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A UNIVERSAL INEQUALITY FOR RIESZ POTENTIALS ON DOMAINS ON N-SPHERES

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In this short note we give a universal inequality between the largest eigenvalue of the Riesz potentials and non-zero Neumann eigenvalue for domains on hemispheres of $\mathbb{S}^n$.

1. Introduction

Inequalities for the eigenvalues of the Laplace operator have been extensively studied during the past several decades. Pólya[9], Szegö[14], Payne[8], Weinberger[7] have done outstanding work on comparison theorems for the eigenvalues of the Laplace operator with variety of boundary conditions. Many of the classical results have been generalized to the case of Laplace-Beltrami operator on Riemannian manifolds (see[2], [4], [6], [5] and [12]). Very recently Rozenblum, Ruzhansky and Suragan established isoperimetric inequalities for the Riesz operators [11]. In this short note we address a question raised by J. Anderson, D. Khavinson and V. Lomonosov in [1] regarding the universal inequalities of the norm of certain potential operators.

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2. Main Results

For a bounded domain Ω with sufficiently smooth boundary, by $\lambda_1^N(\Omega)$ and $\lambda_1^D(\Omega)$ we denote the least non-zero Neumann eigenvalue and the least Dirichlet eigenvalue of Laplace operator on Ω, respectively.

Let $\Omega \subset \mathbb{R}^n$, $n = 3, 4, \cdots$ be a domain with piecewise smooth boundary. The Newtonian potential $\mathcal{N}_\Omega$ is the integral transform on $L^2(\Omega, dy)$ defined by

$$(\mathcal{N}_\Omega f)(x) = \frac{1}{(n-2)\omega_n} \int_\Omega \frac{f(y)}{\|x-y\|^{n-2}} dy,$$

where ω_n is the surface area of the unit sphere $\mathbb{S}^{n-1}$. In [1], the authors ask the following question:

"Does there exist a universal constant C such that

$$\|\mathcal{N}_\Omega\| \leq C \frac{1}{\lambda_1^D(\Omega)} \quad \text{for all bounded domains } \Omega \text{ with smooth boundary?"}$$

We provide a universal inequality for the Riesz potentials and in particular for the Newtonian potentials on domains in $n-$spheres in terms if the least non-zero eigenvalue of the Neumann Laplacian. Our result reveals more than what Anderson-Khavinson-Lomonosov[1] had suspected and gives a comparison between the norm of Riesz potentials and the least non-zero Neumann eigenvalue on any two domains of equal Riemannian measure. The similar results for the logarithmic and the Newtonian potentials in the Euclidean setting will be subject of investigation on a forthcoming article.

For a bounded domain $\Omega \subset \mathbb{S}^n$ and $0 < \alpha < n$, the Riesz transform $\mathcal{R}_{\alpha,\Omega}$: $L^2(\Omega, dy) \to L^2(\Omega, dy)$ is defined by

$$\mathcal{R}_{\alpha,\Omega} f(x) = \int_\Omega \frac{f(y)}{d^\alpha(x,y)} dy, \;\; f \in L^2(\Omega, dy),$$

where dy is the Riemannian measure on $\mathbb{S}^n$ and $d(x,y)$ is the geodesic distance between points x and y.

Let $\mathscr{C}$ be the class of all domains with smooth boundaries in $\mathbb{S}^n$ which are *proper subset* of only one hemisphere of $\mathbb{S}^n$.

Theorem 2.1. *For any two domains Ω and $\tilde{\Omega}$ belonging to the class $\mathscr{C}$ with $|\Omega| = |\tilde{\Omega}|$ the following inequality holds*

$$\|\mathcal{R}_{\alpha,\Omega}\| \leq c_{\alpha,n} \frac{1}{\lambda_1^N(\tilde{\Omega})},$$

where $c_{\alpha,n}$ only depends on α and n. In particular,

$$\|\mathcal{R}_{\alpha,\Omega}\| \leq c_{\alpha,n} \frac{1}{\lambda_1^N(\Omega)}.$$

Proof. Let Ω^* denote the geodesic ball(cap) in $\mathbb{S}^n$ with $|\Omega^*| = |\Omega|$. By the Rayleigh-Faber-Krahn inequality for the Riesz potentials (see [13]) we have

$$0 < \|\mathcal{R}_{\alpha,\Omega}\| \leq \|\mathcal{R}_{\alpha,\Omega^*}\|. \quad (1)$$

Since Ω^* is a geodesic ball then the quantity $\lambda_1^N(\Omega^*)\|\mathcal{R}_{\alpha,\Omega^*}\|$ depends only on n and α. If we denote this quantity with $c_{\alpha,n}$, then by the Szegö-Weinberger type inequality of Ashbaugh and Levine [3],

$$\|\mathcal{R}_{\alpha,\Omega^*}\| = c_{\alpha,n}\frac{1}{\lambda_1^N(\Omega^*)} \leq c_{\alpha,n}\frac{1}{\lambda_1^N(\tilde{\Omega})}. \quad (2)$$

The desired inequity follows from (1) and (2). □

We immediately obtain the following inequality for the Newtonian potentials.

Corollary 2.2. *For any two domains Ω and $\tilde{\Omega}$ belonging to the class $\mathscr{C}$ with $|\Omega| = |\tilde{\Omega}|$ the following inequality holds*

$$\|\mathcal{N}_\Omega\| \leq C_n\frac{1}{\lambda_1^N(\tilde{\Omega})},$$

where C_n only depends on n. In particular,

$$\|\mathcal{N}_\Omega\| \leq C_n\frac{1}{\lambda_1^N(\Omega)}.$$

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