CHARACTERIZATION OF THE ABSOLUTELY SUMMING OPERATORS IN A BANACH SPACE USING \( \mu \)-APPROXIMATE \( l_1 \) SEQUENCES

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In this paper we will give a characterization of 1-absolutely summing operators using \( \mu \)-approximate \( l_1 \) sequences. Exactly if \( (x_n)_{n=1}^{\infty} \) is \( \mu \)-approximate \( l_1 \), basic and normalized sequence in Banach space \( X \) then every bounded linear operator \( T \) from \( X \) into Banach space \( Y \) is 1-absolutely summing if and only if \( Y \) is isomorphic to Hilbert space.

Introduction.

In the following we will denote by \( X \) a Banach space with norm \( ||.|| \).

Some notations which are useful in the sequel.

Definition 1. [3] Let \( (x_i)_{i \in \mathbb{N}} \) be a sequence of unit vectors in a Banach space \( X \) (where \( I = \{1, 2, ..., n\} \) or \( I = \mathbb{N} \)), and let \( \mu \geq 0 \). We say that \( (x_i) \) is a \( \mu \)-approximate \( l_1 \) system if

\[
\left\| \sum_{i \in A} \pm x_i \right\| \geq k(A) - \mu
\]

for all finite sets \( A \subset I \) and for all choices of signs, where \( k(A) \) denotes the cardinal number of the set \( A \).

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Definition 2. [1] An operator \( T \in L(X, Y) \) is called \( p \)-absolutely summing if there is a constant \( K \) so that, for every choice of an integer \( n \) and vectors \( (x_i)_{i=1}^n \) in \( X \), we have

\[
\left( \sum_{i=1}^n \|Tx_i\|^p \right)^{\frac{1}{p}} \leq K \sup_{\|x^*\| \leq 1} \left( \sum_{i=1}^n |x^*(x_i)|^p \right)^{\frac{1}{p}}
\]

All other notations are like as in [1].

Theorem 1. [4] Every bounded linear operator \( T \) from \( l_1 \) into \( l_2 \) is absolutely summing and \( \pi_1(T) \leq KG \|T\| \).

Theorem 2. [5] (Ideal property of \( p \)-summing operators) Let \( 1 \leq p < \infty \) and let \( v \in \Pi_p(X, Y) \), then the composition of \( v \) with any bounded linear operator is \( p \)-summing.

Results.

Lemma 3. Let \( (x_i)_{i=1}^n \) be a sequence of unit vectors in Banach space \( X \). Then for any finite number of scalars \( \{a_1, a_2, \ldots, a_n\} \), the following is true

\[(1) \quad \|a_1 \cdot x_1 + \cdots + a_n \cdot x_n\| \leq \max_{1 \leq i \leq n} \{|a_i|\}\|x_1 + \cdots + x_n\|\]

Proof. In the sequel we will prove the above fact using the mathematical induction and it’s enough to prove it for two terms. Let us consider vectors \( x \) and \( y \) from \( X \) and \( a, b \) scalars such that \( a > b \), then from Hahn-Banach Theorem there exists \( x^* \in X^* \) such that

\[\|x^*\| = 1\]

and

\[x^*(a \cdot x + b \cdot y) = \|a \cdot x + b \cdot y\| .\]

On the other hand

\[|x^*(x + y)| \leq ||x^*|| \cdot ||x + y||\]

From the above relations we will have

\[|ax^*(x) + ax^*(y)| \leq |a| \cdot ||x + y|| \Rightarrow |ax^*(x) + ax^*(y) + bx^*(y) - bx^*(y)|\]
\[ \leq |a| \cdot |x + y| \]

Respectively

(2) \[ \|ax + by\| + (a - b)x^*(y) \leq |a| \cdot |x + y| \]

In the following we will distinguish two cases

I) \(0 < x^*(y) < 1\), then relation (1) follows directly from (2)

II) \(-1 < x^*(y) < 0\), then from relation (2) we will have this estimate

\[ \|ax + by\| - |(a - b)||x^*(y)| \leq \|ax + by\| + (a - b)x^*(y) \]

from which again it follows that (1) is valid.

**Lemma 4.** Let \((x_n)_{n \in \mathbb{N}}\) be sequence of normalized and \(\mu\)-approximate \(l_1\) vectors in Banach space \(X\). Then the relation

\[ \left\| \sum_{i=1}^{n} a_i x_i \right\| \geq K \sum_{i=1}^{n} |a_i| \]

is true for any finite sequence \((a_i)\) of scalars and \(K\) positive constant.

**Proof.** Let us start from the relation

\[ \left\| \sum_{i=1}^{n} a_i x_i \right\| = \left\| \sum_{i=1}^{n} |a_i| \cdot \text{sgn}(a_i) \cdot x_i \right\| \]

Then from Hanh-Banach Theorem there exists a functional \(f \in X^*\), such that

\[ f \left( \sum_{i=1}^{n} |a_i| \cdot \text{sgn}(a_i) \cdot x_i \right) = \left\| \sum_{i=1}^{n} |a_i| \cdot \text{sgn}(a_i) \cdot x_i \right\| \]

and \(\|f\| = 1\). From this it follows that

\[ \sum_{i=1}^{n} |a_i| \cdot f(\text{sgn}(a_i) \cdot x_i) = \left\| \sum_{i=1}^{n} |a_i| \cdot y_i \right\| \]

where \(y_i = \text{sgn}(a_i) \cdot x_i\). On the other hand, let us consider \(|a_i| \neq 0\), \(\forall i \in \{1, 2, \cdots, n\}\)

\[ \left\| \sum_{i=1}^{n} \pm x_i \right\| = \left\| \sum_{i=1}^{n} \pm a_i \cdot x_i \cdot \frac{1}{a_i} \right\| = \]
\[ \left\| \sum_{i=1}^{n} |a_i| \cdot x_i \cdot \frac{\text{sgn}(a_i)}{\pm a_i} \right\| \leq \max_{1 \leq i \leq n} \frac{1}{\pm a_i} \cdot \left\| \sum_{i=1}^{n} |a_i| |y_i| \right\| \]

(from lemma 3)

\[ \leq \max_{1 \leq i \leq n} \frac{1}{|a_i|} \cdot \sum_{i=1}^{n} |a_i| \cdot f(y_i) \leq \max_{1 \leq i \leq n} \frac{1}{|a_i|} \cdot \max_{1 \leq i \leq n} |a_i| \cdot \sum_{i=1}^{n} f(y_i) \]

(again from lemma 3) so it is true that

\[ \left\| \sum_{i=1}^{n} \pm x_i \right\| \leq M \cdot \sum_{i=1}^{n} f(y_i) \]

where \( M = \max_{1 \leq i \leq n} \frac{1}{|a_i|} \cdot \max_{1 \leq i \leq n} |a_i| \).

Now we will have this estimate

\[ M \cdot \sum_{i=1}^{n} f(y_i) \geq \left\| \sum_{i=1}^{n} \pm x_i \right\| \geq n - \mu \]

The last relation is possible if and only if

\[ f(y_i) \geq \frac{1 - \delta_i}{M}, \quad \forall i \in \{1, 2, \ldots, n\}, \sum_{i=1}^{n} \delta_i = \mu \]

and \( 0 < \delta_i < 1 \). From this it follows

\[ f(y_i) \geq \frac{1 - \delta_i}{M} \geq \frac{1 - \delta}{M}, \quad \text{where } \delta = \max_{1 \leq i \leq n} \delta_i. \]

Finally

\[ \left\| \sum_{i=1}^{n} a_i x_i \right\| = \sum_{i=1}^{n} |a_i| \cdot f(y_i) \geq \sum_{i=1}^{n} |a_i| \cdot \frac{1 - \delta}{M} = K \cdot \sum_{i=1}^{n} |a_i| \]

where \( K \) is constant and \( K = \frac{1 - \delta}{M} \).

**Theorem 5.** Let \((x_n)_{n \in \mathbb{N}}\) be a normalized, basic sequence in \( X \) that is a \( \mu \)-approximate \( l_1 \) system, too. Then every bounded linear operator from \( X \) into \( l_2 \) is 1-absolutely summing.
Proof. Let $H$ be the operator defined from $l_1$ into $X$ as follows

$$H : x = \sum_i a_i e_i \rightarrow \sum_i a_i x_i$$

from the above it follows that $H$ is bijective and bounded with its inverse. Boundedness follows from

$$\|Hx\| = \left\| \sum_i a_i x_i \right\| \leq \sum_i |a_i| \leq \frac{1}{K} \left\| \sum_i a_i x_i \right\| = \frac{1}{K} \|x\|$$

(from lemma 4). $H$ is onto, let $y = \sum_i b_i x_i$ any element from $X$, then $z = \sum_i b_i e_i$ belongs to $l_0^1$, indeed,

$$\left\| \sum_i b_i e_i \right\| = \sum_i |b_i| < \frac{1}{K} \left\| \sum_i b_i x_i \right\| < \infty,$$

from which it also follows that $H(z) = y$. From the above it follows that $H^{-1}$ also is bounded: let $t = \sum_i t_i x_i \in X$, then

$$\|H^{-1}t\| = \left\| H^{-1}\left(\sum_i t_i x_i\right) \right\| = \left\| \sum_i t_i e_i \right\| = \sum_i |t_i| \leq \frac{1}{K} \left\| \sum_i t_i x_i \right\|$$

$$= \frac{1}{K} \|t\|$$

Let us denote by $T$ any bounded linear operator from Banach space $X$ into $l_2$, then operator $K = T \cdot H$ is defined from $l_1$ into $l_2$ and is bounded, so 1-absolutely summing (from Theorem 1). Finally using the ideal properties of operators in Theorem 2 and the fact that $K \cdot H^{-1} = T$, it follows that $T$ is an absolutely summing operator.

Lemma 6. Let $x_n \in X$ be a normalized, basic sequence in $X$ that is, a $\mu$-approximate $l_1$ system, too. Then $(x_n)_{n \in \mathbb{N}}$ is an unconditional basic sequence in $X$.

Proof. It’s enough to prove that for any $x$, $y$ and any finite disjoint subsets $A, B \in \mathbb{N}$ relation

$$\|x + y\| \sim \|x - y\|$$

is true for $x \in \text{span} \{x_i : i \in A\}$ and $y \in \text{span} \{x_j : i \in B\}$, where $a \sim b$ means that there exists constant $c_1$ and $c_2$ such that $c_1 \cdot a \leq b \leq c_2 \cdot a$ (see [7]). From the definition of $\mu$-approximate $l_1$ sequences it follows that

$$\|x + y\| = \left\| \sum_{i \in A} a_i x_i + \sum_{i \in B} b_i x_i \right\| \geq \left\| \sum_{i \in A} a_i x_i \right\| - \left\| \sum_{i \in B} b_i x_i \right\| \geq \ldots$$
\[ K \left( \sum_{i \in A} |a_i| \right) - \sum_{i \in B} |b_i| \]

\[ \|x + y\| \leq \sum_{i \in A} |a_i| + \sum_{i \in B} |b_i| ; \]

from the other hand

\[ \|x - y\| = \left\| \sum_{i \in A} a_i x_i - \sum_{i \in B} b_i x_i \right\| \geq \left\| \sum_{i \in A} a_i x_i \right\| - \left\| \sum_{i \in B} b_i x_i \right\| \geq \]

\[ K \left( \sum_{i \in A} |a_i| \right) - \sum_{i \in B} |b_i| \]

and

\[ \|x + y\| \leq \sum_{i \in A} |a_i| + \sum_{i \in B} |b_i| \]

from the above relations it follows that \((x_n)_{n \in \mathbb{N}}\) is an unconditional sequence in \(X\).

**Theorem 7.** Let \((x_n)_{n \in \mathbb{N}}\) be normalized, basic and \(\mu\)-approximate \(l_1\) sequence in \(X\), such that every bounded linear operator \(T\) from \(X\) into \(Y\) is \(1\)-absolutely summing. Then \(X\) is isomorphic to \(l_1\) and \(Y\) is isomorphic to Hilbert space.

**Proof.** From Lemma 6 it follows that \((x_n)_{n \in \mathbb{N}}\) is an unconditional basis in \(X\). Now the proof of Theorem is similar to that of Theorem 4.2 in [6].

**Theorem 8.** Let \(X\) and \(Y\) be two infinite dimensional Banach spaces and \((x_n)_{n \in \mathbb{N}}\) basic, normalized and \(\mu\)-approximate \(l_1\) sequence in \(X\). Then every bounded linear operator \(T\) from \(X\) into \(Y\) is \(1\)-absolutely summing if and only if \(Y\) is isomorphic to a Hilbert space.

**Proof.** The forward direction follows from Theorem 7 and converse direction from Theorem 5.

**Proposition 9.** Let \((x_n)_{n \in \mathbb{N}}\) be basic, normalized and \(\mu\)-approximate \(l_1\) sequence of vectors in \(X\). Regardless of the measure \(\mu\), every bounded linear operator \(T\) from \(X\) into \(L_2(\mu)\), is \(1\)-absolutely summing.

**Proof.** Let us show that \(X\) is an \(L_{1,\nu}\) space for some \(\nu\). For any finite dimensional subspace \(E\) of \(X\), let us say that \(\dim E = n\), \(E = \text{span}\{x_i : i = 1, \ldots, n\}\). There exist a finite dimensional subspace \(F\) of \(X\), such that
$F = \text{span}\{x_i : i = 1, \ldots, n + 1\}, E \subset F$ and an isomorphism $H : x = \sum_{i=1}^{n+1} a_i x_i \in F \to \sum_{i=1}^{n+1} a_i e_i \in l_1^1$ such that $\|H\| \cdot \|H^{-1}\| \leq \nu$. Hence

$$\|Hx\| = \left\| \sum_i a_i x_i \right\| \leq \sum_i |a_i| \leq \frac{1}{K} \left\| \sum_i a_i x_i \right\| = \frac{1}{K} \|x\|$$

so $\|H\| \leq \frac{1}{K}$ and a similar estimate $\|H^{-1}\| \leq \frac{1}{K}$ holds, where $K$ is like as in Lemma 4. $\nu = \left(\frac{1}{K}\right)^2 > 1$, because

$$M = \max_{1 \leq i \leq n} \frac{1}{|a_i|} \cdot \max_{1 \leq i \leq n} |a_i| = \frac{\max_{1 \leq i \leq n} |a_i|}{\min_{1 \leq i \leq n} |a_i|} \geq 1,$$

$K = \frac{1}{\min_{1 \leq i \leq n} |a_i|} \leq 1$ and with this was proved that $X$ is an $L_{1,\nu}$-space. Now proof of the Theorem follows from Theorems 3.1, 3.2 and 3.4 in [5].

**Proposition 10.** Let $(x_n)_{n \in \mathbb{N}}$ be basic, normalized and $\mu$-approximate $l_1$ sequence of vectors in $X$. Then every infinite dimensional subspace $Y$ of $X$ is isomorphic to $X$ and complemented in $X$.

**Proof.** Let $H$ be an operator defined from the Banach space $X$ into the space $l_1$ by

$$H : x = \sum_i a_i x_i \to \sum_i a_i e_i.$$ 

This operator is invertible (exactly as in Theorem 5). Let $Y$ be any infinite dimensional subspace of $X$ and let us denote by $Y_1 = H(Y)$, a subspace of $l_1$. From the decomposition method of Pelczynski (see [2]) it follows that

$$l_1 = Y_1 \oplus B$$

for some Banach space $B$. Let $x \in X$, then $H(x) = y \in l_1$ and $y$ has unique representation

$$y = a + b$$

for suitable $a \in Y_1$ and $b \in B$. From this there is a $a_1 \in Y, H(a_1) = a$

$$y = H(a_1) + b \Rightarrow H^{-1}(y) = H^{-1}(H(a_1) + H^{-1}(b)) \Rightarrow$$

$$x = a_1 + H^{-1}(b)$$
and the last representation of \( x \) is unique, because if we will use another one \( x = a'_1 + H^{-1}(b') \), then \( H(x) = H(a'_1) + b' \Rightarrow \\
(5) \ y = H(a'_1) + b' \\
But relation (5) is in contradiction with relation (3). So every \( x \in X \) has unique representation through space \( Y \), and we can use notation \\
\( X = Y \oplus C \)

for some Banach space \( C \), with \( Y \) isomorphic to \( X \). \( H(Y) = Y_1 \) is isomorphic to \( l_1 \); let us denote by \( A \) that isomorphism between them, then \( A(l_1) = AH(X) = Y_1 \Rightarrow AH(X) = H(Y) \) and from this follows that \( H^{-1} \cdot A \cdot H \) is isomorphism between spaces \( X \) and \( Y \), with which was proved proposition.

**Corollary 11.** Let \((x_n)_{n \in \mathbb{N}}\) be basic, normalized and \( \mu \)-approximate \( l_1 \) sequence of vectors in \( X \). Then \( X \) is a prime space.

**Proof.** of corollary follows directly from the above proposition.

**REFERENCES**


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