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REAL CURVES WITH FIXED GONALITY AND EMPTY REAL LOCUS

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Here we give two existence theorems for smooth genus g real curves with fixed gonality and empty real locus.

1. Introduction.

For any a smooth and connected projective curve X of genus $g \geq 0$ let $X(\mathbb{R})$ denote its set of real points and $n(X)$ the set number of the connected component of $X(\mathbb{R})$. Hence $X(\mathbb{R})$ is the disjoint union of $n(X)$ circles. Set $a(X) = 1$ if $X(\mathbb{C})\backslash X(\mathbb{R})$ is connected and $a(X) = 0$ if $X(\mathbb{C})\backslash X(\mathbb{R})$ is not connected, i.e. if $X(\mathbb{C})\backslash X(\mathbb{R})$ has two connected components. The topological pair $(X(\mathbb{C}), X(\mathbb{R}))$ is uniquely determined by the pair of integers $(n(X), a(X))$ and such a pair of integers (n, a) is associated to some smooth real genus g curve if and only if either $a = 0$, $n \equiv g + 1 \pmod 2$ and $1 \leq n \leq g + 1$ or $a = 1$ and $0 \leq n \leq g$ ([1], Prop. 3.1). There are two real types of smooth real genus zero curves: $\mathbb{P}^1_{\mathbb{R}}$ (i.e. the one with a non-zero real point, i.e. the one whose real part is a circle) and the one, N, such that $N(\mathbb{R}) = \emptyset$ ([1]).

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Theorem 1. *Fix integers d, g such that $d \geq 2$, $g > 0$ and $d \equiv g+1 \pmod 2$. Then there exists a smooth and connected genus g real curve X such that $X(\mathbb{R}) = \emptyset$ and a real degree d morphism $f : X \to N$.*

We will also give a sketch of a proof of the following result.

Theorem 2. *Fix integers g, d such that $g \geq 0$, $d \geq 2$, and d is even. Then there exist aa smooth and connected genus g real curve X such that $X(\mathbb{R}) = \emptyset$ and a real degree d morphism $f : X \to \mathbb{P}^1_{\mathbb{R}}$.*

Obviously, the assumption " d even " in Theorem 2 is necessary, because $f^{-1}(P)(\mathbb{R}) \neq \emptyset$ for any $P \in \mathbb{P}^1_{\mathbb{R}}(\mathbb{R})$ not in the branching set of f if $\deg(f)$ is odd.

Remark 1. Fix an integer $k \geq 2$, a finite set $S \subset \mathbb{P}^1(\mathbb{C})$ with $\sharp(S) \geq 2$ and for any $P \in S$ an integer $a_P > 0$ and a_P integers $n_{P,i} \geq 2$, $1 \leq i \leq a_P$. Set $g := 1-k+(\sum_{P\in S}\sum_{i=1}^{a_P} n_{P,i}-1)/2$. By Riemann Existence Theorem g is an integer and there is a finite non-empty subset of pairs (X, f) such that X is a smooth and connected genus g complex curve, f is a degree k holomorphic map which is unramified over $X \backslash f^{-1}(S)$ and such that for every $P \in S$ exactly a_P of the points of $f^{-1}(P)$ are ramification points of f, say $Q_{P,i}$, $1 \leq i \leq a_P$, and f has ramification order $n_{P,i} - 1$. Hence we may count the dimension of such set varying S among the subsets of $\mathbb{P}^1(\mathbb{C})$ with fixed cardinality. To compute the dimension of the isomorphic classes of such pairs (X, f) we will also use that dim $(\mathrm{Aut})(\mathbb{P}^1)) = 3$ and that the group $\mathrm{Aut}(\mathbb{P}^1)$ acts 3-transitively on $\mathbb{P}^1(\mathbb{C})$. In particular we get that we have the maximal possible dimension if and only if $a_P = 1$ and $n_{P,i} = 2$ for all P, i.

Proof of Theorem 1. By assumption the integer $d+g-1$ is even. Hence there is a σ-invariant effective divisor $D \subset N$ such that $\deg(D) = d+g-1$. Fix a general such divisor D. By Remark 1 there is a smooth and connected real curve X and a degree d real morphism $f : X \to N$ such that D is the ramification divisor of f. Since $N(\mathbb{R}) = \emptyset$ and f is real, $X(\mathbb{R}) = \emptyset$. The curve X has genus g by Riemann - Hurwitz formula.

Sketch of the proof of Theorem 2. First we assume g odd. Set $S := \mathbb{P}^1 \times \mathbb{P}^1$ with the real structure σ obtained as the product real structure of $\mathbb{P}^1_{\mathbb{R}}$. Let $a > 0$ be the minimal even integer such that $g \leq ad - a - d + 1$. Hence $0 \leq ad - a - d + 1 - g \leq 2d - 2$. Since d is even and g is odd, the integer $ad - a - d + 1 - g$ is even. By the adjunction formula we have $\omega_S \cong \mathcal{O}_S(-2, -2)$, $\omega_C \cong \mathcal{O}_C(a-2, d-2)$ and $p_a(C) = ad - a - d + 1$ for every $C \in |\mathcal{O}_S(a, d)|$. Set $x := (ad - a - d + 1)/2$ and take x general $P_i \in S$, $1 \leq i \leq x$. Set

$E := \bigcup_{i=1}^{x} P_i \cup \bigcup_{i=1}^{x} \sigma(P_i)$ and $Z := \bigcup_{i=1}^{x} 2P_i \cup \bigcup_{i=1}^{x} \sigma(2P_i)$. Hence E and Z are real. Since $a + d \geq x$, it is easy to check that $h^1(S, \mathcal{I}_Z(a, d)) = 0$ and that a general curve $C \in |\mathcal{I}_Z(a, d)|$ is integral and with an ordinary node at each point of $E \cup \sigma(E)$ as only singularities. Since both a and d are even and $E \cup \sigma(E)$ is real, we may find C as above with $C(\mathbb{R}) = \emptyset$ (use bihomogeneous coordinates on S). Take as X the normalization of C. Now assume g even. We work on the Hirzebruch surface F_1 (i.e. the blowing - up of $\mathbb{P}^2$ at one real point) with its unique real structure. Take the standard basis h, f of $\mathrm{Pic}(F_1)$ with $h^2 = -1$, $h \cdot f = 1$ and $f^2 = 0$. We have $\omega_{F_1} \cong \mathcal{O}_{F_1}(-2h - 3f)$. Let a be the minimal integer such that $a \equiv 1, 2 \pmod 4$ such that $ad - (a^2 + a)/2 - d - 1 \geq g$. Hence $0 \leq 1 + ad - (a^2 + a)/2 - g \leq 3d - 1$ and $1 + ad - (a^2 + a)/2 - g$ is even. Copy the proof of the case g odd.

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