CALDERÓN’S REPRODUCING FORMULAS FOR THE SPHERICAL MEAN \(L^2\)-MULTIPLIER OPERATORS

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First we study the spherical mean \(L^2\)-multiplier operators on \([0, +\infty] \times \mathbb{R}^n\). Next, we give for these operators Calderón’s reproducing formulas and best approximation formulas.

1. Introduction

In the Euclidean case the multiplier operator \(T_m\) associated with a bounded function \(m\) on \(\mathbb{R}^n\) is defined by \(\hat{T}_m \hat{f} = m \hat{f}\), where \(\hat{f}\) denotes the classical Fourier transform. Many authors [5, 9, 24] have been interested to extend the \(L^p\) Fourier-multipliers on several hypergroups and to show similarly its \(L^p\)-boundedness. Recently, these operators are studied in [25] where the author established some applications (Calderón’s reproducing formulas, best approximation formulas and extremal functions...).

The spherical mean operator \(R\) is defined, for a function \(f\) on \(\mathbb{R} \times \mathbb{R}^n\), even with respect to the first variable [15], by

\[
R(f)(r,x) = \int_{S^n} f(r\eta, x + r\xi) d\sigma_n(\eta, \xi), \quad (r, x) \in \mathbb{R} \times \mathbb{R}^n,
\]

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where $S^n$ is the unit sphere of $\mathbb{R} \times \mathbb{R}^n$ and $d\sigma_n$ is the surface measure on $S^n$ normalized to have total measure one.

The dual of the spherical mean operator $\mathcal{R}$ is defined by

$$\mathcal{R}(g)(r,x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} g(\sqrt{r^2 + |x-y|^2}, y) dy.$$ 

The spherical mean operator $\mathcal{R}$ and its dual have many important physical applications, namely in image processing of so-called synthetic aperture radar (SAR) data [6, 7, 23, 28], or in the linearized inverse scattering problem in acoustics [4].

The Fourier transform $\mathcal{F}$ associated with the spherical mean operator is defined for every integrable function $f$ on $[0, +\infty[ \times \mathbb{R}^n$ with respect to the measure $d\nu_{n+1}$, by

$$\forall (s,y) \in \mathcal{Y}, \quad \mathcal{F}(f)(s,y) = \int_0^{+\infty} \int_{\mathbb{R}^n} f(r,x) \mathcal{R}(\cos(s.)e^{-i\langle \cdot | \cdot \rangle})(r,x) d\nu_{n+1}(r,x),$$

where $d\nu_{n+1}$ is the measure defined on $[0, +\infty[ \times \mathbb{R}^n$ by

$$d\nu_{n+1}(r,x) = \frac{r^n}{2^{\frac{n-1}{2}} \Gamma(n+\frac{1}{2})} dr \otimes \frac{dx}{(2\pi)^{\frac{n}{2}}},$$

$\|\cdot\|_{p,\nu_{n+1}}$ its norm, and $\mathcal{Y}$ is the set given by

$$\mathcal{Y} = \mathbb{R} \times \mathbb{R}^n \cup \{(ir,x), (r,x) \in \mathbb{R} \times \mathbb{R}^n, |r| \leq |x|\}. \quad (1.1)$$

Many harmonic analysis results related to the Fourier transform $\mathcal{F}$ have already been proved by Dziri, Jlassi, Nessibi, Rachdi and Trimche [3, 8, 15, 18] or also by Peng and Zhao [17, 30]. Recently, Baccar, Omri and Rachdi [2] studied the generalized Fock spaces associated with the spherical mean operator $\mathcal{R}$, and Msehli, Rachdi and Omri [13, 14, 16] established several uncertainty principles for the Fourier transform $\mathcal{F}$.

Let $m$ be a function in the Lebesgue space $L^2(d\nu_{n+1})$. We define the spherical mean $L^2$-multiplier operators on $[0, +\infty[ \times \mathbb{R}^n$, for regular functions

$$T_{m,\varepsilon}f = \mathcal{F}^{-1}((m_\varepsilon \circ \theta) \mathcal{F}(f)), \quad \varepsilon > 0,$$

where $m_\varepsilon$ is the function given by

$$m_\varepsilon(r,x) = m(\varepsilon r, \varepsilon x), \quad (1.2)$$
and $\theta$ is the bijective function, defined on the set

$$\Upsilon_+ = [0, +\infty \times \mathbb{R}^n \cup \{(is, y) \ ; \ (s, y) \in [0, +\infty \times \mathbb{R}^n; \ s \leq |y|\}$$

by,

$$\theta(s, y) = (\sqrt{s^2 + |y|^2}, y). \quad (1.3)$$

Our purpose in this work is to study the multiplier $T_{m, \varepsilon}$, for which we shall prove an analogue of the Calderón’s reproducing formulas by using the theory of the Fourier transform $\mathcal{F}$ and the convolution product $\ast$.

Next, we use the theory of reproducing kernels to give best approximation of these operators and a Calderón’s reproducing formula of the associated extremal function. This paper is organized as follows, in the second section we recall some harmonic analysis results related to the spherical mean operator $\mathcal{R}$ and its associated Fourier transform $\mathcal{F}$.

In the third section we study the spherical mean $L^2$-multiplier operators $T_{m, \varepsilon}$, and for these operators we establish Calderón’s reproducing formulas.

The last section of this paper is devoted to giving best approximation for every function $m \in L^\infty (d\nu_{n+1})$ of the operators $T_{m, \varepsilon}$.

2. The spherical mean operator

In [15], Nessibi, Rachdi and Trimèche showed that for every $(\mu, \lambda) \in \mathbb{C} \times \mathbb{C}^n$, the function $\varphi_{(\mu, \lambda)}$ defined on $\mathbb{R} \times \mathbb{R}^n$ by

$$\varphi_{(\mu, \lambda)}(r, x) = \mathcal{R} \left( \cos(\mu.) e^{-i|\lambda|} \right)(r, x), \quad (2.1)$$

is the unique infinitely differentiable function on $\mathbb{R} \times \mathbb{R}^n$, even with respect to the first variable, satisfying the following system

$$\begin{cases}
\frac{\partial u}{\partial x_j}(r, x_1, \ldots, x_n) = -i\lambda_j u(r, x_1, \ldots, x_n), & 1 \leq j \leq n, \\
\ell_{n+1} u(r, x_1, \ldots, x_n) - \Delta u(r, x_1, \ldots, x_n) = -\mu^2 u(r, x_1, \ldots, x_n), \\
u(0, \ldots, 0) = 1, \\
\frac{\partial u}{\partial r}(0, x_1, \ldots, x_n) = 0, & (x_1, \ldots, x_n) \in \mathbb{R}^n,
\end{cases}$$

where $\ell_{n+1}$ is the Bessel operator, defined by $\ell_{n+1} = \frac{d^2}{dr^2} + \frac{n}{r} \frac{d}{dr}$, and $\Delta$ denotes the usual Laplacian operator defined by $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$. The authors proved also
that the eigenfunction \( \varphi_{(\mu, \lambda)} \) defined by relation (2.1), is explicitly given by

\[
\forall (r, x) \in \mathbb{R} \times \mathbb{R}^n, \quad \varphi_{(\mu, \lambda)}(r, x) = j_{\alpha - \frac{1}{2}}(r \sqrt{\mu^2 + |\lambda|^2}) e^{-i|\lambda|x},
\]

(2.2)

where \( j_{\alpha - \frac{1}{2}} \) is the modified Bessel function defined by

\[
j_{\alpha - \frac{1}{2}}(z) = 2^{\alpha} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n}{2}}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma\left(\frac{n+1}{2} + k\right)} (\frac{z}{2})^{2k}, \quad z \in \mathbb{C},
\]

and \( J_{\alpha - \frac{1}{2}} \) is the Bessel function of the first kind and index \( \frac{n-1}{2} \) (see [1, 11] and [29]).

The modified Bessel function \( j_{\alpha - \frac{1}{2}} \) has the following integral representation

\[
\forall z \in \mathbb{C}, \quad j_{\alpha - \frac{1}{2}}(z) = \frac{2 \Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{n}{2}\right)} \int_0^1 (1 - t^2)^{\frac{n}{2} - 1} \cos(zt) dt.
\]

(2.3)

Relation (2.3) shows in particular that, for every \( z \in \mathbb{C} \) and for every \( k \in \mathbb{N} \), we have

\[
\left| j^{(k)}_{\alpha - \frac{1}{2}}(z) \right| \leq e^{\text{Im}(z)}.
\]

From the properties of the modified Bessel function \( j_{\alpha - \frac{1}{2}} \), we deduce that the eigenfunction \( \varphi_{(\mu, \lambda)} \) is bounded on \( \mathbb{R} \times \mathbb{R}^n \) if, and only if, \( (\mu, \lambda) \) belongs to the set \( \Upsilon \) given by relation (1.1), and in this case

\[
\sup_{(r, x) \in \mathbb{R} \times \mathbb{R}^n} \left| \varphi_{(\mu, \lambda)}(r, x) \right| = 1.
\]

(2.4)

In the following we shall define the translation operators, the convolution product and the Fourier transform \( \mathcal{F} \) associated with the operator \( \mathcal{R} \). For this we denote by

- \( \mathcal{B}_\Upsilon \) the \( \sigma \)-algebra defined on \( \Upsilon \) by,

\[
\mathcal{B}_\Upsilon = \{ \theta^{-1}(B) \mid B \in \mathcal{B}_{\text{Bor}}([0, +\infty[ \times \mathbb{R}^n) \},
\]

where \( \theta \) is the function, given by relation (1.3).

- \( \gamma_{n+1} \) the measure defined on \( \mathcal{B}_\Upsilon \), by, \( \gamma_{n+1}(B) = \nu_{n+1}(\theta(B)) \).

- \( L^p(d\gamma_{n+1}), \; p \in [1, +\infty) \) the Lebesgue space of measurable functions \( f \) on \( \Upsilon \), such that \( \|f\|_{L^p(d\gamma_{n+1})} < +\infty \).

We have the following properties (see [15] and [26])
i) For every nonnegative measurable function \( g \) on \( \Upsilon_+ \), we have
\[
\int_{\Upsilon_+} \int g(\mu, \lambda) \, d\gamma_{n+1}(\mu, \lambda) = \frac{1}{2 \pi^{\frac{n-1}{2}} \Gamma(\frac{n+1}{2})} \left( \int_0^{+\infty} \int_{\mathbb{R}^n} g(\mu, \lambda)(\mu^2 + |\lambda|^2)^{\frac{n-1}{2}} \, d\mu \, d\lambda \right.
\]
\[
+ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\lambda| \, g(i\mu, \lambda)(|\lambda|^2 - \mu^2)^{\frac{n-1}{2}} \, d\mu \, d\lambda \bigg). 
\]

ii) For every nonnegative measurable function \( f \) on \([0, +\infty[ \times \mathbb{R}^n\) (respectively integrable on \([0, +\infty[ \times \mathbb{R}^n\) with respect to the measure \( d\nu_{n+1} \), \( fo\theta \) is a measurable nonnegative function on \( \Upsilon_+ \), (respectively integrable on \( \Upsilon_+ \) with respect to the measure \( d\gamma_{n+1} \)) and we have
\[
\int_{\Upsilon_+} \int (fo\theta)(\mu, \lambda) \, d\gamma_{n+1}(\mu, \lambda) = \int_0^{+\infty} \int_{\mathbb{R}^n} f(r,x) \, d\nu_{n+1}(r,x). 
\]
Moreover, the function \( f \) belongs to \( L^p(d\nu_{n+1}) \), \( p \in [1, +\infty[ \) if and only if \( fo\theta \) belongs to \( L^p(d\gamma_{n+1}) \) and we have
\[
\|f\|_{p,\nu_{n+1}} = \|fo\theta\|_{p,\gamma_{n+1}}. \tag{2.5}
\]

According to Rachdi, Nessibi and Trimèche (see [15, 26] and [27]), we have the following definition and properties for the translation operator associated with the spherical mean operator

**Definition 2.1.** i) For every \((r,x)\) in \([0, +\infty[ \times \mathbb{R}^n\), the translation operator \( T_{(r,x)} \) associated with the spherical mean operator is defined on \( L^p(d\nu_{n+1}) \), \( p \in [1, +\infty[ \), by
\[
T_{(r,x)}(f)(s,y) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi} \Gamma(\frac{n}{2})} \int_0^{\pi} f(\sqrt{r^2 + s^2 + 2rs \cos \theta}, x + y) \sin^{n-1}(\theta) \, d\theta. 
\]

ii) The convolution product of measurable functions \( f \) and \( g \) on \([0, +\infty[ \times \mathbb{R}^n\), is defined by
\[
\forall (r,x) \in [0, +\infty[ \times \mathbb{R}^n; \; f * g(r,x) = \int_0^{+\infty} \int_{\mathbb{R}^n} T_{(r,x)}(f)(s,-y)g(s,y) \, d\nu_{n+1}(s,y), 
\]
whenever the integral of the right-hand side is defined.

For every \((r,x)\) in \([0, +\infty[ \times \mathbb{R}^n\), and by a standard change of variables, we have
\[
\forall (s,y) \in [0, +\infty[ \times \mathbb{R}^n, \; T_{(r,x)}(f)(s,y) = \frac{1}{2 \pi^{\frac{n-1}{2}} \Gamma(\frac{n+1}{2})} \int_0^{+\infty} f(t,x+y) \mathcal{H}_n(r,s,t) \, dt, 
\]
where the kernel $\mathcal{W}_n$, is given by

$$
\mathcal{W}_n(r,s,t) = \frac{\Gamma\left(\frac{n+1}{2}\right)^2}{2^{n+1}\Gamma\left(\frac{n}{2}\right)\sqrt{\pi}} \frac{(r+s)^{n+1}(t^2 - (r-s)^2)^{\frac{n}{2} - 1}}{(rst)^{n-1}} \mathbf{1}_{[r-s,r+s]}(t).
$$

Also, the coming properties are satisfied

- For every $f \in L^p(d\nu_{n+1})$, $p \in [1, +\infty]$, and $(r,x) \in [0, +\infty) \times \mathbb{R}^n$, the function $\mathcal{T}_{(r,x)}(f)$ belongs to $L^p(d\nu_{n+1})$ and we have

$$
\|\mathcal{T}_{(r,x)}(f)\|_{p,\nu_{n+1}} \leq \|f\|_{p,\nu_{n+1}}. \tag{2.6}
$$

- Let $p,q,r \in [1, +\infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Then for every $f \in L^p(d\nu_{n+1})$ and $g \in L^q(d\nu_{n+1})$, the function $f * g$ belongs to the space $L^r(d\nu_{n+1})$, and we have the following Young’s inequality

$$
\|f * g\|_{r,\nu_{n+1}} \leq \|f\|_{p,\nu_{n+1}} \|g\|_{q,\nu_{n+1}}.
$$

In the following, we shall define the Fourier transform $\mathcal{F}$ connected with the spherical mean operator, and we recall some of its properties that we need in the next sections.

**Definition 2.2.** The Fourier transform $\mathcal{F}$ associated with the spherical mean operator is defined on $L^1(d\nu_{n+1})$ by [15]

$$
\forall (\mu, \lambda) \in \Upsilon ; \quad \mathcal{F}(f)(\mu, \lambda) = \int_0^{+\infty} \int_{\mathbb{R}^n} f(r,x) \phi(\mu, \lambda)(r,x) d\nu_{n+1}(r,x),
$$

where $\phi(\mu, \lambda)$ is the eigenfunction given by relation (2.2), and $\Upsilon$ is the set defined by relation (1.1).

Then, according to [15], we have

For every $f, g \in L^1(d\nu_{n+1}),$

$$
\mathcal{F}(f * g) = \mathcal{F}(f) \mathcal{F}(g),
$$

and $(\mu, \lambda) \in \Upsilon$

$$
\mathcal{F}(\mathcal{T}_{(r,x)}(f))(\mu, \lambda) = \phi(\mu, \lambda)(r,x) \mathcal{F}(f)(\mu, \lambda). \tag{2.7}
$$

Moreover, relation (2.4) implies that the Fourier transform $\mathcal{F}$ is a bounded linear operator from $L^1(d\nu_{n+1})$ into $L^\infty(d\gamma_{n+1})$, and that for every $f \in L^1(d\nu_{n+1})$, we have

$$
\|\mathcal{F}(f)\|_{\infty,\nu_{n+1}} \leq \|f\|_{1,\nu_{n+1}}. \tag{2.8}
$$
For every positive real number $\varepsilon$ and for every $m \in L^p(d\nu_{n+1}), p \in [1, +\infty]$, the function $m_\varepsilon$ defined by relation (1.2), belongs to $L^p(d\nu_{n+1})$ and we have

$$\|m_\varepsilon\|_{p,\nu_{n+1}} = \frac{1}{\varepsilon^{2n+1}}\|m\|_{p,\nu_{n+1}}.$$  \hfill (2.9)

In [15], Rachdi, Nessibi and Trimèche, established the following inversion formula and Plancherel theorem for the Fourier transform $\mathcal{F}$.

**Theorem 2.3** (Inversion formula). Let $f \in L^1(d\nu_{n+1})$ such that $\mathcal{F}(f) \in L^1(d\gamma_{n+1})$, then for almost every $(r,x) \in \mathbb{R} \times \mathbb{R}^n$

$$f(r,x) = \int_{Y_+} \mathcal{F}(f)(\mu, \lambda)\Phi(\mu, \lambda)(r,x)d\gamma_{n+1}(\mu, \lambda).$$

**Theorem 2.4** (Plancherel theorem). The Fourier transform $\mathcal{F}$ can be extended to an isometric isomorphism from $L^2(d\nu_{n+1})$ onto $L^2(d\gamma_{n+1})$. In particular, for every $f \in L^2(d\nu_{n+1})$

$$\|\mathcal{F}(f)\|_{2,\gamma_{n+1}} = \|f\|_{2,\nu_{n+1}}.$$  \hfill (2.10)

**Corollary 2.5.** For all functions $f$ and $g$ in $L^2(d\nu_{n+1})$, we have

$$\int_0^{+\infty} \int_{\mathbb{R}^n} f(r,x)\overline{g(r,x)}d\nu_{n+1}(r,x) = \int_{Y_+} \mathcal{F}(f)(\mu, \lambda)\overline{\mathcal{F}(g)(\mu, \lambda)}d\gamma_{n+1}(\mu, \lambda).$$

**Remark 2.6.** (i) For every $f, g \in L^2(d\nu_{n+1})$, the function $f \ast g$ belongs to the space $C_{c,0}(\mathbb{R} \times \mathbb{R}^n)$ consisting of continuous functions $h$ on $\mathbb{R} \times \mathbb{R}^n$, even with respect to the first variable and such that $\lim_{r^2+|x|^2 \to +\infty} h(r,x) = 0$.

Moreover,

$$f \ast g = \mathcal{F}^{-1}([\mathcal{F}(f)\mathcal{F}(g)],$$  \hfill (2.11)

where $\mathcal{F}^{-1}$ is the mapping defined on $L^1(d\gamma_{n+1})$ by

$$\mathcal{F}^{-1}(g)(r,x) = \int_{Y_+} g(\mu, \lambda)\overline{\Phi(\mu, \lambda)(r,x)}d\gamma_{n+1}(\mu, \lambda).$$

(ii) Let $f, g \in L^2(d\nu_{n+1})$, the function $f \ast g$ belongs to $L^2(d\nu_{n+1})$ if and only if $\mathcal{F}(f)\mathcal{F}(g)$ belongs to $L^2(d\gamma_{n+1})$, and we have

$$\mathcal{F}(f \ast g) = \mathcal{F}(f)\mathcal{F}(g).$$

(iii) Let $f, g \in L^2(d\nu_{n+1})$, then

$$\|\mathcal{F}(f)\mathcal{F}(g)\|_{2,\gamma_{n+1}} = \|f \ast g\|_{2,\nu_{n+1}}.$$  \hfill (2.11)

(iv) For every $g \in L^1(d\gamma_{n+1})$, $\mathcal{F}^{-1}(g)$ belongs to $L^\infty(d\nu_{n+1})$, and we have

$$\|\mathcal{F}^{-1}(g)\|_{\infty,\nu_{n+1}} \leq \|g\|_{1,\gamma_{n+1}}.$$
3. The Spherical mean $L^2$-multiplier operators

In this section we study the spherical mean $L^2$-multiplier operators on $[0, +\infty]\times\mathbb{R}^n$ and for these operators we establish Calderón’s reproducing formulas.

**Definition 3.1.** Let $m$ be a function in $L^2(d\nu_{n+1})$ and let $\varepsilon$ be a positive real number. The spherical mean $L^2$-multiplier operators is defined for regular functions $f$ on $[0, +\infty]\times\mathbb{R}^n$, by

$$\forall (r, x) \in [0, +\infty]\times\mathbb{R}^n, \quad T_{m, \varepsilon}f(r, x) = \mathcal{F}^{-1}((m_\varepsilon \circ \theta)\mathcal{F}(f))(r, x),$$

(3.1)

where $m_\varepsilon$ is the function given by relation (1.2) and $\theta$ is the function defined by (1.3).

**Proposition 3.2.** (i) For every $m \in L^2(d\nu_{n+1})$, and $f \in L^1(d\nu_{n+1})$, the function $T_{m, \varepsilon}f$ belongs to $L^2(d\nu_{n+1})$, and we have

$$\|T_{m, \varepsilon}f\|_{2, \nu_{n+1}} \leq \frac{1}{\varepsilon^{2n+1}} \|m\|_{2, \nu_{n+1}} \|f\|_{1, \nu_{n+1}}.$$ 

(ii) For every $m \in L^2(d\nu_{n+1})$, and $f \in L^2(d\nu_{n+1})$, then $T_{m, \varepsilon}f$ belongs to $L^\infty(d\nu_{n+1})$, and we have

$$T_{m, \varepsilon}f(r, x) = \int_{\mathbb{R}^n} (m_\varepsilon \circ \theta)(\mu, \lambda)\mathcal{F}(f)(\mu, \lambda)\phi_{(\mu, \lambda)}(r, x)d\gamma_{n+1}(\mu, \lambda),$$

and

$$\|T_{m, \varepsilon}f\|_{\infty, \nu_{n+1}} \leq \frac{1}{\varepsilon^{2n+1}} \|m\|_{2, \nu_{n+1}} \|f\|_{2, \nu_{n+1}}.$$ 

(iii) For every $m \in L^\infty(d\nu_{n+1})$, and $f \in L^2(d\nu_{n+1})$, the function $T_{m, \varepsilon}f$ belongs to $L^2(d\nu_{n+1})$, and we have

$$\|T_{m, \varepsilon}f\|_{2, \nu_{n+1}} \leq \|m\|_{\infty, \nu_{n+1}} \|f\|_{2, \nu_{n+1}}.$$ 

**Proof.** (i) From relations (2.5), (2.8), (3.1), and Theorem 2.4, the function $T_{m, \varepsilon}$ belongs to $L^2(d\nu_{n+1})$, and we have

$$\|\mathcal{F}(T_{m, \varepsilon}f)\|_{2, \gamma_{n+1}} = \|(m_\varepsilon \circ \theta)\mathcal{F}(f)\|_{2, \gamma_{n+1}} \leq \|(m_\varepsilon \circ \theta)\|_{2, \gamma_{n+1}} \|\mathcal{F}(f)\|_{\infty, \gamma_{n+1}} \leq \|m_\varepsilon\|_{2, \nu_{n+1}} \|f\|_{1, \nu_{n+1}}.$$ 

Then, the result follows from (2.9), and Theorem 2.4.

(ii) Using (2.5), (3.1), and Remark 2.6 (iv), for every $m \in L^2(d\nu_{n+1})$, and $f \in L^2(d\nu_{n+1})$, the function $T_{m, \varepsilon}f$ belongs to $L^\infty(d\nu_{n+1})$, and we have

$$\|T_{m, \varepsilon}f\|_{\infty, \nu_{n+1}} \leq \|(m_\varepsilon \circ \theta)\mathcal{F}(f)\|_{1, \gamma_{n+1}}.$$
From Hölder’s inequality, relation (2.9), and Theorem 2.4, we obtain
\[
\|T_{m,\epsilon} f\|_{\infty, \nu_{n+1}} \leq \|(m_\epsilon \circ \theta)\|_{2, \nu_{n+1}} \|F(f)\|_{2, \nu_{n+1}}
\]
\[
= \|m_\epsilon\|_{2, \nu_{n+1}} \|f\|_{2, \nu_{n+1}},
\]
\[
= \frac{1}{\epsilon^{2n+1}} \|m\|_{2, \nu_{n+1}} \|f\|_{2, \nu_{n+1}}.
\]

Part (iii) follows from (2.5), (3.1), and Theorem 2.4.

\[\square\]

**Remark 3.3.** According to relation (2.10), for every \(m \in L^2(d\nu_{n+1})\) and \(f \in L^2(d\nu_{n+1})\), we can write the spherical mean \(L^2\)-multiplier as
\[
\forall (r,x) \in [0, +\infty] \times \mathbb{R}^n, \quad T_{m,\epsilon} f(r,x) = \mathcal{F}^{-1} (m_\epsilon \circ \theta) * f(r,x).
\]

**Theorem 3.4.** Let \(m\) be a function in \(L^2(d\nu_{n+1})\), satisfying the admissibility condition
\[
\int_0^{+\infty} |m_\epsilon \circ \theta(\mu, \lambda)|^2 \frac{d\epsilon}{\epsilon} = 1, \quad (\mu, \lambda) \in \varUpsilon.
\]

(i) **Plancherel formula:** For every \(f \in L^2(d\nu_{n+1})\), we have
\[
\|f\|_{2, \nu_{n+1}}^2 = \int_0^{+\infty} \|T_{m,\epsilon} f\|_{2, \nu_{n+1}}^2 \frac{d\epsilon}{\epsilon}.
\]

(ii) **First Calderó’s formula:** Let \(f\) be a function in \(L^1(d\nu_{n+1})\), such that \(\mathcal{F}(f)\) in \(L^1(d\gamma_{n+1})\), we have
\[
f(r,x) = \int_0^{+\infty} (T_{m,\epsilon} f * \mathcal{F}^{-1} (m_\epsilon \circ \theta))(r,x) \frac{d\epsilon}{\epsilon}, \quad \text{a.e. (r,x) \in [0, +\infty] \times \mathbb{R}^n}.
\]

**Proof.** (i) From relations (2.11) and (3.2), we have
\[
\int_0^{+\infty} \|T_{m,\epsilon} f\|_{2, \nu_{n+1}}^2 \frac{d\epsilon}{\epsilon} = \int_0^{+\infty} \int_0^{+\infty} \int_{\mathbb{R}^n} |(\mathcal{F}^{-1}(m_\epsilon \circ \theta) * f)(r,x)|^2 d\nu_{n+1}(r,x) \frac{d\epsilon}{\epsilon}
\]
\[
= \int_0^{+\infty} \int_{\mathbb{R}^n} |m_\epsilon \circ \theta(r,x) \mathcal{F}(f)(r,x)|^2 d\gamma_{n+1}(r,x) \frac{d\epsilon}{\epsilon}
\]
\[
= \int \int_{\varUpsilon} |\mathcal{F}(f)(r,x)|^2 \left(\int_0^{+\infty} |m_\epsilon \circ \theta(r,x)|^2 \frac{d\epsilon}{\epsilon}\right) d\gamma_{n+1}(r,x).
\]

The result follows from Theorem 2.4, and (3.3).

(ii) Let \(f\) in \(L^1(d\nu_{n+1})\). According to Proposition 3.2 (i), relation (2.6), and Corollary 2.5, we have
\[
\int_0^{+\infty} (T_{m,\epsilon} f * \mathcal{F}^{-1} (m_\epsilon \circ \theta))(r,x) \frac{d\epsilon}{\epsilon}
\]
\[
= \int_0^{+\infty} \left[ \int_{\mathbb{R}^n} T_{m,\epsilon} f(s,y) \mathcal{F}(r-x)(\mathcal{F}^{-1}(m_\epsilon \circ \theta))(s,y) d\nu_{n+1}(s,y) \right] \frac{d\epsilon}{\epsilon}
\]
\[
= \int_0^{+\infty} \left[ \int_{\mathbb{R}^n} \mathcal{F}(T_{m,\epsilon} f)(s,y) \mathcal{F}(r-x)(\mathcal{F}^{-1}(m_\epsilon \circ \theta))(s,y) d\gamma_{n+1}(s,y) \right] \frac{d\epsilon}{\epsilon}.
\]
Using (2.7), we obtain
\[
\int_{0}^{+\infty} \left( T_{m,\epsilon} f \ast \mathcal{F}^{-1}(m_{\epsilon} \circ \theta) \right) (r, x) \frac{d\epsilon}{\epsilon}
\]
\[
= \int_{0}^{+\infty} \left[ \int \int_{\mathcal{Y}_{+}} \mathcal{F}(f)(s, y) \phi(s, y)(r, x) \left| m_{\epsilon} \circ \theta(s, y) \right|^{2} d\gamma_{n+1}(s, y) \right] \frac{d\epsilon}{\epsilon}.
\]

Since,
\[
\int_{0}^{+\infty} \left[ \int \int_{\mathcal{Y}_{+}} \left| \mathcal{F}(f)(s, y) \phi(s, y)(r, x) \left| m_{\epsilon} \circ \theta(s, y) \right|^{2} d\gamma_{n+1}(s, y) \right] \frac{d\epsilon}{\epsilon}
\]
\[
\leq \int \int_{\mathcal{Y}_{+}} \left| \mathcal{F}(f)(s, y) \right| d\gamma_{n+1}(s, y).
\]

Then, the result follows from Fubini’s theorem, relation (3.3), and Theorem 2.3.

\[
\Box
\]

\textbf{Lemma 3.5.} Let \( m \in L^{2}(d\nu_{n+1}) \cap L^{\infty}(d\nu_{n+1}) \), satisfy the admissibility condition (3.3). For every \( 0 < \xi < \delta < \infty \), the function
\[
K_{\xi, \delta}(\mu, \lambda) = \int_{\xi}^{\delta} \left| m_{\epsilon} \circ \theta(\mu, \lambda) \right|^{2} \frac{d\epsilon}{\epsilon},
\]

belongs to \( L^{2}(d\gamma_{n+1}) \), and we have
\[
\| K_{\xi, \delta} \|_{2, \gamma_{n+1}}^{2} \leq \ln \left( \frac{\delta}{\xi} \right)^{\frac{2n+1}{2n+1}} \| m \|_{2, \nu_{n+1}}^{2} \| m \|_{\infty, \nu_{n+1}}^{2}.
\]

\textbf{Proof.} Using Hölder’s inequality for the measure \( \frac{d\epsilon}{\epsilon} \), we get for every \( (\mu, \lambda) \in \mathcal{Y} \)
\[
| K_{\xi, \delta}(\mu, \lambda) |^{2} \leq \ln \left( \frac{\delta}{\xi} \right) \int_{\xi}^{\delta} \left| m_{\epsilon} \circ \theta(\mu, \lambda) \right|^{4} \frac{d\epsilon}{\epsilon}.
\]

From (2.5), and (2.9), we obtain
\[
\| K_{\xi, \delta} \|_{2, \gamma_{n+1}}^{2} \leq \ln \left( \frac{\delta}{\xi} \right) \int_{\xi}^{\delta} \left[ \int \int_{\mathcal{Y}_{+}} \left| m_{\epsilon} \circ \theta(\mu, \lambda) \right|^{4} d\gamma_{n+1}(\mu, \lambda) \right] \frac{d\epsilon}{\epsilon}
\]
\[
\leq \ln \left( \frac{\delta}{\xi} \right)^{\frac{2n+1}{2n+1}} \| m \|_{2, \nu_{n+1}}^{2} \| m \|_{\infty, \nu_{n+1}}^{2} < \infty.
\]

\[
\Box
\]
Theorem 3.6. Second Calderón’s formula. Let \( m \in L^2(d\nu_{n+1}) \cap L^\infty(d\nu_{n+1}) \), satisfy the admissibility condition (3.3). Then for every \( f \in L^2(d\nu_{n+1}) \) and \( 0 < \xi < \delta < \infty \), the function

\[
f^{\xi,\delta}(r,x) = \int_\xi^\delta (T_{m,\epsilon}f \ast F^{-1}(m_{\epsilon} \circ \theta))(r,x) \frac{d\epsilon}{\epsilon},
\]

belongs to \( L^2(d\nu_{n+1}) \) and satisfies

\[
\lim_{(\xi,\delta) \to (0^+,+\infty)} \|f^{\xi,\delta} - f\|_{2,\nu_{n+1}} = 0.
\]

Proof. From Proposition 3.2 (iii), (2.6), (2.7) and Corollary 2.5, we have

\[
f^{\xi,\delta}(r,x) = \int_\xi^\delta \left[ \int_0^{+\infty} \int_{\mathbb{R}^n} T_{m,\epsilon}f(s,y) \mathcal{F}^{-1}(m_{\epsilon} \circ \theta)(s,y) d\nu_{n+1}(s,y) \right] \frac{d\epsilon}{\epsilon}
\]

By Fubini-Tonelli’s theorem, Hölder’s inequality, relation (2.4) and Lemma 3.5, we get

\[
\int_\xi^\delta \left[ \int_{Y_+} \left| \mathcal{F}(f)(s,y) \varphi(s,y)(r,x) \right| m_{\epsilon} \circ \theta(s,y) \right]^2 d\gamma_{n+1}(s,y) \frac{d\epsilon}{\epsilon}
\]

\[
\leq \int_{Y_+} \left| \mathcal{F}(f)(s,y) \right| \mathcal{K}_{\xi,\delta}(s,y) d\gamma_{n+1}(s,y)
\]

\[
\leq \sqrt{\ln\left(\frac{\delta}{\xi}\right)} \frac{\xi^{-(2n+1)} - \delta^{-(2n+1)}}{2n+1} \|f\|_{2,\nu_{n+1}} \|m\|_{2,\nu_{n+1}} \|m\|_{\infty,\nu_{n+1}} < \infty.
\]

Then, from Fubini’s theorem and Theorem 2.3, we obtain

\[
f^{\xi,\delta}(r,x) = \int_{Y_+} \mathcal{F}(f)(s,y) \varphi(s,y)(r,x) \mathcal{K}_{\xi,\delta}(s,y) d\gamma_{n+1}(s,y)
\]

\[
= \mathcal{F}^{-1}(\mathcal{F}(f)\mathcal{K}_{\xi,\delta})(r,x).
\]

On the other hand, from relation (3.3), the function \( \mathcal{K}_{\xi,\delta} \) belongs to \( L^\infty(d\gamma_{n+1}) \), from this fact and Theorem 2.4, the function \( f^{\xi,\delta} \in L^2(d\nu_{n+1}) \), and we have

\[
\mathcal{F}(f^{\xi,\delta}) = \mathcal{F}(f)\mathcal{K}_{\xi,\delta}.
\]

Using the previous result and Theorem 2.4, we get

\[
\|f^{\xi,\delta} - f\|_{2,\nu_{n+1}}^2 = \int_{Y_+} \left| \mathcal{F}(f)(\mu,\lambda) \right|^2 (\mathcal{K}_{\xi,\delta}(\mu,\lambda) - 1)^2 d\gamma_{n+1}(\mu,\lambda).
\]

The relation (3.4) follows from \( \lim_{(\xi,\delta) \to (0^+,+\infty)} \mathcal{K}_{\xi,\delta}(\mu,\lambda) = 1 \), and the dominated convergence theorem. \( \square \)
4. The extremal function related to spherical mean $L^2$-multiplier operators

In this section, by using the theory of extremal function and reproducing Kernel of Hilbert space [19–22], we study the extremal function associated to the spherical mean $L^2$-multiplier operators. The main result of this section can be stated as follows.

**Definition 4.1.** Let $\sigma$ be a positive function on $\Upsilon$ satisfying:

$$\sigma(\mu, \lambda) \geq 1, \quad (\mu, \lambda) \in \Upsilon,$$

and

$$\frac{1}{\sigma} \in L^1(d\gamma_{n+1}).$$

We define the space $\Omega_\sigma([0, +\infty[ \times \mathbb{R}^n)$, by

$$\Omega_\sigma([0, +\infty[ \times \mathbb{R}^n) = \{ f \in L^2(d\nu_{n+1}), \sqrt{\sigma} \mathcal{F}(f) \in L^2(d\gamma_{n+1}) \}.$$ 

The space $\Omega_\sigma([0, +\infty[ \times \mathbb{R}^n)$ provided with inner product

$$\langle f, g \rangle_\sigma = \int \int_{\Upsilon_+} \sigma(\mu, \lambda) \mathcal{F}(f)(\mu, \lambda) \overline{\mathcal{F}(g)(\mu, \lambda)} d\gamma_{n+1}(\mu, \lambda),$$

and the norm $\|f\|_\sigma = \sqrt{\langle f, f \rangle_\sigma}$ is a Hilbert space.

**Proposition 4.2.** Let $m \in L^\infty(d\nu_{n+1})$. For every $f \in \Omega_\sigma([0, +\infty[ \times \mathbb{R}^n)$, the operators $T_{m, \varepsilon}$ are bounded linear operators from $\Omega_\sigma([0, +\infty[ \times \mathbb{R}^n)$ into $L^2(d\nu_{n+1})$, and we have

$$\|T_{m, \varepsilon}f\|_{2, \nu_{n+1}} \leq \|m\|_{\infty, \nu_{n+1}} \|f\|_\sigma.$$

**Proof.** Let $f \in \Omega_\sigma([0, +\infty[ \times \mathbb{R}^n)$. According to Proposition 3.2(iii), the operator $T_{m, \varepsilon}f$ belongs to $L^2(d\nu_{n+1})$, and

$$\|T_{m, \varepsilon}f\|_{2, \nu_{n+1}} \leq \|m\|_{\infty, \nu_{n+1}} \|f\|_{2, \nu_{n+1}}.$$ 

By relation (4.1), we have $\|f\|_\sigma^2 \geq \int \int_{\Upsilon_+} |\mathcal{F}(f)(\mu, \lambda)|^2 d\gamma_{n+1}(\mu, \lambda)$, which gives the result. $\square$

**Definition 4.3.** Let $\rho > 0$, and let $m \in L^\infty(d\nu_{n+1})$, we denote by $\langle \cdot, \cdot \rangle_{\sigma, \rho}$ the inner product defined on the space $\Omega_\sigma([0, +\infty[ \times \mathbb{R}^n)$ by

$$\langle f, g \rangle_{\sigma, \rho} = \int \int_{\Upsilon_+} (\rho \sigma(\mu, \lambda) + |m_\varepsilon \circ \theta(\mu, \lambda)|^2) \mathcal{F}(f)(\mu, \lambda) \overline{\mathcal{F}(g)(\mu, \lambda)} d\gamma_{n+1}(\mu, \lambda),$$

and the norm $\|f\|_{\sigma, \rho} = \sqrt{\langle f, f \rangle_{\sigma, \rho}}$. (4.3)
Lemma 4.4. Let \((s, y) \in [0, +\infty \times \mathbb{R}^n]\). Then

(i) The function
\[
\Lambda_{(s,y)} : (\mu, \lambda) \mapsto \frac{\varphi_{(\mu, \lambda)}(s,y)}{\rho \sigma(\mu, \lambda) + |m_{\epsilon} \circ \theta(\mu, \lambda)|^2},
\]
belongs to \(L^1(d\gamma_{n+1}) \cap L^2(d\gamma_{n+1})\).

(ii) The function
\[
\Phi_{(s,y)} : (\mu, \lambda) \mapsto \frac{m_{\epsilon} \circ \theta(\mu, \lambda) \varphi_{(\mu, \lambda)}(s,y)}{\rho \sigma(\mu, \lambda) + |m_{\epsilon} \circ \theta(\mu, \lambda)|^2},
\]
belongs to \(L^1(d\gamma_{n+1}) \cap L^2(d\gamma_{n+1})\).

Where \(\varphi_{(\mu, \lambda)}\) is the function given by relation (2.2).

Proof. The proof of the Lemma follows from relations (2.4), (4.1) and (4.2). \(\square\)

Proposition 4.5. Let \(m \in L^\infty(d\nu_{n+1})\). Then the Hilbert space
\((\Omega_\sigma([0, +\infty[ \times \mathbb{R}^n), \langle \cdot, \cdot \rangle_{\sigma, \rho})\) has the following reproducing Kernel
\[
K_{\sigma, \rho}((r,x), (s,y)) = \int \int_{\mathbb{R}^+} \frac{\varphi_{(\mu, \lambda)}(r, x)\varphi_{(\mu, \lambda)}(s, y)}{\rho \sigma(\mu, \lambda) + |m_{\epsilon} \circ \theta(\mu, \lambda)|^2} d\gamma_{n+1}(\mu, \lambda),
\]
that is

(i) For every \((s, y) \in [0, +\infty \times \mathbb{R}^n], the function \((r,x) \mapsto K_{\sigma, \rho}((r,x), (s,y))\) belongs to \(\Omega_\sigma([0, +\infty \times \mathbb{R}^n]).\)

(ii) For every \(f \in \Omega_\sigma([0, +\infty \times \mathbb{R}^n], and (s,y) \in [0, +\infty \times \mathbb{R}^n], we have the reproducing property,
\[
\langle f, K_{\sigma, \rho}((\cdot, \cdot), (s,y)) \rangle_{\sigma, \rho} = f(s,y).
\]

Proof. From Lemma 4.4 (i), the function \(K_{\sigma, \rho}\) is well defined and by Theorem 2.3, we have
\[
K_{\sigma, \rho}((r,x), (s,y)) = \mathcal{F}^{-1}(\Lambda_{(s,y)})(r,x), \quad (r,x) \in [0, +\infty \times \mathbb{R}^n].
\]

By Theorem 2.4, it follows that the function \(K_{\sigma, \rho}((\cdot, \cdot), (s,y))\) belongs to \(L^2(d\nu_{n+1})\), and we have
\[
\mathcal{F}(K_{\sigma, \rho}((\cdot, \cdot), (s,y)))(\mu, \lambda) = \Lambda_{(s,y)}(\mu, \lambda), \quad (\mu, \lambda) \in \mathcal{Y}.
\]

Then by relations (2.4), (4.2) and (4.5), we obtain
\[
\|K_{\sigma, \rho}((\cdot, \cdot), (s,y))\|_{2, \sigma}^2 \leq \frac{1}{\rho^2} \| \frac{1}{\sigma} \|_{1, \gamma_{n+1}}.
\]
This proves that for every \((s, y) \in [0, +\infty[ \times \mathbb{R}^n\), the function \(K_{\sigma, \rho}((\ldots), (s, y))\) belongs to \(\Omega_\sigma([0, +\infty[ \times \mathbb{R}^n)\).

(ii) From (4.3) and (4.5), we obtain

\[
\langle f, K_{\sigma, \rho}((\ldots), (s, y)) \rangle_{\sigma, \rho} = \int \int_{Y_+} \mathcal{F}(f)(\mu, \lambda) \varphi(\mu, \lambda)(s, y) d\gamma_{n+1}(\mu, \lambda).
\]

On the other hand, from relation (4.2) the function \(\frac{1}{\sqrt{\sigma}}\) belongs to \(L^2(d\gamma_{n+1})\), hence for every \(f \in \Omega_\sigma([0, +\infty[ \times \mathbb{R}^n)\), the function \(\mathcal{F}(f)\) belongs to \(L^1(d\gamma_{n+1})\). From this result and Theorem 2.3, we obtain

\[
\langle f, K_{\sigma, \rho}((\ldots), (s, y)) \rangle_{\sigma, \rho} = f(s, y).
\]

This completes the proof of the Proposition.

\[\square\]

**Theorem 4.6.** Let \(m \in L^\infty(d\nu_{n+1})\) and \(\epsilon > 0\), for every \(h \in L^2(d\nu_{n+1})\) and \(\rho > 0\), there exists a unique function \(f_{\rho, h, \epsilon}^*\), where the infimum

\[
\inf_{f \in \Omega_\sigma} \{ \rho \| f \|_\sigma^2 + \| h - T_{m, \epsilon} f \|_{2, \nu_{n+1}}^2 \},
\]

(4.6)

is attained. Moreover the extremal function \(f_{\rho, h, \epsilon}^*\) is given by

\[
f_{\rho, h, \epsilon}^*(s, y) = \int_0^{+\infty} \int_{\mathbb{R}^n} h(r, x) V_{\sigma, \rho}((r, x), (s, y)) d\nu_{n+1}(r, x),
\]

(4.7)

where \(V_{\sigma, \rho}((r, x), (s, y)) = \int \int_{Y_+} \frac{m_e \circ \theta(\mu, \lambda) \varphi(\mu, \lambda)(r, x) \varphi(\mu, \lambda)(s, y)}{\rho \sigma(\mu, \lambda) + |m_e \circ \theta(\mu, \lambda)|^2} d\gamma_{n+1}(\mu, \lambda)\).

**Proof.** The existence and unicity of the extremal function \(f_{\rho, h, \epsilon}^*\) satisfying relation (4.6) is given by [10, 12, 21]. On the other hand From Proposition 4.2 and 4.5, we have

\[
f_{\rho, h, \epsilon}^*(s, y) = \langle h, T_{m, \epsilon}(K_{\sigma, \rho})((\ldots), (s, y)) \rangle_{\nu_{n+1}},
\]

(4.8)

where \(\langle \cdot, \cdot \rangle_{\nu_{n+1}}\) denoted the inner product of \(L^2(d\nu_{n+1})\), and \(K_{\sigma, \rho}\) is the Kernel given by relation (4.4). According to Proposition 3.2 (ii), (4.5) and (4.8), we obtain

\[
V_{\sigma, \rho}((r, x), (s, y)) = \int \int_{Y_+} \frac{m_e \circ \theta(\mu, \lambda) \varphi(\mu, \lambda)(r, x) \varphi(\mu, \lambda)(s, y)}{\rho \sigma(\mu, \lambda) + |m_e \circ \theta(\mu, \lambda)|^2} d\gamma_{n+1}(\mu, \lambda).
\]

\[\square\]

**Theorem 4.7.** Let \(m \in L^\infty(d\nu_{n+1})\) and \(h \in L^2(d\nu_{n+1})\). The extremal function \(f_{\rho, h, \epsilon}^*\) belongs to \(\Omega_\sigma([0, +\infty[ \times \mathbb{R}^n)\), and we have

\[
\| f_{\rho, h, \epsilon}^* \|_{\sigma}^2 \leq \frac{1}{4\rho} \| h \|_{2, \nu_{n+1}}^2.
\]
Proof. Let \((s, y) \in [0, +\infty[ \times \mathbb{R}^n\). From Lemma 4.4 (ii) and Theorem 2.3, we have

\[ V_{\sigma, \rho}((r, x), (s, y)) = \mathcal{F}^{-1}(\Phi_{(s,y)})(r, x). \]

By Theorem 2.4, it follows that the function \(V_{\sigma, \rho}((\ldots), (s, y))\) belongs to \(L^2(d\nu_{n+1})\) and using Corollary 2.5, we get

\[
\begin{align*}
    f^*_{\rho, h, \epsilon}(s, y) &= \int \int_{Y_+} \mathcal{F}(h)(\mu, \lambda) \Phi_{(s,y)}(\mu, \lambda) d\gamma_{n+1}(\mu, \lambda) \\
    &= \int \int_{Y_+} \mathcal{F}(h)(\mu, \lambda) \frac{m_\epsilon \circ \theta(\mu, \lambda) \varphi(\mu, \lambda)(s, y)}{\rho \sigma(\mu, \lambda) + |m_\epsilon \circ \theta(\mu, \lambda)|^2} d\gamma_{n+1}(\mu, \lambda).
\end{align*}
\]

On the other hand, the function \((\mu, \lambda) \mapsto \mathcal{F}(h)(\mu, \lambda) \frac{m_\epsilon \circ \theta(\mu, \lambda)}{\rho \sigma(\mu, \lambda) + |m_\epsilon \circ \theta(\mu, \lambda)|^2}\) belongs to \(L^1(d\gamma_{n+1}) \cap L^2(d\gamma_{n+1})\), then by Theorem 2.3, we have

\[
    f^*_{\rho, h, \epsilon}(s, y) = \mathcal{F}^{-1}\left( \mathcal{F}(h) \frac{m_\epsilon \circ \theta(\ldots)}{\rho \sigma(\ldots) + |m_\epsilon \circ \theta(\ldots)|^2} \right)(s, y).
\]

From Theorem 2.4, it follows that, the function \(f^*_{\rho, h, \epsilon}\) belongs to \(L^2(d\nu_{n+1})\), and we have for every \((\mu, \lambda) \in Y,\)

\[
    \left| \mathcal{F}(f^*_{\rho, h, \epsilon})(\mu, \lambda) \right|^2 = \left| \mathcal{F}(h)(\mu, \lambda) \frac{m_\epsilon \circ \theta(\mu, \lambda)}{\rho \sigma(\mu, \lambda) + |m_\epsilon \circ \theta(\mu, \lambda)|^2} \right|^2 \leq \frac{1}{4\rho \sigma(\mu, \lambda)} \left| \mathcal{F}(h)(\mu, \lambda) \right|^2,
\]

thus, from Theorem 2.4 and Definition 4.1, we obtain

\[
    \|f^*_{\rho, h, \epsilon}\|_{\sigma}^2 \leq \frac{1}{4\rho} \|h\|_{2, \nu_{n+1}}^2.
\]

\[ \square \]

Theorem 4.8. Third Calderón’s formula Let \(m \in L^\infty(d\nu_{n+1})\), and \(f \in \Omega_\sigma([0, +\infty[ \times \mathbb{R}^n\). The extremal function \(f^*_{\rho, \epsilon}\) given by

\[
    f^*_{\rho, \epsilon}(s, y) = \int_0^{+\infty} \int_{\mathbb{R}^n} T_m \epsilon f(r, x) V_{\sigma, \rho}((r, x), (s, y)) d\nu_{n+1}(r, x),
\]

satisfies

(i)

\[
    \lim_{\rho \to 0^+} \|f^*_{\rho, \epsilon} - f\|_\sigma = 0. \quad (4.10)
\]

(ii)

\[
    \lim_{\rho \to 0^+} f^*_{\rho, \epsilon} = f, \quad \text{uniformly.}
\]
Proof. (i) Let $f \in \Omega_{\sigma}([0, +\infty[ \times \mathbb{R}^n)$, $h = T_{m, \varepsilon} f$, and $f_{\rho, \varepsilon}^* = f_{\rho, \varepsilon}^*$. From Proposition 4.2, the function $h$ belongs to $L^2(d\gamma_{n+1})$. Applying Definition 3.1, and relation (4.9), we obtain

$$\mathcal{F}(f_{\rho, \varepsilon}^*)(\mu, \lambda) = \mathcal{F}(f)(\mu, \lambda) \frac{|m_\varepsilon \circ \theta(\mu, \lambda)|^2}{\rho \sigma(\mu, \lambda) + |m_\varepsilon \circ \theta(\mu, \lambda)|^2}$$

Thus, it follows that for every $(\mu, \lambda) \in \Upsilon$

$$\mathcal{F}(f_{\rho, \varepsilon}^* - f)(\mu, \lambda) = -\rho \sigma(\mu, \lambda) \mathcal{F}(f)(\mu, \lambda) \frac{\rho^2 \sigma^3(\mu, \lambda) |\mathcal{F}(f)(\mu, \lambda)|^2}{(\rho \sigma(\mu, \lambda) + |m_\varepsilon \circ \theta(\mu, \lambda)|^2)^2} d\gamma_{n+1}(\mu, \lambda).$$

Then, the result follows from the fact

$$\frac{\rho^2 \sigma^3(\mu, \lambda) |\mathcal{F}(f)(\mu, \lambda)|^2}{(\rho \sigma(\mu, \lambda) + |m_\varepsilon \circ \theta(\mu, \lambda)|^2)^2} \leq \sigma(\mu, \lambda) |\mathcal{F}(f)(\mu, \lambda)|^2,$$

and the dominated convergence theorem.

(ii) By relation (4.2), the function $\frac{1}{\sqrt{\sigma}}$ belongs to $L^2(d\gamma_{n+1})$, hence for $f \in \Omega_{\sigma}([0, +\infty[ \times \mathbb{R}^n)$, the function $\mathcal{F}(f)$ belongs to $L^1(d\gamma_{n+1})$. Then, from (4.11) and Theorem 2.3, we get

$$f_{\rho, \varepsilon}(s, y) - f(s, y) = \int_{\Upsilon} -\rho \sigma(\mu, \lambda) \mathcal{F}(f)(\mu, \lambda) \frac{\rho \sigma(\mu, \lambda) |\mathcal{F}(f)(\mu, \lambda)|^2}{\rho \sigma(\mu, \lambda) + |m_\varepsilon \circ \theta(\mu, \lambda)|^2} \phi(\mu, \lambda)(s, y) d\gamma_{n+1}(\mu, \lambda).$$

By using the dominated convergence theorem and the fact

$$\frac{|-\rho \sigma(\mu, \lambda) \mathcal{F}(f)(\mu, \lambda) \phi(\mu, \lambda)(s, y)|}{\rho \sigma(\mu, \lambda) + |m_\varepsilon \circ \theta(\mu, \lambda)|^2} \leq |\mathcal{F}(f)(\mu, \lambda)|,$$

we deduce that

$$\lim_{\rho \to 0^+} \sup_{(s, y) \in [0, +\infty[ \times \mathbb{R}^n} |f_{\rho, \varepsilon}(s, y) - f(s, y)| = 0.$$  

Which completes the proof of the Theorem. \qed
REFERENCES


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