq p \in \mathbb{P}} (0) \right) \right\} \\ & \bigcup \left\{ (0) \oplus \mathbb{Z}_q \mid q \in \mathbb{P}, q \neq p \right\}. \end{aligned}$$

and

$$Min_{\mathbb{Z}}(M) = \left\{ (1/p + \mathbb{Z}) \oplus \left(\oplus_{q \neq p \in \mathbb{P}} (0) \right), (0) \oplus \mathbb{Z}_q \mid q \in \mathbb{P}, q \neq p \right\}$$

by Remark 2.23 (a). This shows that part (b) in Corollary 2.22 is not valid in general if the word "max-spectral" is replaced with "spectral".

Example 2.24. (a) $Min_{\mathbb{Z}}\left(\bigoplus_{i=1}^n \mathbb{Z}_{p_i} \right)$ is a Max-spectral topological space by Remark 2.23 (a) and Remark 2.20.

(b) $Min_{\mathbb{Z}}(\mathbb{Z}_p \oplus \mathbb{Z}_p)$ is not a max-spectral topological space because $\mathbb{Z}_p \oplus (0)$ and $(0) \oplus \mathbb{Z}_p$ are minimal submodules of the $\mathbb{Z}$-module $\mathbb{Z}_p \oplus \mathbb{Z}_p$ with $Ann_{\mathbb{Z}}(\mathbb{Z}_{|} \oplus (0)) = Ann_{\mathbb{Z}}((0) \oplus \mathbb{Z}_p) = p\mathbb{Z}$, while $\mathbb{Z}_p \oplus (0) \neq (0) \oplus \mathbb{Z}_p$. Thus $Min_{\mathbb{Z}}(\mathbb{Z}_p \oplus \mathbb{Z}_p)$ is not Max-spectral by Corollary 2.22 (a).

Let M be an R-module such that $Min_R(M)$ is a Max-spectral topological space. For a submodule N of M, it is natural to ask the following question: Is $Min_R(N)$ a Max-spectral topological space? In Proposition 2.25 (b), we give a positive answer to this question under some additional conditions.

Proposition 2.25. *Let M be an R-module and let N be a submodule of M. Then the following hold. Let $Min_R(M)$ be a Max-spectral space. Then $Min_R(N)$ is a Max-spectral space in the following cases:*

(a) The subspace $Min_R(N)$ of $Min_R(M)$ is closed.

(b) R is a ring such that the intersection of every infinite collection of maximal ideals of R is zero (for example, when R is PID or one dimensional Noetherian domain).

Proof. (a) By part (a) and Remark 2.20.

(b) This follows from Proposition 2.4, Remark 2.20, and part (a). □

The next theorem is an important result about an R-module M for which $Min_R(M)$ is Max-spectral.

Theorem 2.26. *Let M be a Min-injective R-module. Then $Min_R(M)$ is a Max-spectral topological space in each of the following cases.*

(a) M is Min-surjective.

(b) $Im(\phi)$ is quasi-compact, where $\phi : Min_R(M) \to Max(\overline{R})$ is the natural map of $Min_R(M)$.

(c) $Ann_R(M)$ is a maximal ideal of R.

(d) $Min_R(M)$ is a finite set.

(e) $Max(R)$ is a finite set.

(f) $Max(\overline{R})$ is Noetherian, in particular when R is Noetherian.

(g) The intersection of every infinite of maximal ideals of R is zero, in particular when R is PID or one dimensional Noetherian domain.

(h) The descending chain condition for Socle submodules of M holds.

Proof. (a) This is clear because the natural map of $Min_R(M)$ is a homeomorphism by Lemma 2.3.

(b) By Lemma 2.3 (a), $\phi\mid_{Im(\phi)}$ is a homeomorphism because

$$V^m(\overline{Ann_R(N)}) \cap Im(\phi) = \phi(\phi^{-1}(V^m(\overline{Ann_R(N)}))) = \phi(V^{sm}(N)).$$

Hence, $(Min_R(M), \tau^{sm})$ is a Max-spectral space by Remark 2.20 and Proposition 2.21.

(c) We claim that $Min_R(M)$ has at most one element. To see this let $S_1, S_2 \in Min_R(M)$. Then since $Ann_R(M)$ is a maximal ideal, we have $Ann_R(M) = Ann_R(S_1) = Ann_R(S_2)$. It follows that $S_1 = S_2$ because M is Min-injective. The claim follows from Remark 2.20 and Proposition 2.21.

(d) This follows from Remark 2.20 and Proposition 2.21.

(e) Follows from part (d). (Note that $Min_R(M)$ is a finite set by hypothesis.)

(f) Since every subspace of a Noetherian topological space is Noetherian, $Im(\phi)$ is a Noetherian topological space and hence the claim follows part (b).

(g) By Proposition 2.4, Remark 2.20, and Proposition 2.21.

(h) By Theorem 2.9. □

A family $(M_i)_{i\in I}$ of R-modules is said to be *second-compatible* if for all $i \neq j$ in I, there doesn't exist a prime ideal p in R with $Spec^s_p(M_i)$ and $Spec^s_p(M_j)$ both nonempty [4, Definition 3.14]. (We recall that if M is an R-module, then $Spec^s_p(M) = \{S \in Spec^s(M) \mid Ann_R(S) = p\}$.)

Remark 2.27. (a) Let $(M_i)_{i\in I}$ be a family of R-modules and let $M = \bigoplus_{i\in I} M_i$. Then M is an X^s-injective R-module if and only if $(M_i)_{i\in I}$ is a family of second-compatible X^s-injective R-modules [4, Theorem 3.15].

(b) Let $M = \bigoplus_{p\in\mathbb{P}} \mathbb{Z}_p$, where $\mathbb{P}$ is the set of all prime integers. Then by Remark 2.23 (a) and part (a), M is an X^s-injective and $Spec^s_{\mathbb{Z}}(M) = \{\mathbb{Z}_p \oplus \big(\oplus_{p\neq q\in\mathbb{P}}(0)\big) \mid p \in \mathbb{P}\}$. On the other hand we see that $(Spec^s_{\mathbb{Z}}(M), \tau^s)$ is irreducible which contains no generic point. Hence $(Spec^s_{\mathbb{Z}}(M), \tau^s)$ is not spectral by Hochster characterizations. This shows that the words "Min-injective", "$Min_R(M)$", and "Max-spectral" in part (g) of Theorem 2.26, can not be replaced with "X^s-injective", "$Spec^s_R(M)$", and "spectral", respectively.

Corollary 2.28. *Let M be an R-module. Then $Min_R(M)$ is a Max-spectral topological space in each of the following cases.*

(a) M is secondful and X^s-injective.

(b) M is a comultiplication R-module with a finite length.

(c) M is X^s-injective and R is PID.

Proof. This follows from parts (a) and (g) of Theorem 2.26 and taking into account the following fact from [1, Example 3.10].
Fact. Let denote the class of comultiplication, X^s-injective, Min-injective, finite length, secondful, and Min-surjective modules respectively by A_1, A_2, A_3, A_4, A_5, and A_6, then

(i) $A_1 \subseteq A_2 \subseteq A_3$ and $A_4 \subseteq A_5$.

(ii) If M is X^s-injective, then $A_5 \subseteq A_6$. □

Acknowledgements

The authors would like to thank the referee for the careful reading of our manuscript and helpful comments that considerably improved this article.

REFERENCES

[1] H. Ansari-Toroghy and F. Farshadifar, *The Zariski topology on the second spectrum of a module*, Algebra Colloq. (4) 21 (2014), 671–688.

[2] H. Ansari-Toroghy and F. Farshadifar, *On the dual notion of prime submodules*, Algebra Colloq. 19 (Spec 1)(2012), 1109–1116.

[3] H. Ansari-Toroghy and F. Farshadifar, *The dual notion of multiplication modules*, Taiwanese J. Math. (4) 11 (2007), 1189–1201.

[4] H. Ansari-Toroghy and S. S. Pourmortazavi, *A module whose second spectrum has the surjective or injective natural map*, Eur. J. Pure Appl. Math. (2) 10 (2017), 211–230.

[5] F. Halter-Koch, *Ideal Systems: An Introduction to Multiplicative Ideal Theory*, Vol. 211, CRC Press, 1998.

[6] S. Ceken, M. Alkan, and P. F. Smith, *The dual notion of the prime radical of a module*, Journal of Algebra 392 (2013), 265–275.

[7] F. Farshadifar, *Modules with Noetherian second spectrum*, Journal of Algebra and related topics. (1) 1 (2013), 19–30.

[8] M. Hochster, *Prime ideal structure in commutative rings*, Trans. Amer. Math. Soc. 142 (1969,) 43–60.

[9] J. Ohm and R.L. Pendelton, *Rings with Noetherian spectrum*, Duke Math. J. 35 (1968), 631–640.

[10] S. Yassemi, *The dual notion of prime submodules*, Arch. Math. (Brno) 37 (2001), 273–278.

H. ANSARI-TOROGHY
Department of pure Mathematics, Faculty of mathematical Sciences
University of Guilan
P. O. Box 41335-19141 Rasht, Iran
e-mail: `ansari@guilan.ac.ir`

S. S. POURMORTAZAVI
Department of pure Mathematics, Faculty of mathematical Sciences
University of Guilan
P. O. Box 41335-19141 Rasht, Iran
e-mail: `mortazavi@phd.guilan.ac.ir`