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LOW CODIMENSION STRATA OF THE SINGULAR LOCUS OF MODULI OF LEVEL CURVES

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We further analyze the moduli space of stable curves with level structure provided by Chiodo and Farkas in [2]. Their result builds upon Harris and Mumford analysis of the locus of singularities of the moduli space of curves and shows in particular that for levels 2, 3, 4, and 6 the locus of noncanonical singularities is completely analogous to the locus described by Harris and Mumford, it has codimension 2 and arises from the involution of elliptic tails carrying a trivial level structure. For the remaining levels (5, 7, and beyond), the picture also involves components of higher codimension. We show that there exists a component of codimension 3 for levels $\ell = 5$ and $\ell \geqslant 7$ with the only exception of level 12. We also show that there exists a component of codimension 4 for $\ell = 12$.

1. Introduction

Let $\mathcal{M}_g$ be the moduli space of smooth curves of genus g. We denote by $\overline{\mathcal{M}}_g$ the compactification of $\mathcal{M}_g$. Its objects are nodal curves satisfying Deligne-Mumford stability condition. Recall that to a stable curve one can attach a graph, the so-called stable graph. A graph is called stable if for all of its vertices v, the following inequality holds $2g_v - 2 + \ell_v > 0$, where ℓ_v is the valence, i.e

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the number of entering edges and g_v is the geometric genus of the normalization of the irreducible component corresponding to v.
Let $\mathcal{R}_{g,\ell}$ be the moduli space of level structures of all triples (C, L, ϕ), where C is a smooth curve of genus g equipped with a line bundle L and a trivialization morphism $\phi\colon L^{\otimes \ell} \xrightarrow{\sim} \mathcal{O}$. We refer to these as level-ℓ curves. We consider $\overline{\mathcal{R}}_{g,\ell}$, a compactification of the moduli space $\mathcal{R}_{g,\ell}$. Similarly to the moduli space of stable curves $\overline{\mathcal{M}}_g$ (see [6]), the locus $\overline{\mathcal{R}}_{g,\ell} \setminus \mathcal{R}_{g,\ell}$ can be described by the dual graph of nodal curves. We determine its vertices, V by connected components and its edges, E by nodes of nodal curves. The dual graph Γ is a stable graph decorated by $M = \{m_e\}_{e\in E}$ ($\mathbb{Z}_\ell$-valued 1-cochain from the set of branches of each node of C) lying in the kernel of the cochain homomorphism $\partial\colon C^1(\Gamma, \mathbb{Z}_\ell) \longrightarrow C^0(\Gamma, \mathbb{Z}_\ell)$. This means that M adds up to zero modulo ℓ along each circuit of Γ.
Recall that $\overline{\mathcal{M}}_g$ is locally isomorphic to $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)/\mathrm{Aut}(\mathsf{C})$ where $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)$ is the deformation space of the stack-theoretic curve. Note that a level structure $(\mathsf{C}, \mathsf{L}, \phi)$ is smooth if and only if each element of $\mathrm{Aut}(\mathsf{C})$ operates on $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)$ as a product of quasireflections (i.e., an automorphism whose fixed locus is hyperplane).
We consider the automorphisms which are given by twisting each node. If we specify a coefficient in $\mathbb{Z}_\ell$ for every edge, then every choice of coefficient determines an automorphism a of the stack-theoretic curve C. Note that the automorphisms preserve the line bundle if and only if the action of the automorphism a on M lies in the image of $\delta\colon C^0(\Gamma, \mathbb{Z}_\ell) \longrightarrow C^1(\Gamma, \mathbb{Z}_\ell)$ (see [2]).
Then we consider the existence of noncanonical singularities of $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)/\mathrm{Aut}(\mathsf{C})$. It is known that the noncanonical singularities occur if and only if there exists a junior automorphism (i.e., an automorphism less than one) on $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)$ over the group of automorphism mod out by quasireflections [2]. Using this fact, we prove that there exists a codimension 3 locus of noncanonical singularities for all levels $\ell = 5$ and $\ell \geqslant 7$ with the only exception of level 12. We also show that there exists a codimension 4 locus of noncanonical singularities for $\ell = 12$.

2. Preliminaries

Assume that k is an algebraically closed field. Assume k is $\mathbb{C}$, the field of complex number. Let us recall some useful definitions.

Definition 2.1. Let V be the set of connected components of normalization of stack-theoretic curve, C. Let E be the set of nodes of C. The pair (V, E) is called the dual graph of normalization of C, i.e. C^{v} (see for instance [1]).

Now for every nodal curve we are able to draw its dual graph. We define $C^0(\Gamma, \mathbb{Z}_\ell) = \{a : V \to \mathbb{Z}\} = \bigoplus_{v\in V} \mathbb{Z}$ as the set of $\mathbb{Z}$-valued functions on V. We define $C^1(\Gamma,\mathbb{Z}_\ell) = \{b : \mathbb{E} \to \mathbb{Z} \mid b(\bar{e}) = -b(e)\}$ (i.e. $\mathbb{E}$ is the set of branches of each node of C) as the set of antisymmetric $\mathbb{Z}_\ell$-valued functions on $\mathbb{E}$, where $\bar{e}$ and e are oriented edges with opposite orientations.

Let us recall that $\delta\colon C^0(\Gamma,\mathbb{Z}_\ell) \to C^1(\Gamma,\mathbb{Z}_\ell)$ is defined by sending a to δa, with $\delta a(e) = a(e_+) - a(e_-)$ and the map $\partial\colon C^1(\Gamma,\mathbb{Z}_\ell) \longrightarrow C^0(\Gamma,\mathbb{Z}_\ell)$ is defined by sending b to ∂b, with $\partial b(v) = \Sigma_{e\in E} b(e)$ [2].

Let a be an automorphism of stack-theoretic curve C and M be the multiplicity cochain, then we can define a as a multiple of $\gcd(M,\ell)$, where a is defined as follows:

$$\mathsf{a} \odot M = \frac{aM}{\gcd(M,\ell)},$$

(see [2, pages 35 and 36]).
We recall that, $\mathrm{age}(\mathsf{a}) = \Sigma_{e\in E} \left\{ \frac{\mathsf{a}(e)}{\ell} \right\}$.

Definition 2.2. Let $\mathrm{Aut}_C(\mathsf{C})$ be the group of automorphisms of C. We say that an automorphism a in $\mathrm{Aut}_C(\mathsf{C})$, which operates nontrivially on the curve, is junior on C if $0 < \mathrm{age}(\mathsf{a}) < 1$, (see [2, Definition 2.35]).

According to the Reid–Shepherd-Barron–Tai criterion, the scheme theoretic quotient V/G, where G is a finite group operates on V without quasireflection, has a noncanonical singularity at the origin if and only if the image of age_V intersects $]0,1[$, see [3–5]. As above, the point is that, the quotient $\mathbb{C}^{3g-3}/\mathrm{Aut}(\mathsf{C})$ has a noncanonical singularities if and only if there exists an element $\mathsf{a} \in \mathrm{Aut}_C(\mathsf{C})$ which is junior on C.

Theorem 2.3. *([2]) There exist a junior ghost* a *if the following conditions are satisfied:*

(i) $\mathrm{age}(\mathsf{a}) < 1$ *(i.e.,* a *is junior);*

(ii) $M = \Sigma_{i\in I} K_i$*, where* I *is a finite set of circuits (i.e.,* $M \in \mathrm{Ker}\,\partial$*);*

(iii) $\mathsf{a} \odot M(K) \equiv 0$ *for any circuit* K *(i.e.* $\mathsf{a} \odot M \in \mathrm{Im}\,\delta$*).*

3. Classifying the noncanonical singularities

In this section our aim is to analyse the existence of strata in the locus of noncanonical singularities of codimension 3. First of all, we show that there is a codimension 3 locus of noncanonical singularities for all levels 5,7, and higher except $\ell = 12$. This is done in three steps.

Step1. Let ℓ be a prime number bigger than 3. Consider a level curve whose dual graph has multiplicity M. Let $m_1 = m_3 = n$ and $m_2 = 2n$, where n can be chosen in $\mathbb{Z}/\ell$ (see Figure 1). Let $a_1 = a_3 = 1$ and $a_2 = \dfrac{\ell-1}{2}$ in a. Hence the sum of the values of $a \odot M$ along every circuit is zero modulo ℓ. Indeed $(a_1 \odot m_1) + (a_2 \odot m_2)$ and $(a_1 \odot m_1) - (a_3 \odot m_3)$ add up to zero modulo ℓ.

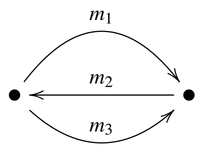

Figure 1. Graph M with two vertices and three edges.

On the other hand, since $\ell > 3$, $\mathrm{age}(\mathsf{a}) = \dfrac{a_1}{\ell} + \dfrac{a_2}{\ell} + \dfrac{a_3}{\ell} = \dfrac{\ell+3}{2\ell} < 1$.

Notice that if we find a junior a our problem for a given ℓ, then every multiple can also be solved similarly. In fact, if for a given ℓ we find m_1, m_2 and m_3 in M, as well as a_1, a_2 and a_3 in a which satisfy in our conditions, then we can set $\ell' = k\ell$, $m_1' = km_1, m_2' = km_2$ and $m_3' = km_3$ in M, also $a_1' = ka_1, a_2' = ka_2$ and $a_3' = ka_3$ in a for every integer k. Hence the sum along every circuit is zero modulo ℓ' and $(a_1' \odot m_1') + (a_2' \odot m_2')$, $(a_1' \odot m_1') - (a_3' \odot m_3')$ add up to zero modulo ℓ'.

According to the above argument, there exists a codimension 3 locus of noncanonical singularities for prime numbers $\ell > 3$. This settles the cases $\ell \neq 2^a 3^b$. We now focus on $\ell = 2^a 3^b$.

Step 2: if $\ell = 8$, take $m_1 = 1, m_2 = 3, m_3 = 2$ and $a_1 = a_2 = a_3 = 2$. Then, there exists a junior ghost automorphism.

If $\ell = 9$, take $m_1 = 1, m_2 = 2, m_3 = 1$ and $a_1 = a_3 = 1$ and $a_2 = 4$. Then, there exists a junior ghost automorphism.

Step 3. [2] show that there is no junior ghost a for any stable graph for which $M \in \mathrm{Ker}\ \partial$ and $\mathsf{a} \odot M \in \mathrm{Im}\ \delta$ for $\ell = 2, 3, 4, 6$. We also show that for $\ell = 12$ if there are only 3 edges. To do this we provide a table. In the first row, we write the possible values of M, i.e., $0, \dots, \ell - 1$. In the first column, we write the possible values of a, i.e., $0, \dots, \ell - 1$ such that a may take at an edge e of multiplicity M. We take $\mathsf{a}(e) = i$ if i satisfies the compatibility condition $\gcd(M, \ell) \mid i$. Then, we fill the slot in the ith row and jth column of the table with the corresponding values of $\mathsf{a} \odot M$ if and only if $\mathsf{a} = i$ is compatible with

$M = j$. For clarity we run the check for $\ell = 6$ as well.

$\ell = 6$	0	1	2	3	4	5
0	0	0	0	0	0	0
1		1				5
2		2	2		4	4
3		3		3		3
4		4	4		2	2
5		5				1

$\ell = 12$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1		1				5		7				11
2		2	2			10		2			10	10
3		3		3		3		9		9		9
4		4	4		4	8		4	8		8	8
5		5				1		11				7
6		6	6	6		6	6	6		6	6	6
7		7				11		1				5
8		8	8		8	4		8	4		4	4
9		9		9		9		3		3		3
10		10	10			2		10			2	2
11		11				7		5				1

Table 2. Multiplication tables for $\odot$ and ℓ= 6 and 12.

Notice that we fill the table with the corresponding values of a $\odot M$, if a is a multiple of M. Without loss of generality, we arrange a_1, a_2 and a_3 in unordered 3-tuple (a_1, a_2, a_3) and consider all possible cases such that $a_1 + a_2 + a_3 < \ell = 12$. Then, the following cases are obtained:

(1,2,3), (1,2,4), (1,2,5), (1,2,6), (1,2,7), (1,2,8), (1,3,4), (1,3,5), (1,3,6), (1,3,7), (1,4,5), (1,4,6), (2,3,4), (2,3,5), (2,3,6), (1,1,1), (1,2,1), (1,3,1), (1,4,1), (1,5,1), (1,6,1), (1,7,1), (1,8,1), (1,9,1), (2,1,2), (2,2,2), (2,3,2), (2,4,2), (2,5,2), (2,6,2), (2,7,2), (3,1,3), (3,2,3), (3,3,3), (3,4,3), (3,5,3), (4,1,4), (4,2,4), (4,3,4), (5,1,5).

On the other hand the action of automorphism a on M should be in the image of δ. Therefore, we look for two numbers in the table such that they are equal and we also look for the two numbers in the table such that the sum of them is $\ell = 12$ simultaneously. Then, the following cases of automorphism a remain:

(1,1,1), (1,5,1), (1,7,1), (2,2,2), (3,3,3), (5,1,5).

If take $(1,1,1)$, then we can choose m_1, m_2 and m_3 as the following list:
$(m_1 = 1, m_2 = 11, m_3 = 1)$, $(m_1 = 11, m_2 = 1, m_3 = 11)$, $(m_1 = 7, m_2 = 5, m_3 = 7)$ and $(m_1 = 5, m_2 = 7, m_3 = 5)$.
If take $(1,5,1)$ or $(3,3,3)$ or $(5,1,5)$, then we can choose m_1, m_2 and m_3 as the following list:
$(m_1 = 1, m_2 = 7, m_3 = 1)$, $(m_1 = 7, m_2 = 1, m_3 = 7)$, $(m_1 = 5, m_2 = 11, m_3 = 5)$ and $(m_1 = 11, m_2 = 5, m_3 = 11)$.
Finally If take $(1,7,1)$ or $(2,2,2)$, then we can choose m_1, m_2 and m_3 as the following list:
$(m_1 = 1, m_2 = 5, m_3 = 1), (m_1 = 5, m_2 = 1, m_3 = 5), (m_1 = 7, m_2 = 11, m_3 = 7)$ and $(m_1 = 11, m_2 = 7, m_3 = 11)$.
For these remaining cases, $m_1 + m_3 \neq m_2$. It means that M doesn't lie in the Ker ∂. This implies the claim that there is no junior ghost a for any stable graph for which $M \in$ Ker ∂ and $\mathsf{a} \odot M \in$ Im δ for $\ell = 12$.

Now we want to show there exists a codimension 4 locus of noncanonical singularities for $\ell = 12$. Hence it suffices to take $m_1 = 1, m_2 = 5, m_3 = m_4 = 2$ and $a_1 = a_2 = a_3 = a_4 = 2$. Then, there exists a junior ghost automorphism.

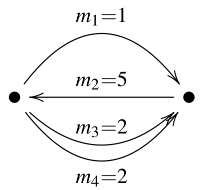

Figure 3. Graph M with two vertices and four edges.

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