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SURFACES OF GENERAL TYPE WITH VANISHING GEOMETRIC GENUS FROM DOUBLE PLANES

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We show how to construct some old and new surfaces of general type with vanishing geometric genus from double planes, by computing explicit equations of their branch curves.

1. Introduction

The first example of a double plane X whose smooth model is a surface Y of general type with vanishing geometric genus has been given by Campedelli in 1932 (see [11]): its branch curve has degree 10 and has six $[3,3]$-points $p_1,\dots,p_6$, such that there is no conic passing through $p_1,\dots,p_6$ (a $[3,3]$-point is a triple point with infinitely near another triple point, cf. §2 for notation and definitions). This double plane has bigenus $P_2(Y) = h^0(Y, \mathcal{O}_Y(2K_Y)) = 3$.

Campedelli proposed also the construction of a reduced curve of degree 10 with five $[3,3]$-points $p_1,\dots,p_5$ and a point p_6 of multiplicity 4, such that there is no conic through $p_1,\dots,p_6$. A curve with these properties has been explicitly constructed only much later by Oort and Peters in [24], following an idea due to Vik. S. Kulikov. The resulting double plane has $p_g = 0$ and $P_2 = 2$.

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We say that the double planes branched along curves of degree 10 with singularities as above are *of Campedelli type* $C_{6,0}$ (the former one) and $C_{5,1}$ (the latter one).

Just one year before Campedelli, Godeaux constructed the first example of surface of general type with $p_g = 0$ (see [17]). He considered a quotient of a quintic surface in $\mathbb{P}^3$ by a freely acting cyclic group of order 5 of projective transformations. Some years ago, in [23], Murakami showed that the Godeaux surface is birationally equivalent to a double plane of type $C_{5,1}$, branched over an irreducible curve, constructed by Stagnaro in [31].

The first examples of surfaces of general type with $p_g = 0$ and $4 \leq P_2 \leq 7$, given by Burniat in the 1960's as bidouble covers of $\mathbb{P}^2$ branched along suitable 9 lines, are birationally equivalent to double planes too; the same is true for Inoue's surface, the first one of general type with $p_g = 0$ and $P_2 = 8$ (cf. [19] and [21]).

Although surfaces of general type with $p_g = 0$ have been studied by several mathematicians like D. Mumford, E. Bombieri, Y. Miyaoka, A. Beauville, I. Dolgachev, F. Catanese, M. Reid, R. Barlow, J. Keum, Y. Lee, M. Mendes Lopes, R. Pardini, D. Naie, C. Ciliberto, C. Werner, P. Supino, Vik. S. Kulikov, R. Pignatelli, I. Bauer, C. Rito, J. Park, H. Park, D. Shin, with different techniques and from many points of view, their classification is not yet known. We cannot report here about all relevant contributions during the last decades; we will mention only few results, mainly related to double planes (see [3] for a recent survey).

A smooth minimal surface Y of general type with $p_g(Y) = h^2(Y, \mathcal{O}_Y) = 0$ has $q(Y) = h^1(Y, \mathcal{O}_Y) = 0$, $\chi(\mathcal{O}_Y) = 1$ and $K_Y^2 \geq 1$, or equivalently $P_2(Y) \geq 2$. On the other hand, Miyaoka-Yau inequality implies that $K_Y^2 \leq 9$, i.e. $P_2(Y) \leq 10$, and it has been shown by Dolgachev, Mendes Lopes and Pardini in [15] that there exists no double plane whose smooth minimal model is a surface of general type with $p_g = 0$ and $P_2 = 10$.

Some years ago, in [25], Rita Pardini classified double planes of general type whose smooth model has $p_g = 0$ and $P_2 = 9$ (see Theorem 3.1 later) by means of quotients of a product of two curves under a suitable group action, a construction originally due to Beauville. However Pardini did not give equations of the branch curves of these double planes. Pardini herself says that "it seems very difficult to construct the plane models directly" (p. 112 in [25]).

The other extremal case $P_2 = 2$ has been classified in [8]. It turns out that double planes of general type with $p_g = 0$ and $P_2 = 2$ are birationally equivalent to double planes either of Campedelli type $C_{5,1}$ or branched over a reduced curve $B = C + L_1 + L_2$, where L_1, L_2 are lines, such that B has degree 14, multiplicity 6 at the point $L_1 \cap L_2$, two $[5,5]$-points $p_i \in L_i$, $i = 1,2$, one $[3,3]$-point and two points of multiplicity 4. We say that the latter double plane is of *Du Val type* $DV_{2;1,3}$, because it is a degeneration of examples originally described by Du Val (cf. Example 2.2 and Definition 2.3). The existence of double planes of type $DV_{2;1,3}$ is proved in that paper, together with some explicit examples of their branch curves.

The subsequent case $P_2 = 3$ has been studied and essentially classified in [10], although the problem of the existence of double planes with $p_g = 0, P_2 = 3$ and bicanonical map not composed with the involution has been left open.

Therefore, in this paper, we deal with double planes of general type with $p_g = 0$ and $4 \leq P_2 \leq 9$.

Nowadays, the better way to construct surfaces of general type with $p_g = 0$ is to consider quotients of the product of two curves by the action of a finite group acting on each curve, see e.g. [2] for a list of surfaces constructed in that way. However, it is not clear how to find explicit equations of the branch curve of double planes appearing in that list.

Other very interesting classification results have been proved by Borrelli in [5], concerning surfaces of general type with non-birational bicanonical map. In his classification, a central role is played by *Du Val double planes*, see Example 2.2.

In this paper we find the explicit equations of the branch curves of some double planes of Du Val types $DV_{6;0,0}$, $DV_{4;1,1}$, $DV_{4;0,2}$, $DV_{3;2,1}$ and $DV_{2;3,1}$, see Definition 2.3 for notation. Note that the bicanonical map of the smooth model of a Du Val double plane is composed with the involution induced by the double plane structure.

Furthermore, we construct other branch curves of double planes of general type with vanishing geometric genus, whose degree and configuration of singularities were previously unknown. In these examples the bicanonical map of the smooth model is not composed with the involution.

This paper is organized as follows: after introducing notation in §2, in the following sections we will give several examples of surfaces of general type with $p_g = 0$, considering in each section a different value of P_2, backwards from 9 to 4. As suggested by the referee, in the appendix we list the explicit equations of the plane curves we construct so that they can be copied and pasted in a computer algebra system.

Let us briefly explain our approach to construct the examples. Regarding Du Val double planes, we already know the degree and the number and type of singularities of the branch curve. One immediately sees that the singular points of the branch curve cannot be general in the plane, because they have to impose dependent conditions to curves of suitable degree. On the other hand, these points cannot be too special, e.g. they cannot lie on a conic, otherwise the geometric genus is > 0.

In order to construct new examples, our approach is threefold:

(a) try to split the branch curve in components which should be easier to construct;

(b) try to construct the branch curve invariant under a finite-order linear automorphism of $\mathbb{P}^2$;

(c) if (a) & (b) does not work, apply a Cremona transformation, e.g. a quadratic one, and try again (a) & (b).

There are two linear automorphisms of $\mathbb{P}^2$ which turn out to be useful: the first one is the automorphism $\iota_2\colon \mathbb{P}^2 \to \mathbb{P}^2$ of order 2 given, in homogeneous coordinates x, y, z on $\mathbb{P}^2$, by

$$\iota_2(x, y, z) = (x, -y, z); \tag{1}$$

the second one is the automorphism $\iota_5\colon \mathbb{P}^2 \to \mathbb{P}^2$ of order 5 given by

$$\iota_5(x, y, z) = (\varepsilon x, \varepsilon^2 y, z), \tag{2}$$

where ε is a 5^{th} root of unity, say $\varepsilon = e^{2\pi\sqrt{-1}/5}$. The latter automorphism has three isolated fixed points, namely the coordinate points. The former one has the line $y = 0$ of fixed points and the isolated fixed point $(0, 1, 0)$.

Almost all computations in this paper were performed by using Maple [20]. In particular, we defined some procedures in Maple which help to construct plane curves with prescribed singularities. The interested reader may contact the authors in order to get them. Note that similar procedures have been defined for the computer algebra system Magma by Rito in [28].

2. Notation, definitions and preliminaries

A singular point p of type $[m, n]$, or briefly a $[m, n]$-point, of a curve C on a surface is a point of multiplicity m with infinitely near, in the first neighbourhood, a point of multiplicity n. For example, a *tacnode* is a point of type $[2, 2]$. If $C \subset \mathbb{P}^2$ and $n = m$, the unique line which has intersection multiplicity larger than m with C at p is called the *proper* tangent to C at p.

A *double plane* is a double cover of $\mathbb{P}^2$, i.e. a finite morphism $\pi\colon X \to \mathbb{P}^2$ of degree 2, and it is uniquely determined by its branch curve $B \subset \mathbb{P}^2$. If $f(x, y) = 0$ is the affine equation of B, where x, y are affine coordinates in $\mathbb{P}^2$, then X is birationally equivalent to the surface in $\mathbb{P}^3$ with affine equation $z^2 = f(x, y)$.

One may assume, with no loss in generality, that X is normal, or equivalently that B is reduced.

If B is singular, then X is singular too and there exists a birational morphism $\sigma\colon S \to \mathbb{P}^2$ such that the normalization of the fibred product $Y = S \times_{\mathbb{P}^2} X$ is smooth and the induced map $\rho\colon Y \to S$ is a double cover. Indeed, one can proceed as follows: (1) blow-up $\mathbb{P}^2$ at a singular point of the branch curve B, (2) normalize the double cover branched over the total transform of B, which is again a double cover, say $Y' \to S'$ branched over a curve B'. If B' is still singular, repeat steps (1) and (2) replacing $\mathbb{P}^2$ with S' and B with B'. After finitely many steps, one gets rid of all the singularities (cf. e.g. [9], [18]). In such a way, one gets the so-called *canonical resolution* $\rho\colon Y \to S$ of the double plane $\pi\colon X \to \mathbb{P}^2$. If $D \subset S$ is the smooth branch curve of ρ, then D is an *even* divisor—i.e. D is divisible by 2 in $\mathrm{Pic}(S)$—and the projection formula says that $\rho_*\mathcal{O}_Y \cong \mathcal{O}_S \oplus \mathcal{O}_S(-D/2)$, thus the geometric genus of Y is

$$p_g(Y) = h^0(Y,K_Y) = h^0(S,K_S+D/2)+h^0(S,K_S) = h^0(S,K_S+D/2) \qquad (3)$$

and the second plurigenus, also known as the *bigenus*, of Y is

$$P_2(Y) = h^0(Y,2K_Y) = h^0(S,2K_S+D)+h^0(S,2K_S+D/2). \qquad (4)$$

More generally, for any positive integer m, the m-plurigenus of Y is

$$P_m(Y) = h^0(Y,mK_Y) = h^0(S,mK_S+mD/2)+h^0(S,mK_S+(m-1)D/2). \quad (5)$$

Equivalently, one can directly compute the adjoint and pluri-adjoint curves in the way explained in detail by the second author in [32] and [33].

Another birational invariant of surfaces is the irregularity:

$$q(Y) = h^1(Y,K_Y) = h^1(S,-D/2) = p_g(Y) - p_a(D/2), \qquad (6)$$

where $p_a(D/2) = D(D+2K_S)/8+1$ is the arithmetic genus of $D/2$ on S. By abusing notation, we often write $P_m(X)$ and $q(X)$ instead of $P_m(Y)$ and $q(Y)$.

The double cover structure on Y determines an involution $\iota\colon Y \to Y$, i.e. an automorphism of order 2. One says that a rational map $\varphi\colon Y \dashrightarrow Z$, for some Z, is *composed with the involution* ι if and only if $\varphi \circ \iota = \varphi$.

From now on, we always assume that Y is a surface of general type and $p_g(Y) = 0$. With these assumptions, it is well-known that (cf., e.g., Corollary 3.6 in [8]):

Proposition 2.1. *The bicanonical map φ_{2K_Y} is composed with the involution ι if and only if $h^0(S,2K_S+D/2) = 0$.* □

Let us recall some significant examples of double planes.

Example 2.2 (Du Val)**.** Suppose that there exist a reduced curve $B = C + L_1 + \cdots + L_d$, $d \geq 1$, such that

(i) $L_1, \dots, L_d$ are lines through a fixed point p_0;

(ii) C is a curve of degree $10 + d$ having multiplicity $d + 2$ at p_0 and $[4,4]$-points $p_i \in L_i$, where L_i is the proper tangent, for each $i = 1, \dots, d$.

If C has exactly such singularities at $p_0, \dots, p_d$, and no more, one sees that the smooth minimal model of the double plane branched along B is a surface of general type with $p_g = 6 - d$, $q = 0$, $K^2 = 8$ and non-birational bicanonical map (assuming that there is no conic through $p_1, \dots, p_6$ if $d = 6$). The bicanonical map is indeed composed with the involution induced by the double plane structure.

These examples have been originally described by Du Val in [16] and have been called *Du Val ancestors* in the paper [12] by Ciliberto.

Note that the invariants of the double plane do not change even if B acquires *non-essential* singularities, namely double points or triple points with no infinitely near triple point. On the other hand, if B acquires essential singularities, then p_g and K^2 of the smooth minimal model of the double plane decrease.

Definition 2.3. Let us say that a double plane X is of *Du Val type* $(d;t,q)$, or briefly of type $DV_{d;t,q}$, if X is branched along a reduced curve $B = C + L_1 + \cdots + L_d$ as in Example 2.2, such that:

- C has further t points $p_{d+1}, \dots, p_{d+t}$ of type $[3,3]$,
- C has further $q = 6 - d - t$ points $p_{d+t+1}, \dots, p_{d+t+q=6}$ of multiplicity 4,
- there is no conic through $p_1, \dots, p_6$.

In order to distinguish surfaces of general type, another invariant is the torsion subgroup of the Picard group. For double planes, the subgroup Tors_2 of 2-torsion elements can be easily computed by means of the following:

Lemma 2.4 (Beauville, Lemme 2 in [4])**.** *Let $V \to W$ be a double cover between two smooth surfaces V and W, branched over the smooth curve $B_1 + \cdots + B_n$, where $B_1, \dots, B_n$ are irreducible. Suppose that* $\mathrm{Pic}(W)$ *has no 2-torsion element. Define a group homomorphism ψ: $(\mathbb{Z}/2\mathbb{Z})^{\oplus n} \to (\mathbb{Z}/2\mathbb{Z}) \otimes \mathrm{Pic}(W)$ by $\psi(\varepsilon_1, \dots, \varepsilon_n) = \sum_{i=1}^n \varepsilon_i B_i$. Then* $\mathrm{Tors}_2(V) \cong \ker(\psi)/\langle(1, \dots, 1)\rangle$.

Notation 2.5. We denote the linear system of the plane curves, say of degree d with multiplicity m_0 at a point p_0 and with a point p_1 of type $[m_1, m_1']$, as follows:

$$|\, dL - m_0 p_0 - m_1 p_1 - m_1' p_1' \,|,$$

where p_1' is the infinitely near point to p_1 of multiplicity m_1'.

3. Double planes of general type with $p_g = 0$ and $P_2 = 9$

These double planes have been classified by Rita Pardini in [25]:

Theorem 3.1 (Pardini)**.** *A double plane, whose smooth model is a surface of general type with $p_g = 0$ and $P_2 = 9$, is birationally equivalent to a double plane of two types, I and II. Type I is Du Val type $DV_{6;0,0}$, such that the curve C of degree 16 either is irreducible or has two irreducible components of degree 8. Type II double planes are branched along a curve $B = C + L_1 + \cdots + L_5$, where $L_1, \ldots, L_5$ are lines through a fixed point p_0 and C is an irreducible curve of degree 21 with the following singularities:*

- *a point of multiplicity 9 at p_0,*
- *a $[6,6]$-point $p_i \in L_i$, $i = 1, \ldots, 5$, where L_i is the proper tangent,*

such that the canonical resolution of the double plane is obtained by exactly 11 blowing-ups and $|5L - p_0 - \sum_{i=1}^{5}(2p_i + p_i')| = \emptyset$, where p_i' is the infinitely near point to p_i in the direction of the line L_i.

Conversely, if there exists such a curve B, then the smooth model of the double plane branched along B is a surface of general type with $p_g = 0$ and $P_2 = 9$. □

However Pardini did not give equations of the branch curves of these double planes. Pardini herself says that "it seems very difficult to construct the plane models directly" (p. 112 in [25]).

In this section we find the equation of the branch curve of two double planes, one of type II and the other of type I_a in Pardini's notation, i.e. of type $DV_{6;0,0}$ such that the curve C of degree 16 is the union of two curves of degree 8.

Example 3.2. We now find the equation of an irreducible curve C_{21} of degree 21, with the properties described in Theorem 3.1, whose equation is invariant under the automorphism $\iota_5 \colon \mathbb{P}^2 \to \mathbb{P}^2$ defined in (2).

Let f be a polynomial of degree 21 such that $f(\varepsilon x, \varepsilon^2 y, z) = f(x,y,z)$, where ε is a 5$^{\text{th}}$ root of unity, so f is ι_5-invariant. We impose to the curve $C_{21} \colon f = 0$ the following singularities:

- the point $p_0 = (0,0,1)$ of multiplicity 9;
- the point $p_5 = (1,1,1)$ of type $[6,6]$, where the proper tangent is the line $L_5 \colon y = x$, passing through p_0 and p_5.

This forces C_{21} to have five points $p_i = (\varepsilon^i, \varepsilon^{2i}, 1)$, $i = 1, \ldots, 5$, of type $[6,6]$, where the proper tangent line to C_{21} at p_i is $L_i \colon y = \varepsilon^i x$.

Since there exists a projectivity which fixes the three fixed points of ι_5 and maps p_5 to any other point (x,y,z) of $\mathbb{P}^2$ with $xyz \neq 0$, our choice of the coordinates of p_5 causes no loss in generality.

There are 51 monomials of degree 21 in 3 variables which are ι_5-invariant. The point p_0 of multiplicity 9 imposes only 9 (instead of 45) conditions, because p_0 is a fixed point of ι_5. Since a point of multiplicity 6 imposes 21 conditions, one would expect to find no curve like C_{21} in such a way.

Quite surprisingly, it turns out that this C_{21} exists, as one may check by using a computer algebra software, like e.g. Maple or Magma. The affine equation of C_{21} is written in the appendix, see A.1.

Proposition 3.3. *The curve C_{21} of degree 21 found in Example 3.2 is irreducible and, together with the five lines $L_i\colon\ y = \varepsilon^i x$, $i = 1,\dots,5$, is the branch curve of a double plane, whose smooth model is a surface of general type with $p_g = 0$ and $P_2 = 9$.*

Proof. Maple is able to check that C_{21} is irreducible in the algebraic closure of $\mathbb{Q}$, so it is irreducible also over $\mathbb{C}$. Then one checks that C_{21} has exactly the imposed singularities, and not worse, and that $|5L - p_0 - \sum_{i=1}^{5}(2p_i + p_i')| = \emptyset$. The rest of the statement follows by Pardini's Theorem 3.1 (cf. [25, Theorem 5.2]). $\square$

Remark 3.4. The singularity of C_{21} at p_0 is not ordinary, indeed C_{21} has 7 branches locally around p_0. Nonetheless, it is resolved by just one blowing-up, because the strict transform of C_{21} has a flex at the point infinitely near to p_0 in the direction of the line $x = 0$, where the flex tangent is the exceptional curve.

Question 3.5. Is there any geometric reason which explains the existence of such unexpected ι_5-invariant polynomial of degree 21?

Let us now see how to find some explicit equation of a branch curve of Pardini's type I_a, i.e. of type $DV_{6;0,0}$ with curve C_{16} of degree 16 split in two curves C_8 and C_8' of degree 8.

Example 3.6. We were not able to find equations of C_8 and C_8' which are invariant under either ι_2 as in (1) or ι_5 as in (2). So we perform a quadratic transformation α and we consider curves C_{10}, C_{10}' of degree 10 with the following singularities:

- a point of multiplicity 6 at p_0,
- a tacnode q_i, $i = 1,\dots,6$, such that the proper tangent H_i to C_{10} and to C_{10}' at q_i passes also through p_0,

- a further point q_7 of multiplicity 4,

so that α maps C_{10}, C'_{10} respectively to curves C_8, C'_8 in such a way that $C_8 + C'_8 +$ 6 lines is a branch curve of a Du Val double plane of type $(6;0,0)$.

We want the equation of C_{10} to be ι_2-invariant, where ι_2 is as in (1). In particular we impose p_0 to be a fixed point of ι_2, namely $(0,0,1)$, and q_7 to be the isolated fixed point of ι_2, i.e. $(0,1,0)$. Then we choose three points q_1, q_2, q_3 in the plane and we require $q_{7-i} = \iota_2(q_i)$, $i = 1,2,3$.

Note that ι_2-invariant polynomials of degree 10 in x,y,z depend on 36 homogeneous parameters and we are imposing:

- 12 (and not 21) conditions for p_0 to be of multiplicity 6,
- 4 (and not 10) conditions for q_7 to be of multiplicity 4,
- 6 conditions for q_i, $i = 1,2,3$, to be a tacnode, with H_i as tacnodal tangent.

Thus one expects to find a pencil of such curves of degree 10.

By using a computer algebra software, one may check that this naive expectation turns out to be right.

Indeed, we chose $q_1 = (1,1,1)$, $q_2 = (2,1,1)$ and $q_3 = (-1,3,1)$, so that $q_4 = (1,-1,1)$, $q_5 = (2,-1,1)$ and $q_6 = (-1,-3,1)$, and we find a pencil of curves of degree 10 whose general member is irreducible and it has exactly the imposed singularities, and no worse. By applying the quadratic Cremona transformation

$$\alpha \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \qquad \alpha(x,y,z) = ((x-y)(x-z), y(z-x), z(y-x)).$$

with fundamental points p_0, q_1, q_7, we find a pencil $C_8 + tC'_8$ of curves of degree 8, where C_8, C'_8 have the affine equations written in the appendix, see A.2.

The singular points of C_8 and C'_8 are $p_0 = (0,0,1)$ of multiplicity 4, $p_1 = (0,1,0)$, $p_2 = (2,3,1)$, $p_3 = (4,3,2)$, $p_4 = (-1,1,1)$ and $p_5 = (3,1,-3)$ which are tacnodes with proper tangent respectively

$$L_1 \colon x = 0,\ L_2 \colon 3x - 2y = 0,\ L_3 \colon 3x - 4y = 0,\ L_4 \colon x + y = 0,\ L_5 \colon x - 3y = 0;$$

moreover, C_8 and C'_8 have a point p_6 of type $[2,2]$ which is infinitely near to p_0 in the direction of the line $L_6 \colon x - 2y = 0$ in such a way that the intersection multiplicity with L_6 at p_0 is 8.

Proposition 3.7. *Let $C_8, C'_8, L_1, \ldots, L_6$ be as in the previous example. Then $B = C_8 + C'_8 + L_1 + \cdots + L_6$ is the branch curve of a Du Val double plane X of type $DV_{6;0,0}$, in particular its smooth minimal model is a surface of general type with $p_g = 0$ and $P_2 = 9$.*

Proof. Maple is able to check that the curves C_8, C'_8 are irreducible in the algebraic closure of $\mathbb{Q}$, so they are irreducible also over $\mathbb{C}$. Then one checks that $C_8 + C'_8$ has exactly the wanted singularities, and not worse, and that there is no conic through $p_0, p_1, \dots, p_5$. Therefore X is a double plane of Pardini's type I_a by Pardini's Theorem 3.1, cf. [25, Theorem 5.2]. □

We refer the readers, interested to more details about the classification of double planes with these invariants, to Pardini's paper [25].

4. A double plane of general type with $p_g = 0$ and $P_2 = 8$

In this section we construct a double plane, whose smooth model is a surface of general type with $p_g = 0$ and $P_2 = 8$, which has a new configuration of degree and singularities of the branch curve. In particular the bicanonical map of its smooth model is not composed with the involution induced by the double plane structure. The referee informed us that a double plane with the same invariants and properties has been constructed by Rito in the unpublished paper [28]. Note however that our construction is different from that of Rito, in particular our branch curve is ι_2-invariant.

Example 4.1. We now find an equation of a reduced curve $B = C_6 + C_{14} + L_1 + L_2 + L_3 + L_4$, where $L_1, \dots, L_4$ are lines through a fixed point p_0 and B is a curve of degree 24 with the following singularities:

- a point of multiplicity 12 at p_0,
- a point $p_i \in L_i$, $i = 1, \dots, 4$, of type $[7,7]$, where L_i is the proper tangent,
- a further $[5,5]$-point p_5, where we denote by L_5 its proper tangent.

So we require the curve C_{14} of degree 14 to have the following singularities:

- a point of multiplicity 6 at p_0,
- a $[4,4]$ point $p_i \in L_i$, $i = 1, \dots, 4$, where L_i is the proper tangent,
- a further $[4,4]$-point p_5, with L_5 as proper tangent,

and the curve C_6 to be a sextic with the following properties:

- a double point at p_0;
- a tacnode at p_i, $i = 1, \dots, 4$, where L_i is the tacnodal tangent;
- passing simply through p_5, with L_5 as tangent line.

Setting ι_2 the automorphism of $\mathbb{P}^2$ of order 2 defined in (1), we require B to be ι_2-invariant, namely the equations of C_6 and of C_{14} are required to be ι_2-invariant, whereas $\iota_2(L_1) = L_3$ and $\iota_2(L_2) = L_4$.

We choose p_0 and p_5 to be fixed points of ι_2, e.g. $p_0 = (0,0,1)$ and $p_5 = (1,0,0)$, and L_5 to be the line $z = 0$. In such a way we are imposing to sextic ι_2-invariant polynomials in x,y,z, which depend on 16 parameters, the following conditions:

- 2 (and not 3) for p_0 to be double;
- 1 (and not 2) for passing through p_5 with tangent line L_5;
- 6 for p_i, $i = 1,2$, to be a tacnode, with L_i as tacnodal tangent.

Thus we expect to find one ι_2-invariant sextic with these singularities.

Similarly, ι_2-invariant polynomials in x,y,z of degree 14 depend on 64 parameters and we are imposing to these curves the following conditions:

- 12 (instead of 21) for p_0 to have multiplicity 6;
- 10 (instead of 20) for p_5 to be a point of type $[4,4]$ with proper tangent L_5;
- 20 for p_i, $i = 1,2$, to be a point of type $[4,4]$ with proper tangent L_i.

Therefore we expect to find a pencil of such ι_2-invariant curves of degree 14.

These naive expectations turn out to be right.

Let us choose $p_1 = (2,1,1)$, $p_2 = (1,2,1)$, so that $p_3 = \iota_2(p_1) = (2,-1,1)$ and $p_4 = \iota_2(p_2) = (1,-2,1)$. Hence $L_1\colon x = 2y$, $L_2\colon 2x = y$, $L_3\colon x = -2y$, $L_4\colon 2x = -y$. By using Maple, we found such an irreducible sextic plane curve C_6, with affine equation written in the appendix (see A.3) and we found a pencil of plane curves of degree 14 with the above singularities and such that its general member is irreducible, e.g. the curve C_{14} written in the appendix, see A.3. One can check that C_6 and C_{14} have exactly the prescribed singularities.

Proposition 4.2. *The smooth model Y of the double plane branched along the curve $B = C_6 + C_{14} + L_1 + \cdots + L_4$ of the previous Example 4.1 is a surface of general type with $p_g = 0$ and $P_2 = 8$. Furthermore* $\mathrm{Tors}_2(Y) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 4}$.

Proof. The branch curve is $B \in |24L - 12p_0 - \sum_{i=1}^{4} 7(p_i + p_i') - 5(p_5 + p_5')|$, where the coordinate of the points are given in Example 4.1 and p_i', $i = 1,\dots,5$, is the infinitely near point to p_i in the direction of the line L_i. According to (3), the canonical linear system of Y is determined by

$$\left| 5L - p_0 - \sum_{i=1}^{4}(2p_i + p_i') - 2p_5 - p_5' \right| + \sum_{i=1}^{4} L_i$$

that turns out to be empty: quintic polynomials in x,y,z depend on 21 parameters and one checks that they have to satisfy 21 independent linear conditions. Therefore, one has $p_g(Y)=0$.

According to (4), the second summand of the bigenus of Y is determined by $\sum_{i=1}^4 L_i + |2L-\sum_{i=1}^5 p_i|$, where the last linear system is the unique conic $y^2+3xz-7z^2=0$. By Proposition 2.1, it follows that the bicanonical map of Y is not composed with the involution induced by the double plane structure.

The first summand of (4) is determined by

$$\left|14L-6p_0-\sum_{i=1}^4 4(p_i+p_i')-3(p_5+p_5')\right|+\sum_{i=1}^4 L_i$$

which turns out to have dimension 6, therefore $P_2(Y)=7+1=8$. Two general curves in the above linear system meet in 14 points, off the base points, which implies that the smooth minimal model has $K^2=7$ and it is a surface of general type. Note that the pencil of lines through p_0 pulls back to an hyperelliptic pencil of curves of genus 5 on Y.

It remains to show only the assertion about the 2-torsion. Following the canonical resolution, let S be the blowing-up of $\mathbb{P}^2$ at p_0, p_i, p_i', $i=1,\dots,5$. Let E_i, $i=0,\dots,5$, be the irreducible exceptional curve corresponding to p_i. Note that E_0 is a (-1)-curve, whereas $E_1,\dots,E_5$ are (-2)-curves. By abusing notation, let us denote by C_6, C_{14}, L_i, $i=1,\dots,4$, also their proper transform in S. Then the smooth double cover Y of S branched along $C_6+C_{14}+L_1+\cdots+L_4+E_1+\cdots+E_5$ is a smooth model of the double plane. In $\mathrm{Pic}(S)$ one sees that the following four divisors are even:

$$C_{14}, \qquad L_1+E_1+L_j+E_j, \quad j=2,3,4,$$

and, setting ψ the map defined in Lemma 2.4, one sees that the inverse images of these 4 divisors generate $\ker(\psi)$. □

5. A double plane of general type with $p_g=0$ and $P_2=7$

In this section we construct an example of a double plane, whose smooth model is a surface of general type with $p_g=0$, $P_2=7$ and bicanonical map not composed with the involution, by slightly modifying Example 4.1 of the previous section.

Example 5.1. We will find a reduced curve $B=C_6+C'_{14}+L_1+L_2+L_3+L_4$ where $L_1,\dots,L_4$ are lines through a fixed point P_0 and the curve C'_{14} has degree 14 with the following singularities:

- a point of multiplicity 6 at p_0,

- a $[4,4]$ point $p_i \in L_i$, $i = 1, \dots, 4$, where L_i is the proper tangent,
- a point p_5 of type $[5,3]$, where the infinitely near point p_5' is in the direction of a line denoted by L_5,

and C_6 is a sextic with the following properties:

- a double point at p_0;
- a tacnode at p_i, $i = 1, \dots, 4$, where L_i is the tacnodal tangent;
- passing simply through p_5, with L_5 as tangent line.

In other words, B has the same singularities as in Example 4.1, except at p_5, that is a point of type $[6,4]$ here, whereas it was of type $[5,5]$ there.

As before, we want the equations of C_6 and of C_{14}' to be ι_2-invariant, where ι_2 is the automorphism (1) of $\mathbb{P}^2$, and moreover $\iota_2(L_1) = L_3$ and $\iota_2(L_2) = L_4$.

Let us choose the same points, thus C_6 is the same as in Example 4.1, cf. A.3 in the appendix.

Concerning the curve of degree 14, note that ι_2-invariant polynomials in x, y of degree 14 depend on 64 parameters and we are imposing the following conditions:

- 12 (instead of 21) for p_0 to have multiplicity 6;
- 11 (instead of 21) for p_5 to be a point of type $[5,3]$ with proper tangent L_5;
- 20 for p_i, $i = 1, 2$, to be a point of type $[4,4]$ with proper tangent L_i.

Therefore one expects to find one such ι_2-invariant curve C_{14}' of degree 14. We indeed found the one written in the appendix, see A.4. One can check that C_{14}' have exactly the prescribed singularities.

Proposition 5.2. *The smooth minimal model Y of the double plane branched along the curve $B = C_6 + C_{14}' + L_1 + \cdots + L_4$ of Example 5.1 is a surface of general type with $p_g = 0$ and $P_2 = 7$. Furthermore,* $\mathrm{Tors}_2(Y) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$.

Proof. Just follow the proof of Proposition 4.2: the only difference is that the second summand of the bicanonical linear system (4) is now

$$\left| 14L - 6p_0 - \sum_{i=1}^{4} 4(p_i + p_i') - 4p_5 - 2p_5' \right| + \sum_{i=1}^{4} L_i$$

and it has dimension 5, so the bi-genus of a smooth model of the double plane is $P_2 = 1 + 6 = 7$. Moreover, two general curves in this linear system now meet in 12 points, off the base points, thus the smooth minimal model has $K^2 = 6$.

Concerning the 2-torsion, the divisors $\bar{L}_1 + E_1 + \bar{L}_j + E_j$, $j = 2,3,4$, where E_j is the irreducible exceptional curve corresponding to p_j and $\bar{L}_j$ is the strict transform of L_i on the blown-up surface, generate the kernel of the map ψ in Lemma 2.4. □

6. Two double planes of general type with $p_g = 0$ and $P_2 = 6$

There are two types of Du Val double planes with these invariants, namely $DV_{3;3,0}$ and $DV_{4;1,1}$. The existence of the former type is already known: e.g. Burniat surfaces are birationally equivalent to double planes of this type $DV_{3;3,0}$. So in this section we prove the existence of Du Val type $DV_{4;1,1}$ by giving an example.

We will then construct a surface of general type with $p_g = 0$, $P_2 = 6$ and bicanonical map not composed with the involution, from a double plane with a new configuration of degree and singularities.

Example 6.1. We want to find a reduced curve $B = C_4 + C_{10} + L_1 + L_2 + L_3 + L_4$ where $L_1, \dots, L_4$ are lines through a fixed point p_0, and the curve C_{10} has degree 10 with the following singularities:

- a point of multiplicity 4 at p_0,
- a $[3,3]$ point $p_i \in L_i$, where L_i is the proper tangent, $i = 1, \dots, 4$,
- a tacnode p_5, where we denote by L_5 the tacnodal (i.e., proper) tangent,
- a point p_6 of multiplicity 2,

and C_4 is a quartic with the following properties:

- double points at p_0 and at p_6;
- passing simply through the points p_i, $i = 1, \dots, 5$, where the tangent line to C_4 is L_i.

We require the equations of C_4 and of C_{10} to be ι_2-invariant, whilst $\iota_2(L_1) = L_3$ and $\iota_2(L_2) = L_4$, where ι_2 is the usual automorphism (1) of $\mathbb{P}^2$.

Let us choose $p_0 = (0,0,1)$, $p_1 = (2,1,1)$, $p_2 = (1,2,1)$, $p_3 = \iota_2(P_1) = (2,-1,1)$, $p_4 = \iota_2(P_2) = (1,-2,1)$, $p_5 = (1,0,0)$, $p_6 = (0,1,0)$ and $L_5 \colon z = 0$.

Quartic ι_2-invariant polynomials in x, y, z depend on 9 homogeneous parameters and we are imposing 2 (instead of 3) conditions for p_0 to be double, 1 (instead of 2) condition for passing through p_5 with tangent line L_5, 1 (instead of 3) condition for p_6 to be double, 2 conditions for passing through p_i, $i = 1,2$

with tangent line L_i. In such a way, we find a unique quartic curve C_4, with affine equation

```
8*x^3-15*x^2*y^2-12*x^2+28*x*y^2-12*y^2=0.        (7)
```

Similarly, ι_2-invariant polynomials in x, y of degree 10 depend on 36 parameters. The conditions we are imposing are

- 6 (instead of 10) for p_0 to have multiplicity 4;
- 3 (instead of 6) for p_5 to be a tacnode with tacnodal tangent L_5;
- 1 (instead of 3) for p_6 to be a double point;
- 12 for p_i, $i = 1, 2$, to be a point of type $[3, 3]$ with proper tangent L_i.

We find a pencil of ι_2-invariant curves of degree 10 with the above singularities and such that its general member is irreducible, e.g. the curve C_{10} with affine equation in the appendix, see A.5. One may check that C_4 and C_{10} have exactly the prescribed singularities.

Proposition 6.2. *The smooth minimal model Y of the double plane branched along the curve $B = C_4 + C_{10} + L_1 + \cdots + L_4$ of Example 6.1 is a surface of general type with $p_g = 0$ and $P_2 = 6$. Furthermore* $\mathrm{Tors}_2(Y) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 4}$.

Proof. The branch curve B is reduced, of degree 18, with the following singularities:

- a point of multiplicity 10 at p_0,
- a $[5, 5]$ point $p_i \in L_i$, $i = 1, \dots, 4$, where L_i is the proper tangent,
- a $[3, 3]$-point p_5, where L_5 is the proper tangent,
- a point of multiplicity 4 at p_6,

or equivalently $B \in |18L - 10p_0 - \sum_{i=1}^{4} 5(p_i + p_i') - 3(p_5 + p_5') - 4p_6|$. Since there is no conic through $p_1, \dots, p_6$, one sees that $p_g = 0$ and the double plane branched along B is of Du Val type $DV_{4;1,1}$. In particular, the first summand of the bicanonical linear system (5) is $|3L - 3p_0 - \sum_{i=1}^{4} p_i| = \emptyset$ and Proposition 2.1 implies that the bicanonical map of the smooth model of the double plane is composed with the involution induced by the double plane structure.

The second summand of the bicanonical linear system (5) is

$$\left| 8L - 4p_0 - \sum_{i=1}^{4} 2(p_i + p_i') - p_5 - p_5' - p_6 \right| + \sum_{i=1}^{4} L_i$$

which has dimension 5, thus the bi-genus of a smooth model of the double plane is $P_2 = 0+6 = 6$. Two general curves in the above linear system meet in 10 points, off the base points, which implies that the smooth minimal model has $K^2 = 5$.

It remains to show only the assertion about the 2-torsion. Following the canonical resolution, one blows-up $\mathbb{P}^2$ at $p_0,\dots,p_6$ and at the points p_i', $i = 1,\dots,5$, infinitely near to p_i in the direction of the line L_i. Let E_i, $i = 1,\dots,5$, be the irreducible exceptional curve corresponding to p_i. By abusing notation, let us denote by C_4, C_{10}, L_i, $i = 1,\dots,4$, also their proper transform in the blown-up surface S. Then Y is a smooth double cover branched along $C_4+C_{10}+L_1+\cdots+L_4+E_1+\cdots+E_5$. In $\mathrm{Pic}(S)$ one sees that the following five divisors are even:

$$C_4+\sum_{i=1}^{5} E_i, \qquad L_1+E_1+L_j+E_j, \quad j = 2,3,4,$$

and, setting ψ the map defined in Lemma 2.4, one sees that the inverse image of these 4 divisors generate $\ker(\psi)$. □

We now construct a surface of general type with $p_g = 0$, $P_2 = 6$ and bi-canonical map not composed with the involution, from a double plane with a new configuration of degree and singularities.

Example 6.3. We want to construct a reduced curve $B = C_4+C_6+C_8+L_1+L_2$ where L_1, L_2 are lines, the curve C_8 has degree 8 with the following singularities:

- a double point at $p_0 = L_1 \cap L_2$,
- a $[3,3]$ point $p_i \in L_i$, $i = 1,2$, where L_i is the proper tangent,
- three tacnodes, say p_j, $j = 3,4,5$ (denote by L_j the tacnodal tangent),

C_6 is a sextic with the following properties:

- a double point at p_0;
- a tacnode at p_i, $i = 1,\dots,4$, where L_i is the tacnodal tangent;
- passing simply through p_5, with L_5 as tangent line;

and finally C_4 is a quartic with the following properties:

- a double point at p_0;
- passing simply through p_i, $i = 1,\dots,4$, with L_i as tangent line,
- a tacnode at p_5, with L_5 as tacnodal tangent.

We want B to be ι_2-invariant, where ι_2 is the involution (1) of $\mathbb{P}^2$. So we assume that the equations of C_4, C_6, C_8, L_5 are ι_2-invariant and that $\iota_2(L_1) = L_2$.

Let us choose $p_0 = (0,0,1)$, $p_1 = (1,1,1)$, $p_3 = (2,1,1)$, $p_5 = (1,0,0)$ and $L_5\colon z = 0$, hence $L_1\colon x - y = 0$, $p_2 = \iota_2(p_1) = (1,-1,1)$, $L_2\colon x + y = 0$, and $p_4 = \iota_2(p_3) = (2,-1,1)$.

Quartic ι_2-invariant polynomials in x, y, z depend on 9 homogenous parameters and we are imposing the following conditions:

- 2 (and not 3) for p_0 to be double;
- 3 (and not 6) for p_5 to be a tacnode with tacnodal tangent L_5;
- 2 for passing through p_1 with tangent line L_1,
- 1 for passing through p_3,

so one expects to find one such quartic. This is indeed true and the quartic C_4 has the following affine equation

```
2*x^2-6*x*y^2+3*y^4+y^2=0,
```

and the tangent line to C_4 at p_3 (at p_4, resp.) is $L_3\colon x - 5y + 3z = 0$ (is $L_4\colon x + 5y + 3z = 0$, resp.).

Sextic ι_2-invariant polynomials in x, y, z depend on 16 homogeneous parameters and we are imposing the following conditions:

- 2 (and not 3) for p_0 to be double;
- 1 (and not 2) for passing through p_5 with tangent line L_5;
- 6 for p_i, $i = 1, 3$, to be a tacnode, with L_i as tacnodal tangent.

so one expects to find one such sextic C_6. This is again true and its affine equation is given in the appendix, see A.6.

Finally, ι_2-invariant polynomials in x, y of degree 8 depend on 25 parameters. The conditions we are imposing to these curves of degree 8 are

- 2 (instead of 3) for p_0 to be double,
- 3 (instead of 6) for p_5 to be a tacnode with proper tangent L_5,
- 12 for p_1 to be a point of type $[3,3]$, with L_1 as proper tangent,
- 6 for p_3 to be a tacnode, with L_2 as tacnodal tangent.

Therefore one expects to find a pencil of such ι_2-invariant curves of degree 8.

One indeed finds a pencil of curves of degree 8, whose general element is irreducible, e.g. the curve C_8 with affine equation written in the appendix, see A.6.

Proposition 6.4. *The smooth model Y of the double plane branched along the curve $B = C_4 + C_6 + C_8 + L_1 + L_2$ of Example 6.3 is a surface of general type with $p_g = 0$ and $P_2 = 6$. Furthermore* $\mathrm{Tors}_2(S) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$.

Proof. The branch curve B is reduced, of degree 20, with the following singularities:

- a point of multiplicity 8 at p_0,
- a $[7,7]$ point $p_i \in L_i$, $i = 1,2$, where L_i is the proper tangent,
- three further $[5,5]$-points p_j, $j = 3,4,5$, where L_j is the proper tangent,

or equivalently $B \in |20L - 8p_0 - \sum_{i=1}^{2} 7(p_i + p'_i) - \sum_{j=3}^{5} 5(p_j + p'_j)|$. According to (3), the canonical linear system is given by

$$\left| 5L - p_0 - \sum_{i=1}^{5} (2p_i + p'_i) \right| + L_1 + L_2$$

that turns out to be empty: quintic polynomials depend on 21 homogenous parameters and one checks that the 21 imposed conditions are independent. This means that $p_g = 0$.

The first summand of the bicanonical linear system (5) corresponds to a unique, fixed, curve. By Proposition 2.1, it follows that the bicanonical map of the smooth model of the double plane is not composed with the involution induced by the double plane structure.

The second summand of the bicanonical linear system (5) is

$$\left| 12L - 4p_0 - \sum_{i=1}^{5} 3(p_i + p'_i) \right| + L_1 + L_2$$

which turns out to have dimension 4, therefore the bi-genus of the smooth model of the double plane is $P_2 = 1 + 5 = 6$. Two general curves in the above linear system meet in 10 points, off the base points, which implies that the smooth minimal model Y has $K^2 = 5$. Note that the pencil of lines through P_0 pulls back to an hyperelliptic pencil of curves of genus 5 on Y.

It remains to show only the assertion about the 2-torsion. Following the canonical resolution, one blows up $\mathbb{P}^2$ at $p_0, p_1, \ldots, p_5$ and at the points p'_i, $i = 1, \ldots, 5$, infinitely near to p_i in the direction of the line L_i. Let E_i, $i = 0, \ldots, 5$, be the irreducible exceptional curve corresponding to p_i. By abusing notation, let us denote by C_4, C_6, C_8, L_1, L_2 also their proper transform in the blown-up surface S. Then the smooth double cover Y of S branched along $C_4+C_6+C_8+L_1+L_2+E_1+\cdots+E_5$ is a smooth model of the double plane. In $\mathrm{Pic}(S)$ one sees that the following three divisors are even:

$$C_6+E_5, \quad C_8+E_1+E_2, \quad L_1+E_1+L_2+E_2,$$

and, setting ψ the map defined in Lemma 2.4, one sees that the inverse image of these three divisors generate $\ker(\psi)$. $\square$

Remark 6.5. The double plane branched along C_4+C_6 is of Campedelli type $C_{5,1}$, so its smooth minimal model is a numerically Godeaux surface, i.e. it is of general type with $p_g=0$ and $P_2=2$.

7. Two double planes of general type with $p_g=0$ and $P_2=5$

There are three Du Val types of double planes with $p_g=0$ and $P_2=5$, namely $DV_{4;0,2}$, $DV_{3;2,1}$ and $DV_{2;0,4}$. The existence of the last type $DV_{2;0,4}$ is already known: e.g. Burniat surfaces are birationally equivalent to double planes of this type (cf. also the construction given by the second author in [32, Example 4]).

In this section we construct a double plane of type $DV_{4;0,2}$ by slightly modifying Example 6.1, and then a double plane of type $DV_{3;2,1}$.

Example 7.1. We want to find a reduced curve $B=C_4+C_{10}+L_1+L_2+L_3+L_4$, where $L_1,\ldots,L_4$ are lines through a fixed point p_0, the curve C_{10} has degree 10 with the following singularities:

- a point of multiplicity 4 at p_0,
- a $[3,3]$ point $p_i \in L_i$, where L_i is the proper tangent, $i=1,\ldots,4$,
- a point p_5 of multiplicity 3,
- a point p_6 of multiplicity 2,

and C_4 is a quartic with the following properties:

- double points at p_0 and p_6;
- passing simply through p_i, $i=1,\ldots,4$, where L_i is the tangent line,
- passing simply through p_5.

In other words, the singularities are the same as in Example 6.1, but p_5 is a 4-tuple point for the branch curve here, whereas it was a $[3,3]$-point there.

We again require the equations of C_4 and of C_{10} to be ι_2-invariant, where ι_2 is the usual involution (1) and we choose the same points as in Example 6.1. Imposing the conditions to a generic plane quartic, we find again the same curve C_4 with equation (7).

On the other hand, ι_2-invariant polynomials in x,y of degree 10 depend on 36 parameters and the imposed conditions are the following:

- 6 (instead of 10) for p_0 to have multiplicity 4;
- 4 (instead of 6) for p_5 to be a triple point;
- 1 (instead of 3) for p_6 to be a double point;
- 12 for p_i, $i=1,2$, to be a point of type $[3,3]$ with proper tangent L_i.

We find a unique such curve C_{10} of degree 10, written in the appendix, see A.7. One can check that C_{10} has exactly the imposed singularities. Note that C_4+C_{10} has a double point p_6' infinitely near to p_6 in the direction of the line $z=0$.

Proposition 7.2. *The smooth model Y of the double plane branched along the curve $B=C_4+C_{10}+L_1+\cdots+L_4$ of Example 7.1 is a surface of general type with $p_g=0$ and $P_2=5$. Furthermore* $\mathrm{Tors}_2(Y)\cong(\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$.

Proof. The branch curve B is reduced, of degree 18, with the following singularities:

- a point of multiplicity 10 at p_0,
- a $[5,5]$ point $p_i\in L_i$, $i=1,\dots,4$, where L_i is the proper tangent,
- two points p_5,p_6 of multiplicity 4,

or equivalently $B\in|18L-10p_0-\sum_{i=1}^4 5(p_i+p_i')-4(p_5+p_6)|$. Such as in Example 6.1, one may check that there is no conic through $p_1,\dots,p_6$, so the double plane branched along B is of Du Val type $DV_{4;0,2}$. The only difference with Example 6.1 is that the second summand of the bicanonical linear system (5) is now

$$\left|8L-4p_0-\sum_{i=1}^4 2(p_i+p_i')-2p_5-2p_6\right|+\sum_{i=1}^4 L_i$$

which turns out to have dimension 4, therefore the bi-genus of a smooth model of the double plane is $P_2=5$, and two general curves in the above linear system meet in 8 points, off the base points, so $K^2=4$.

Concerning the 2-torsion, one sees that, with usual notation, the canonical resolution of the double plane is a smooth double cover branched along:

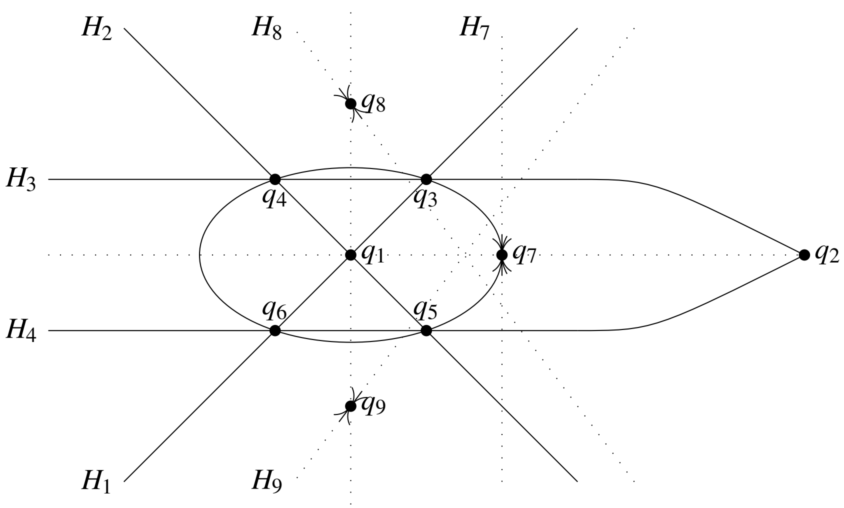
$$C_4+C_{10}+L_1+L_2+L_3+L_4+E_1+E_2+E_3+E_4,$$

and the following three divisors in the blown-up surface

$$L_1+E_1+L_j+E_j, \quad j=2,3,4,$$

are even and they give the generators of $\ker(\psi)$, where ψ is the usual map defined in Lemma 2.4. □

We finally construct a double plane of type $DV_{3;2,1}$, by means of a quadratic transformation in the plane.



Example 7.3. We are going to find a reduced curve $B=C_2+C_6+C_8+H_1+H_2+H_3+H_4$, which is invariant under the involution ι_2 defined by (1), where

- $H_1,\dots,H_4$ are lines forming a quadrilateral with vertices $q_1=H_1\cap H_2$, $q_2=H_3\cap H_4$, $q_3=H_1\cap H_3$, $q_4=H_2\cap H_3$, $q_5=H_2\cap H_4$, $q_6=H_1\cap H_4$;

C_8 is an irreducible curve of degree 8 with the following singularities:

- double points at q_1, q_2 and triple points at q_i, $i=3,4,5,6$;
- a tacnode q_7, with a line H_7 as tacnodal tangent,
- two further tacnodes q_j, $j=8,9$, with a line H_j as tacnodal tangent;

C_6 is an irreducible sextic curve with the following properties:

- double points at q_i, $i = 1,\dots,6$;
- a tacnode q_7, with H_7 as proper tangent,
- passing through q_j, $j = 8,9$, where H_j is tangent line,

and C_2 is a smooth conic with the following properties:

- passing through $q_3,\dots,q_6$,
- passing through q_7 where H_7 is the tangent line,

such that there is no conic through $q_1,q_2,\dots,q_9$.

We require that the equations of C_2,C_6,C_8 are ι_2-invariant, whereas $\iota_2(H_i) = H_{i+1}$, $i = 1,3$. Let us choose $q_1 = (0,0,1)$, $q_2 = (1,0,0)$, $q_3 = (1,1,1)$, $q_4 = (-1,1,1)$, $q_7 = (2,0,1)$ and $q_8 = (0,2,1)$, so that $H_1 : x-y = 0$, $H_3 : y-z = 0$, hence $H_2 = \iota_2(H_1) : x+y = 0$, $H_4 = \iota_2(H_3) : y+z = 0$, $H_7 : x-2z = 0$, $q_5 = \iota_2(q_3) = (1,-1,1)$, $q_6 = \iota_2(q_4) = (-1,-1,1)$ and $q_9 = \iota_2(q_8) = (0,-2,1)$. The proper tangent H_8 to C_8 at q_8 will be determined in a moment (as well as the proper tangent $H_9 = \iota_2(H_8)$ at q_9).

A ι_2-invariant degree-2 polynomial depends on four parameters and we impose it to pass through q_3,q_4,q_7. In this way we find the conic

C_2: `x^2+3y^2-4=0`,

which passes also through q_5,q_6 and such that the tangent line at q_7 is H_7.

Degree-8 ι_2-invariant polynomials in x,y,z depend on 25 homogeneous parameters and we are imposing the following conditions:

- 2 (and not 3) for p_1 and for p_2 to be double;
- 3 (and not 6) for p_7 to be a tacnode, with H_7 as tacnodal tangent;
- 6 for p_i, $i = 3,4$, to be of multiplicity 3;
- 6 for p_8 to be a tacnode, with a given line H_8 as tacnodal tangent.

Thus, if these conditions are independent, one expects to find no such curve of degree 8. Setting $H_8 : x+ty-2tz = 0$, $t \in \mathbb{C}$, to be the proper tangent to C_8 at p_8, by using a computer algebra software one finds out that these conditions are not independent if $t = 3/4$, i.e. if $H_8 : 4x+3y-6z = 0$. Indeed one finds out the curve C_8 of degree 8 with the affine equation written in the appendix, see A.8. One may check that C_8 is irreducible and has exactly the imposed singularities.

Degree-6 ι_2-invariant polynomials in x,y,z depend on 16 homogeneous parameters and we are imposing the following conditions:

- 2 (and not 3) for p_1 and for p_2 to be double;
- 3 (and not 6) for p_7 to be a tacnode, with H_7 as tacnodal tangent;
- 3 for p_i, $i = 3,4$, to be double;
- 2 for passing through p_8, with H_8 as tangent line.

Thus one expects to find one such sextic curve. Indeed one finds out the sextic C_6 with the affine equation written in the appendix, see A.8. One may check that C_6 is irreducible and has exactly the wanted properties.

Proposition 7.4. *Let C_2, C_6 and C_8 be the irreducible curves constructed in the previous example. Then $B = C_2 + C_6 + C_8 + H_1 + H_2 + H_3 + H_4$ is the branch curve of a double plane X, whose smooth minimal model Y is a surface of general type with $p_g = 0$ and $P_2 = 5$. In particular, Y is isomorphic to the smooth minimal model of a double plane of type $DV_{3;2,1}$. Furthermore,* $\mathrm{Tors}_2(Y) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$.

Proof. The quadratic Cremona transformation α with fundamental points q_4, q_5, q_6 is

$$\alpha\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \qquad \alpha(x,y,z) = (f-2x^2, f-2y^2, f-2z^2),$$

where $f = x^2+y^2+z^2+yz+xz+xy$. Then α determines a double plane X_α birationally equivalent to X (whose smooth minimal model is the same as the one obtained from X), whose branch curve is obtained from the total transform of B by removing the components of even multiplicity and taking the reduced part (cf., e.g., [7]).

One checks that the branch curve of X_α is $\bar{B} = C_6' + C_7 + L_1 + L_2 + L_3$ where

- $L_1\colon x - y = 0$ is the strict transform of H_1,
- $L_2\colon x + 3y - 4z = 0$ is the strict transform of C_2,
- $L_3\colon y - z = 0$ is the strict transform of H_3,
- L_1, L_2, L_3 are lines through $p_0 = q_3 = (1,1,1)$,
- C_7 is the strict transform of C_8: it has the affine equation written in the appendix, see A.9, and has the following singularities:
  - a triple point at p_0 and a double point at $p_6 = q_5 = (1,-1,1)$,
  - a tacnode $p_i \in L_i$, $i = 1,2,3$, where L_i is the proper tangent, namely $p_1 = q_6 = (-1,-1,1)$, $p_2 = \alpha(q_7) = (-1,7,5)$, and $p_3 = q_4 = (-1,1,1)$; note that q_1 (q_2, resp.) corresponds to the infinitely near point to p_1 (p_3, resp.) in the direction of the line L_1 (L_3, resp.);

  - further two tacnodes p_j, $j=4,5$, where we denote by L_j the proper tangent, namely $p_4=\alpha(q_8)=(7,-1,5)$, $p_5=\alpha(q_9)=(3,-5,1)$, $L_4\colon 47x-11y-68z=0$ is the tangent line at p_4 to the strict transform of H_8 (that is a conic), $L_5\colon 29x+15y-12z=0$ is the tangent line at p_5 to the strict transform of H_9;

- C_6' is the strict transform of C_6: it has the affine equation written in the appendix, see A.9, and has the following properties:

  - double points at p_0 and at p_6,
  - a tacnode $p_i\in L_i$, $i=1,2,3$, where L_i is the tacnodal tangent,
  - passing simply through p_j, $j=4,5$, with L_j as tangent line.

One sees that there is no conic through $p_1,\dots,p_6$, therefore X_α is a double plane of Du Val type $DV_{3;2,1}$. The branch curve $\bar{B}$ is indeed reduced, of degree 16, with the following singularities:

- a point of multiplicity 8 at p_0,
- a $[5,5]$-point $p_i\in L_i$, $i=1,2,3$, where L_i is the proper tangent,
- two $[3,3]$-points p_j, $j=4,5$, where L_j is the proper tangent,
- a point p_6 of multiplicity 4,

or equivalently $\bar{B}\in|16L-8p_0-\sum_{i=1}^{3}5(p_i+p_i')-\sum_{j=4}^{5}3(p_j+p_j')-4p_6|$.

In particular $p_g(X_\alpha)=0$ and the first summand of the bicanonical linear system (5) is empty, while the second summand of the bicanonical linear system (5) is

$$\left|7L-3p_0-\sum_{i=1}^{3}2(p_i+p_i')-\sum_{i=4}^{5}(p_i+p_i')-2p_6\right|+L_1+L_2+L_3$$

which turns out to have dimension 4, therefore the bigenus is $P_2=0+5=5$.

It remains to show only the assertion about the 2-torsion. Following the canonical resolution, one blows up $\mathbb{P}^2$ at $p_0,p_1,\dots,p_5$ and at the points p_i', $i=1,\dots,5$, infinitely near to p_i in the direction of the line L_i. Let E_i, $i=0,\dots,5$, be the irreducible exceptional curve corresponding to p_i. By abusing notation, let us denote by C_6',C_7,L_1,L_2,L_3 also their proper transform in the blown-up surface S. Then the smooth double cover Y of S branched along $C_6'+C_7+L_1+L_2+L_3+E_1+\cdots+E_5$ is a smooth model of the double plane. In $\mathrm{Pic}(S)$ one sees that the following three divisors are even:

$$C_6'+E_4+E_5,\quad L_1+E_1+L_2+E_2,\qquad L_1+E_1+L_3+E_3,$$

and, setting ψ the map defined in Lemma 2.4, one sees that the inverse image of these three divisors generate $\ker(\psi)$. $\square$

8. A double plane of general type with $p_g = 0$ and $P_2 = 4$

There are three Du Val types of double planes with $p_g = 0$ and $P_2 = 4$, namely $DV_{3;1,2}$, $DV_{2;3,1}$, and $DV_{1;5,0}$. The existence of the last type $DV_{1;5,0}$ is already known: e.g. Burniat surfaces are birationally equivalent to double planes of this type (cf. also the construction given by the second author in [32, Example 1]).

Concerning double planes of general type with these invariants and bicanonical map not composed with the involution, the second author gave an example in [33]: its branch curve is irreducible, of degree 22, with five points of type $[7,7]$.

In this section we construct one example of Du Val double plane of type $DV_{2;3,1}$.

Example 8.1. We want to construct a reduced curve $B = C_2 + C_4 + C_6 + L_1 + L_2$ where L_1, L_2 are lines, the curve C_6 is a sextic with the following properties:

- double points at $p_0 = L_1 \cap L_2$ and at another point $p_6 \in \mathbb{P}^2$,
- a tacnode at $p_i \in L_i$, $i = 1,2$, where L_i is the tacnodal tangent,
- a further tacnode at p_5, and denote by L_5 the tacnodal tangent,
- passing simply through a given point p_j, $j = 3,4$, with a given tangent L_j,

the curve C_4 is a quartic with the following properties:

- double points at p_0 and at p_6;
- passing simply through p_i, $i = 1,\dots,5$, with L_i as tangent line;

and C_2 is a conic passing simply through p_i, $i = 1,\dots,5$, with L_i as tangent line.

We want B to be ι_2-invariant, where ι_2 is the involution (1) of $\mathbb{P}^2$. So we assume that the equations of C_2, C_4, C_6 are ι_2-invariant and that $\iota_2(L_1) = L_2$.

Let us choose $p_0 = (0,0,1)$, $p_1 = (1,1,1)$, $p_3 = (-2,1,1)$, $p_5 = (1,0,0)$, $L_5\colon z = 0$, and $p_6 = (0,1,0)$, hence $L_1\colon x - y = 0$, $p_2 = \iota_2(p_1) = (1,-1,1)$, $L_2\colon x + y = 0$, and $p_4 = \iota_2(p_3) = (2,-1,1)$. We will choose the line L_3 passing through p_3 in a moment (and hence $L_4 = \iota_2(L_3)$ will be determined as well).

If one imposes a ι_2-invariant conic to pass through p_1, with tangent line L_1, and to pass through p_3, one finds the unique conic C_2 with affine equation

```
2*x^2-3*y^2+2*x-1=0,
```

and we choose L_3 to be the tangent line to C_2 at p_3, namely $L_3\colon x + y + 1 = 0$.

Quartic ι_2-invariant polynomials in x,y,z depend on 9 homogenous parameters and we are imposing the following conditions:

- 2 (and not 3) for p_0 to be double;
- 1 (and not 3) for p_6 to be double;
- 1 (and not 2) for passing through p_5 with tangent line L_5,
- 2 for passing through p_i, $i = 1$ and 3, with tangent line L_i,

so one expects to find one such quartic. This is indeed true and the quartic C_4 has affine equation:

```
x^2*y^2+6*x^3-8*x*y^2+9*x^2-8*y^2=0.
```

Sextic ι_2-invariant polynomials in x, y, z depend on 16 homogeneous parameters and we are imposing the following conditions:

- 2 (and not 3) for p_0 to be double;
- 1 (and not 3) for p_6 to be double;
- 3 (and not 6) for p_5 to be a tacnode with tacnodal tangent L_5;
- 6 for p_1 to be a tacnode, with L_1 as tacnodal tangent,
- 2 for passing through p_3, with L_3 as tangent line,

so one expects to find a pencil of such sextics. This is again true and the pencil has irreducible general member, e.g. the curve C_6 with affine equation written in the appendix, see A.10.

Proposition 8.2. *The smooth model Y of the double plane branched along the curve $B = C_2 + C_4 + C_6 + L_1 + L_2$ of Example 8.1 is a surface of general type with $p_g = 0$ and $P_2 = 4$. Furthermore* $\mathrm{Tors}_2(S) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$.

Proof. The branch curve B is reduced, of degree 14, with the following singularities:

- a point of multiplicity 6 at p_0,
- a $[5,5]$ point $p_i \in L_i$, $i = 1, 2$, where L_i is the proper tangent,
- three further $[3,3]$-points p_j, $j = 3, 4, 5$, where L_j is the proper tangent,
- a further point p_6 of multiplicity 4,

or equivalently $B \in |\,14L - 6p_0 - \sum_{i=1}^{2} 5(p_i + p_i') - \sum_{j=3}^{5} 3(p_j + p_j') - 4p_6|$. One may check that there is no conic through $p_1, \ldots, p_6$, therefore the double plane branched along B is of type $DV_{2;3,1}$.

In particular the first summand of the bicanonical linear system (5) is empty, while the second summand of the bicanonical linear system (5) is

$$\left| 6L - 2p_0 - \sum_{i=1}^{2} 2(p_i + p_i') - \sum_{i=3}^{5} (p_i + p_i') - 2p_6 \right| + L_1 + L_2$$

which has dimension 3, as expected.

It remains to show only the assertion about the 2-torsion. Following the canonical resolution, one blows up $\mathbb{P}^2$ at $p_0, p_1, \ldots, p_5$ and at the points p_i', $i = 1, \ldots, 5$, infinitely near to p_i in the direction of the line L_i. Let E_i, $i = 0, \ldots, 5$, be the irreducible exceptional curve corresponding to p_i. By abusing notation, let us denote by C_2, C_4, C_6, L_1, L_2 also their proper transform in the blown-up surface S. Then the smooth double cover Y of S branched along $C_2 + C_4 + C_6 + L_1 + L_2 + E_1 + \cdots + E_5$ is a smooth model of the double plane. In $\mathrm{Pic}(S)$ one sees that the following three divisors are even:

$$C_2 + E_1 + E_2 + E_3 + E_4, \quad C_6 + E_3 + E_4, \quad L_1 + E_1 + L_2 + E_2,$$

and, setting ψ the map defined in Lemma 2.4, one sees that the inverse image of these three divisors generate $\ker(\psi)$. $\square$

A. Explicit equations of the curves

As suggested by the referee, we collect in this appendix the equations of the previously constructed curves in such a way that they can be copied and pasted in a computer algebra system.

A.1. Curve C_{21} of Example 3.2:

```
C_21:=1024*x^20-8448*x^18*y-448*x^17*y^4+31680*x^16*y^2
+2624*x^15*y^5-704*x^15-71280*x^14*y^3-5984*x^13*y^6+3168*x^13*y
+336*x^12*y^9+108768*x^12*y^4+5280*x^11*y^7-5940*x^11*y^2
-2144*x^10*y^10-119856*x^10*y^5-4*x^10+1320*x^9*y^8+5940*x^9*y^3
+6160*x^8*y^11+96888*x^8*y^6+3*x^8*y-20*x^7*y^14-6072*x^7*y^9
-3322*x^7*y^4-9460*x^6*y^12-55935*x^6*y^7+4*x^5*y^15
+4356*x^5*y^10+992*x^5*y^5+8085*x^4*y^13+21560*x^4*y^8
+22*x^3*y^16-1056*x^3*y^11-128*x^3*y^6+7*x^2*y^19-3696*x^2*y^14
-4928*x^2*y^9-8*y^20+704*y^15+512*y^10.
```

A.2. Curves C_8 and C'_8 of Example 3.6:

```
C_8:=295783*x^8-1774698*x^7*y+4876416*x^6*y^2-7674344*x^5*y^3
+7877724*x^4*y^4-5139288*x^3*y^5+1434672*x^2*y^6+1748212*x^7
-10489272*x^6*y+27275832*x^5*y^2-39174848*x^4*y^3+29327280*x^3*y^4
-8276256*x^2*y^5+4040394*x^6-24242364*x^5*y+54846504*x^4*y^2
-57770256*x^3*y^3+28144080*x^2*y^4-5393952*x*y^5+3232180*x^5
-19393080*x^4*y+42010920*x^3*y^2-38756480*x^2*y^3+12899040*x*y^4
+875875*x^4-5255250*x^3*y+11113200*x^2*y^2-9417800*x*y^3
+2410800*y^4.
```

```
C'_8:=2008759*x^8-12052554*x^7*y+33083520*x^6*y^2-51983720*x^5*y^3
+53278428*x^4*y^4-34729560*x^3*y^5+9692784*x^2*y^6+11844148*x^7
-71064888*x^6*y+184747128*x^5*y^2-265222592*x^4*y^3
+198459120*x^3*y^4-55979424*x^2*y^5+27348666*x^6-164091996*x^5*y
+371262456*x^4*y^2-391103184*x^3*y^3+190599120*x^2*y^4
-36570528*x*y^5+21876820*x^5-131260920*x^4*y+284371080*x^3*y^2
-262411520*x^2*y^3+87396960*x*y^4+5932675*x^4-35596050*x^3*y
+75264000*x^2*y^2-63749000*x*y^3+16287600*y^4.
```

A.3. Curves C_6 and C_{14} of Example 4.1:

```
C_6:=-1056*x^5+1584*x^4*y^2+5884*x^4-9072*x^3*y^2-10680*x^3
+468*x^2*y^4+17113*x^2*y^2+6300*x^2-1266*x*y^4-11730*x*y^2
+9*y^6+754*y^4+2025*y^2.
```

```
C_14:=2993362257600*x^6*y^8-642416767200*x^4*y^10
+53053687800*x^2*y^12+184090725*y^14-7984819814400*x^7*y^6
-13919055703200*x^5*y^8+1958710231800*x^3*y^10-137141766900*x*y^12
+7035707257920*x^8*y^4+43389409117680*x^6*y^6
+32051705852520*x^4*y^8-2181287292570*x^2*y^10+84235458420*y^12
-2231066682240*x^9*y^2-39193561109760*x^7*y^4
-113572304790840*x^5*y^6t-46744118053860*x^3*y^8
+1424709059460*x*y^10+148737778816*x^10+10823053669520*x^8*y^2
+105313984353120*x^6*y^4+180069282386525*x^4*y^6
+40500724263940*x^2*y^8-591585025794*y^10-25399242379200*x^7*y^2
-172911772965600*x^5*y^4-167837539931100*x^3*y^6
-18759582044700*x*y^8-1800503781120*x^8+39870873162720*x^6*y^2
+165855228518880*x^4*y^4+83407878318870*x^2*y^6+4127537462580*y^8
+3352212864000*x^7-37472636232000*x^5*y^2-80588270196000*x^3*y^4
-18662023012500*x*y^6-1732665686400*x^6+14426591734800*x^4*y^2
+15283012348800*x^2*y^4+442001830725*y^6.
```

A.4. Curve C'_{14} of Example 5.1:

```
C'_14:=101101919688000*x^6*y^8-20383701645600*x^4*y^10
```

```
+1922970034800*x^2*y^12+9006959025*y^14-266317230896640*x^7*y^6
-486603095289120*x^5*y^8+56665562170680*x^3*y^10
-5048029874340*x*y^12+222658777180224*x^8*y^4
+1506037113840816*x^6*y^6+1192179060770904*x^4*y^8
-51653930702994*x^2*y^10+3095088945444*y^12-59346373747584*x^9*y^2
-1291115775845376*x^7*y^4-4194787908295224*x^5*y^6
-1836667562583156*x^3*y^8+28991534124276*x*y^10
+279682303646608*x^8*y^2+3752974063291832*x^6*y^4
+7022592103892073*x^4*y^6+1627502363091332*x^2*y^8
-14576442803642*y^10+33912213570048*x^9-733773573809088*x^7*y^2
-6643492578764352*x^5*y^4-6735475699940748*x^3*y^6
-740600472575532*x*y^8-145302180401664*x^8
+1399619530052064*x^6*y^2+6645841686875376*x^4*y^4
+3348737123394654*x^2*y^6+151659358326756*y^8+205257431047680*x^7
-1503914467671360*x^5*y^2-3238350710693760*x^3*y^4
-728779344737220*x*y^6-95538066624000*x^6+620580855920400*x^4*y^2
+578900345347800*x^2*y^4+21625708619025*y^6.
```

A.5. Curve C_{10} of Example 6.1:

```
C_10:=1877040*x^6*y^4-8085420*x^4*y^6+1900665*x^2*y^8
+281952*x^7*y^2-5708304*x^5*y^4+20750886*x^3*y^6-4815108*x*y^8
-8169540*x^6*y^2+36190665*x^4*y^4-20938050*x^2*y^6+3049620*y^8
-447136*x^7+23773152*x^5*y^2-89122638*x^3*y^4+21071824*x*y^6
+8207472*x^6-50754384*x^4*y^2+69186141*x^2*y^4-14455128*y^6
-22631040*x^5+97402320*x^3*y^2-22612140*x*y^4+17031600*x^4
-72805500*x^2*y^2+17082900*y^4.
```

A.6. Curves C_6 and C_8 of Example 6.3:

```
C_6:=-32*x^5+76*x^4*y^2+184*x^4-512*x^3*y^2-312*x^3+124*x^2*y^4
+1000*x^2*y^2+144*x^2-356*x*y^4-588*x*y^2+25*y^6+166*y^4+81*y^2.
```

```
C_8:=373*y^6-90*x^2+276*x^2*y^6+63*y^2-369*y^4+90*x*y^2+444*x*y^4
-630*x*y^6-36*x^2*y^2-390*x^2*y^4-3*y^8+72*x^3+576*x^3*y^4
+48*x^3*y^2-40*x^6+48*x^5-300*x^4*y^4+216*x^5*y^2-348*x^4*y^2.
```

A.7. Curve C_{10} of Example 7.1:

```
C_10:=-20750886*x^3*y^6+72805500*x^2*y^2-17082900*y^4-3049620*y^8
+447136*x^7+50754384*x^4*y^2-69186141*x^2*y^4+20938050*x^2*y^6
+5708304*x^5*y^4-97402320*x^3*y^2+8169540*x^6*y^2+14455128*y^6
+4815108*x*y^8+22612140*x*y^4-8207472*x^6-281952*x^7*y^2
-23773152*x^5*y^2-21071824*x*y^6-36190665*x^4*y^4+22631040*x^5
-1900665*x^2*y^8-17031600*x^4-1877040*x^6*y^4+8085420*x^4*y^6
+89122638*x^3*y^4.
```

A.8. Curves C_8 and C_6 of Example 7.3:

```
C_8:=25*x^6*y^2+704*x^6+2304*x^5*y^2-2304*x^5+2001*x^4*y^4
-4700*x^4*y^2+512*x^4-1536*x^3*y^4-1536*x^3*y^2+3072*x^3
-381*x^2*y^6-1128*x^2*y^4+4720*x^2*y^2-1024*x^2-768*x*y^6
+3840*x*y^4-3072*x*y^2-45*y^8+324*y^6-432*y^4-576*y^2.

C_6:=5*x^4*y^2-32*x^4-128*x^3*y^2+128*x^3-106*x^2*y^4+288*x^2*y^2
-128*x^2+128*x*y^4-128*x*y^2+21*y^6-96*y^4+48*y^2.
```

A.9. Curves C_7 and C'_6 in the proof of Proposition 7.4:

```
C_7:=729*x^7-1469*x^6*y+4108*x^5*y^2-2218*x^4*y^3+794*x^3*y^4
+2296*x^2*y^5-1635*x*y^6-1005*y^7+2198*x^6-2138*x^5*y+410*x^4*y^2
+6760*x^3*y^3-4518*x^2*y^4-2102*x*y^5+990*y^6-4157*x^5+10887*x^4*y
-16564*x^3*y^2-6196*x^2*y^3+9129*x*y^4+2101*y^5-11266*x^4
-2484*x^3*y+14256*x^2*y^2-2556*x*y^3-2750*y^4+13681*x^3-9031*x^2*y
-2485*x*y^2+2635*y^3+5380*x^2+2520*x*y-3100*y^2-3600*x+2000*y-1600

C'_6:=27*x^6-160*x^5*y+196*x^4*y^2-202*x^3*y^3+60*x^2*y^4
+148*x*y^5+11*y^6+214*x^5-320*x^4*y+286*x^3*y^2+118*x^2*y^3
-180*x*y^4+42*y^5+97*x^4+522*x^3*y-546*x^2*y^2-94*x*y^3-59*y^4
-714*x^3+522*x^2*y+74*x*y^2-202*y^3-181*x^2-214*x*y+315*y^2
+320*x-160*y+80.
```

A.10. Curve C_6 of Example 8.1:

```
C_6:=17*x^2*y^4+68*x^3*y^2-136*x*y^4+84*x^4-278*x^2*y^2
+296*y^4+84*x^3-152*x*y^2+9*x^2+8*y^2.
```

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