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CHARACTERIZATION OF THE ABSOLUTELY SUMMING OPERATORS IN A BANACH SPACE USING μ-APPROXIMATE l_1 SEQUENCES

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In this paper we will give a characterization of 1-absolutely summing operators using μ-approximate l_1 sequences. Exactly if $(x_n)_{n=1}^{\infty}$ is μ-approximate l_1, basic and normalized sequence in Banach space X then every bounded linear operator T from X into Banach space Y is 1-absolutely summing if and only if Y is isomorphic to Hilbert space

Introduction.

In the following we will denote by X a Banach space with norm $||.||$.
Some notations which are usefull in the sequel.

Definition 1. [3] Let $(x_i)_{i\in\mathbb{N}}$ be a sequence of unit vectors in a Banach space X (where $I = \{1, 2, ..., n\}$or$I = N$), and let $\mu \geq 0$. We say that (x_i) is a μ-approximate l_1 system if

$$\left\| \sum_{i\in A} \pm x_i \right\| \geq k(A) - \mu$$

for all finite sets $A \subset I$ and for all choices of signs, where k(A) denotes the cardinal number of the set A.

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Definition 2. [1] An operator $T \in L(X, Y)$ is called p-absolutely summing if there is a constant K so that, for every choice of an integer n and vectors $(x_i)_{i=1}^n$ in X, we have

$$\left(\sum_{i=1}^{n} \left\| T x_i \right\|^p \right)^{\frac{1}{p}} \leq K \sup_{||x^*|| \leq 1} \left(\sum_{i=1}^{n} |x^*(x_i)|^p \right)^{\frac{1}{p}}$$

All other notations are like as in [1].

Theorem 1. [4] *Every bounded linear operator T from l_1 into l_2 is absolutely summing and $\pi_1(T) \leq K_G ||T||$.*

Theorem 2. [5] (Ideal property of p-summing operators) *Let $1 \leq p < \infty$ and let $v \in \Pi_p(X, Y)$, then the composition of v with any bounded linear operator is p-summing .*

Results.

Lemma 3. *Let $(x_i)_{i=1}^n$ be a sequence of unit vectors in Banach space X. Then for any finite number of scalars $\{a_1, a_2, \ldots, a_n\}$, the following is true*

$$\left\| a_1 \cdot x_1 + \cdots + a_n \cdot x_n \right\| \leq \max_{1 \leq i \leq n} \{ |a_i| \} \left\| x_1 + \cdots + x_n \right\| \tag{1}$$

Proof. In the sequel we will prove the above fact using the mathematical induction and it's enough to prove it for two terms. Let us consider vectors x and y from X and a, b scalars such that $a > b$, then from Hahn-Banach Theorem there exists $x^* \in X^*$ such that

$$\left\| x^* \right\| = 1$$

and

$$x^*(a \cdot x + b \cdot y) = \left\| a \cdot x + b \cdot y \right\|.$$

On the other hand

$$|x^*(x + y)| \leq ||x^*|| \cdot ||x + y||$$

From the above relations we will have

$$|ax^*(x) + ax^*(y)| \leq |a| \cdot ||x + y|| \Rightarrow |ax^*(x) + ax^*(y) + bx^*(y) - bx^*(y)|$$

$$\leq |a| \cdot ||x + y||$$

Respectively

$$\left| \|ax + by\| + (a - b)x^*(y) \right| \leq |a| \cdot ||x + y|| \tag{2}$$

In the following we will distinguish two cases
I) $0 < x^*(y) < 1$,then relation (1) follows directly from (2)
II)$-1 < x^*(y) < 0$,then from relation (2) we will have this estimate

$$||ax + by|| - |(a - b)||x^*(y)| \leq \left| ||ax + by|| + (a - b)x^*(y) \right|$$

from which again it follows that (1) is valid .

Lemma 4. *Let $(x_n)_{n \in \mathbb{N}}$ be sequence of normalized and μ-approximate l_1 vectors in Banach space X. Then the relation*

$$\left\| \sum_{i \leq n} a_i x_i \right\| \geq K \sum_{i \leq n} |a_i|$$

is true for any finite sequence (a_i) of scalars and K positive constant.

Proof. Let us start from the relation

$$\left\| \sum_{i=1}^{n} a_i x_i \right\| = \left\| \sum_{i=1}^{n} |a_i| \cdot sgn(a_i) \cdot x_i \right\|$$

Then from Hanh-Banach Theorem there exists a functional $f \in X^*$, such that

$$f\left(\sum_{i=1}^{n} |a_i| \cdot sgn(a_i) \cdot x_i \right) = \left\| \sum_{i=1}^{n} |a_i| \cdot sgn(a_i) \cdot x_i \right\|$$

and $\| f \| = 1$. From this it follows that

$$\sum_{i=1}^{n} |a_i| \cdot f(sgn(a_i) \cdot x_i) = \left\| \sum_{i=1}^{n} |a_i| y_i \right\|$$

where $y_i = sgn(a_i) \cdot x_i$. On the other hand ,let us consider $|a_i| \neq 0$, $\forall i \in \{1, 2, \cdots, n\}$

$$\left\| \sum_{i=1}^{n} \pm x_i \right\| = \left\| \sum_{i=1}^{n} \pm a_i \cdot x_i \cdot \frac{1}{a_i} \right\| =$$

$$= \left\| \sum_{i=1}^{n} |a_i| \cdot x_i \cdot \frac{sgn(a_i)}{\pm a_i} \right\| \leq \max_{1 \leq i \leq n} \frac{1}{|\pm a_i|} \cdot \left\| \sum_{i=1}^{n} |a_i| y_i \right\|$$

(from lemma 3)

$$= \max_{1 \leq i \leq n} \frac{1}{|a_i|} \cdot \sum_{i=1}^{n} |a_i| \cdot f(y_i) \leq \max_{1 \leq i \leq n} \frac{1}{|a_i|} \cdot \max_{1 \leq i \leq n} |a_i| \sum_{i=1}^{n} f(y_i)$$

(again from lemma 3) so it is true that

$$\left\| \sum_{i=1}^{n} \pm x_i \right\| \leq M \cdot \sum_{i=1}^{n} f(y_i)$$

where $M = \max_{1 \leq i \leq n} \frac{1}{|a_i|} \cdot \max_{1 \leq i \leq n} |a_i|$.

Now we will have this estimate

$$M \cdot \sum_{i=1}^{n} f(y_i) \geq \left\| \sum_{i=1}^{n} \pm x_i \right\| \geq n - \mu$$

The last relation is possible if and only if

$$f(y_i) \geq \frac{1 - \delta_i}{M}, \quad \forall i \in \{1, 2, \cdots, n\}, \sum_{i=1}^{n} \delta_i = \mu$$

and $0 < \delta_i < 1$. From this it follows

$$f(y_i) \geq \frac{1 - \delta_i}{M} \geq \frac{1 - \delta}{M}, \text{ where } \delta = \max_{1 \leq i \leq n} \delta_i.$$

Finally

$$\left\| \sum_{i=1}^{n} a_i x_i \right\| = \sum_{i=1}^{n} |a_i| \cdot f(y_i) \geq \sum_{i=1}^{n} |a_i| \cdot \frac{1 - \delta}{M} = K \cdot \sum_{i=1}^{n} |a_i|$$

where K is constant and $K = \frac{1-\delta}{M}$.

Theorem 5. *Let $(x_n)_{n \in \mathbb{N}}$ be a normalized, basic sequence in X that is,a μ-approximate l_1 system, too. Then every bounded linear operator from X into l_2 is 1-absolutely summing .*

Proof. Let H be the operator defined from l_1 into X as follows

$$H : x = \sum_i a_i e_i \to \sum_i a_i x_i$$

from the above it follows that H is bijective and bounded with it's inverse. Boundedness follows from

$$\|Hx\| = \left\|\sum_i a_i x_i\right\| \le \sum_i |a_i| \le \frac{1}{K}\left\|\sum_i a_i x_i\right\| = \frac{1}{K}\|x\|$$

(from lemma 4). H is onto ,let $y = \sum_i b_i x_i$ any element from X, then $z = \sum_i b_i e_i$ belongs to l_1^0, indeed,

$$\left\|\sum_i b_i e_i\right\| = \sum_i |b_i| < \frac{1}{K}\left\|\sum_i b_i x_i\right\| < \infty,$$

from which it also follows that $H(z) = y$. From the above it follows that H^{-1} also is bounded: let $t = \sum_i t_i x_i \in X$, then

$$\|H^{-1}t\| = \left\|H^{-1}\left(\sum_i t_i x_i\right)\right\| = \left\|\sum_i t_i e_i\right\| = \sum_i |t_i| \le \frac{1}{K}\left\|\sum_i t_i x_i\right\|$$
$$= \frac{1}{K}\|t\|$$

Let us denote by T any bounded linear operator from Banach space X into l_2, then operator $K = T \cdot H$ is defined from l_1 into l_2 and is bounded, so 1-absolutely summing (from Theorem 1). Finaly using the ideal propertis of operators in Theorem 2 and the fact that $K \cdot H^{-1} = T$,it follows that T is an absolutely summing operator.

Lemma 6. *Let $(x_n)_{n\in\mathbb{N}}$ be a normalized, basic sequence in X that is, a μ-approximate l_1 system, too. Then $(x_n)_{n\in\mathbb{N}}$ is an unconditional basic sequence in X.*

Proof. It's enough to prove that for any x, y and any finite disjoint subsets $A, B \in \mathbb{N}$ relation

$$\|x + y\| \sim \|x - y\|$$

is true for $x \in$ span $\{x_i : i \in A\}$ and $y \in$ span $\{x_i : i \in B\}$, where $a \sim b$ means that there exists constant c_1 and c_2 such that $c_1 \cdot a \le b \le c_2 \cdot a$ (see [7]). From the definition of μ-approximate l_1 sequences it follows that

$$\|x + y\| = \left\|\sum_{i\in A} a_i x_i + \sum_{i\in B} b_i x_i\right\| \ge \left\|\sum_{i\in A} a_i x_i\right\| - \left\|\sum_{i\in B} b_i x_i\right\| \ge$$

$$K\Big(\sum_{i\in A}|a_i|\Big)-\sum_{i\in B}|b_i|$$

$$\|x+y\|\leq\sum_{i\in A}|a_i|+\sum_{i\in B}|b_i|;$$

from the other hand

$$\|x-y\|=\Big\|\sum_{i\in A}a_ix_i-\sum_{i\in B}b_ix_i\Big\|\geq\Big\|\sum_{i\in A}a_ix_i\Big\|-\Big\|\sum_{i\in B}b_ix_i\Big\|\geq$$

$$K\Big(\sum_{i\in A}|a_i|\Big)-\sum_{i\in B}|b_i|$$

and

$$\|x+y\|\leq\sum_{i\in A}|a_i|+\sum_{i\in B}|b_i|$$

from the above relations it follows that $(x_n)_{n\in\mathbb{N}}$ is an unconditional sequence in X.

Theorem 7. *Let $(x_n)_{n\in\mathbb{N}}$ be normalized ,basic and μ-approximate l_1 sequence in X, such that every bounded linear operator T from X into Y is 1-absolutely summing. Then X is isomorphic to l_1 and Y is isomorphic to Hilbert space .*

Proof. From Lemma 6 it follows that $(x_n)_{n\in\mathbb{N}}$ is an unconditional basis in X. Now the proof of Theorem is similar to that of Theorem 4.2 in [6].

Theorem 8. *Let X and Y be two infinite dimensional Banach spaces ,and $(x_n)_{n\in\mathbb{N}}$ basic, normalized and μ-approximate l_1 sequence in X. Then every bounded linear operator T from X into Y is 1-absolutely summing if and only if Y is isomorphic to a Hilbert space.*

Proof. The forward direction follows from Theorem 7 and converse direction from Theorem 5.

Proposition 9. *Let $(x_n)_{n\in\mathbb{N}}$ be basic, normalized and μ-approximate l_1 sequence of vectors in X. Regardless of the measure μ, every bounded linear operator T from X into $L_2(\mu)$, is 1-absolutely summing.*

Proof. Let us show that X is an $L_{1,\upsilon}$ space for some υ. For any finite dimensional subspace E of X, let us say that $\dim E = n$, $E = span\{x_i : i = 1,\cdots,n\}$. There exist a finite dimensional subspace F of X, such that

$F = span\{x_i : i = 1, \dots, n+1\}$, $E \subset F$ and an isomorphism $H : x = \sum_{i=1}^{n+1} a_i x_i \in F \to \sum_{i=1}^{n+1} a_i e_i \in l_1^{\dim F}$ such that $\|H\| \cdot \|H^{-1}\| \leq \upsilon$. Hence

$$\|Hx\| = \left\|\sum_i a_i x_i\right\| \leq \sum_i |a_i| \leq \frac{1}{K}\left\|\sum_i a_i x_i\right\| = \frac{1}{K}\|x\|$$

so $\|H\| \leq \frac{1}{K}$ and a similar estimate $\|H^{-1}\| \leq \frac{1}{K}$ holds, where K is like as in Lemma 4. $\upsilon = (\frac{1}{K})^2 > 1$, because

$$M = \max_{1 \leq i \leq n} \frac{1}{|a_i|} \cdot \max_{1 \leq i \leq n} |a_i| = \frac{\max_{1 \leq i \leq n} |a_i|}{\min_{1 \leq i \leq n} |a_i|} \geq 1,$$

$K = \frac{1-\delta}{M} \leq 1$ and with this was proved that X is an $L_{1,\upsilon}$ -space. Now proof of the Theorem follows from Theorems 3.1, 3.2 and 3.4 in [5].

Proposition 10. *Let $(x_n)_{n \in \mathbb{N}}$ be basic, normalized and μ-approximate l_1 sequence of vectors in X. Then every infinite dimensional subspace Y of X is isomorphic to X and complemented in X.*

Proof. Let H be an operator defined from the Banach space X into the space l_1 by

$$H : x = \sum_i a_i x_i \to \sum_i a_i e_i.$$

This operator is invertible (exactly as in Theorem 5). Let Y be any infinite dimensional subspace of X and let us denote by $Y_1 = H(Y)$, a subspace of l_1. From the decomposition method of Pelczynski (see [2]) it follows that

$$l_1 = Y_1 \oplus B$$

for some Banach space B. Let $x \in X$, then $H(x) = y \in l_1$ and y has unique representation

$$y = a + b \tag{3}$$

for suitable $a \in Y_1$ and $b \in B$. From this there is a $a_1 \in Y$, $H(a_1) = a$

$$y = H(a_1) + b \Rightarrow H^{-1}(y) = H^{-1}(Ha_1) + H^{-1}(b) \Rightarrow$$

$$x = a_1 + H^{-1}(b) \tag{4}$$

and the last representation of x is unique, because if we will use another one $x = a_1^{'} + H^{-1}(b^{'})$, then $H(x) = H(a_1^{'}) + b^{'} \Rightarrow$

$$y = H(a_1^{'}) + b^{'} \tag{5}$$

But relation (5) is in contradiction with relation (3). So every $x \in X$ has unique representation through space Y, and we can use notation

$$X = Y \oplus C$$

for some Banach space C, with Y isomorphic to X. $H(Y) = Y_1$ is isomorphic to l_1; let us denote by A that isomorphism between them, then $A(l_1) = AH(X) = Y_1 \Rightarrow AH(X) = H(Y)$ and from this follows that $H^{-1} \cdot A \cdot H$ is isomorphism between spaces X and Y, with which was proved proposition.

Corollary 11. *Let $(x_n)_{n\in\mathbb{N}}$ be basic, normalized and μ-approximate l_1 sequence of vectors in X. Then X is a prime space.*

Proof. of corollary follows directly from the above proposition.

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