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THE LARGE SUM GRAPH RELATED TO COMULTIPLICATION MODULES

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Let R be a commutative ring and M be an R-module. We define the large sum graph, denoted by $\acute{G}(M)$, as a graph with the vertex set of non-large submodules of M and two distinct vertices are adjacent if and only if $N+K$ is a non-large submodule of M. In this article, we investigate the connection between the graph-theoretic properties of $\acute{G}(M)$ and algebraic properties of M when M is a comultiplication R-module.

1. Introduction

Throughout this paper, R will denote a commutative ring with identity and $\mathbb{Z}$ will denote the ring of integers.

In 2009, *the intersection graph of ideals* was considered by Chakrabarty, Ghosh, Mukherjee, and sen [8]. The intersection graph of ideals of rings and submodules of modules have been investigated by several authors (e.g., [12], [1], [13], and [9]).

The small intersection graph related to non-small ideals of a commutative ring was introduced and studied by Ebrahimi Atani, Dolati Pish Hesari, and Khoramdel [10]. This notion was generalized for multiplication modules over a commutative ring by Ansari-Toroghy, Farshadifar, and Mahboobi-Abkenar [6].

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Let M be an R-module. We denote the set of all minimal submodules of M by $Min(M)$ and the sum of all minimal submodules of M by $Soc(M)$. A submodule N of M is called *large* in M and denoted by $N \trianglelefteq M$ in case for every non-zero submodule L of M, $N \cap L \neq 0$. A module M is called a *uniform* module if every non-zero submodule of M is a large submodule of M.

A module M is said to be a *comultiplication* R-module if for every submodule N of M there exists an ideal I of R such that $N = Ann_M(I)$. Also an R-module M is comultiplication module if and only if for each submodule N of M, we have $N = (0 :_M Ann_R(N))$ [3].

In this paper, we introduce and study *the large sum graph* $\acute{G}(M)$ of M, when M is a comultiplication module. This notion can be regarded as a dual notion of the small intersection graph considered in [6].

In section 2, we give the definition of $\acute{G}(M)$ and consider some basic results on the structure of this graph. In Theorems 2.5 and 2.7, we provide some useful characterization about $\acute{G}(M)$. In Theorem 2.9, it is shown that if $\acute{G}(M)$ is connected, then $diam(\acute{G}(M)) \leqslant 2$. Also we prove that if $\acute{G}(M)$ contains a cycle, then $g(\acute{G}(M)) = 3$ (Theorem 2.10). Moreover, it is proved that if $\acute{G}(M)$ is a connected graph, then $\acute{G}(M)$ has no cut vertex (Theorem 2.11). Finally, in section 3, we investigate the clique and dominating number of this graph.

Here we will include some basic definitions from graph theory as needed. For two distinct vertices a and b, $a - b$ means that a and b are adjacent. The *degree* of a vertex a of graph G is the number of edges incident on a and denoted by $deg(a)$. A *regular graph* is *r-regular* (or regular of degree r) if the degree of each vertex is r. If $|V(G)| \geqslant 2$, a *path* from a to b is a series of adjacent vertices $a - v_1 - v_2 - ... - v_n - b$. In a graph G, the distance between two distinct vertices a and b, denoted by $d(a,b)$, is the length of the shortest path connecting a and b. If there is not a path between a and b, $d(a,b) = \infty$. The *diameter* of a graph G is $diam(G) = sup\{d(a,b) \mid a,b \in V(G)\}$. A graph G is called *connected* if for any vertices a and b of G there is a path between a and b. Otherwise, G is *disconnected*. A *cycle* in a graph G is a path that begins and ends in a same vertex. The *girth* of G, denoted by $g(G)$, is the length of the shortest cycle in G. If G has no cycle, we define the girth of G to be infinite. An *r-partite* graph is one whose vertex set can be partitioned into r subsets such that no edge has both ends in any one subset. A *complete* r-partite graph is one each vertex is joined to every vertex that is not in the same subset. The *complete bipartite* (i.e, 2-partite) graph with part sizes m and n is denoted by $K_{m,n}$. A *star* graph is a complete bipartite graph $K_{1,n}$. A *clique* of a graph is its maximal complete subgraph and the number of vertices in the largest clique of a graph G, denoted by $\omega(G)$, is called the *clique number* of G. For a graph $G = (V,E)$ an *open neighbourhood* $N(a)$ of a vertex $a \in V$ is the set of vertices which are adjacent to a. For each

$S \subseteq V$, we set $N(S) := \bigcup_{a\in S} N(a)$ and $N[S] := N(S) \bigcup S$. A set of vertices S in G is a *dominating set*, if $N[S] = V$. The *dominating number*, denoted by $\gamma(G)$, of G is the minimum cardinality of a dominating set of G ([11]). Note that a graph whose vertices-set is empty is a *null graph* and a graph whose edge-set is empty is an *empty graph*.

2. Basic properties of $\acute{G}(M)$

Definition 2.1. Let M be an R-module. We define the *large sum graph* $\acute{G}(M)$ *of* M with all non-large non-zero submodules of M as vertices and two distinct vertices N, K are adjacent if and only if $N + K$ is a non-large submodule of M.

Remark 2.2. Note that $\acute{G}(M)$ is a null graph if and only if M is a uniform module.

A non-zero R-module M is said to be *second* if for each $a \in R$, the endomorphism $M \xrightarrow{a} M$ is either surjective or zero. We recall that every minimal submodule is second [15, Proposition 1.6].

An R-module M is said to be *cocyclic* if $M \subseteq E(R/m)$ for some maximal ideal m of R (Here $E(R/m)$ denotes the injective envelop of R/m). M is cocyclic if and only if $Soc(M)$ is a large and a simple submodule of M [14].

We use the following lemma frequently.

Lemma 2.3. *Let M be a non-zero comultiplication R-module.*

(a) Every non-zero submodule of M contains a minimal submodule of M. In particular, $Min(M) \neq \varnothing$.

(b) Let N be a submodule of M. Then N is a large submodule of M if and only if $Soc(M) \subseteq N$.

(c) M is a uniform module if and only if M is a cocyclic module.

(d) Let N, K be submodules of M and let S be a second submodule of M with $S \subseteq N + K$. Then $S \subseteq N$ or $S \subseteq K$.

(e) If $|Min(M)| = 1$, then M is a uniform module.

Proof. (a) [4, Theorem 3.2].

(b) and (c) are straightforward.

(d) See [5, Theorem 2.6].

(e) This is clear.

□

From now on we suppose that $|Min(M)| \geq 2$.

Proposition 2.4. *Let M be a non-zero comultiplication R-module.*

(a) Let $Min(M) = \{S_i\}_{i\in I}$, where $|I| > 1$, and let Λ be a non-empty proper finite subset of I. Then $\sum_{\lambda\in\Lambda} S_\lambda$ is non-large submodule of M.

(b) Let M be an R-module. Then $\acute{G}(M)$ is a null graph if and only if M is a cocyclic module.

Proof. Use Lemma 2.3. □

In the rest of this paper we assume that M is a non-zero comultiplication R-module and $\acute{G}(M)$ is a non-null graph.

Theorem 2.5. *Let $Min(M) = \{S_1, S_2\}$ such that $\frac{M}{S_1}$ and $\frac{M}{S_2}$ are cocyclic R-modules. Then $\acute{G}(M)$ is an empty graph.*

Proof. First we show $Soc(\frac{M}{S_1})$ is the only simple submodule which is contained in every non-zero submodule of $\frac{M}{S_1}$. To see this, let $\frac{N}{S_1}$ be a non-zero submodule of $\frac{M}{S_1}$. Since $Soc(\frac{M}{S_1})$ is large and simple, we have $\frac{N}{S_1} \cap Soc(\frac{M}{S_1}) \neq 0$ and hence $\frac{N}{S_1} \cap Soc(\frac{M}{S_1}) = Soc(\frac{M}{S_1})$. It follows that $Soc(\frac{M}{S_1}) \subseteq \frac{N}{S_1}$. Clearly, $\frac{S_1+S_2}{S_1}$ is a minimal submodule of $\frac{M}{S_1}$ and so $Soc(\frac{M}{S_1}) = \frac{S_1+S_2}{S_1}$. Similar arguments shows that $Soc(\frac{M}{S_2}) = \frac{S_1+S_2}{S_2}$. Now we claim that there is no vertex $K \neq S_1, S_2$. To show this, let K be vertex of $\acute{G}(M)$. Then by Lemma 2.3 (a), $S_1 \subseteq K$ or $S_2 \subseteq K$. We may assume that $S_1 \subseteq K$. Consequently, we have $\frac{S_1+S_2}{S_1} = Soc(\frac{M}{S_1}) \subseteq \frac{K}{S_1}$. Thus $Soc(M) \subseteq K$, a contradiction by Lemma 2.3 (b). Hence $\acute{G}(M)$ is an empty graph. □

Remark 2.6. In Theorem 2.5, the condition "$\frac{M}{S_1}$ and $\frac{M}{S_2}$ are cocyclic modules" can not be omitted. For example, let $M = \mathbb{Z}_{18}$ (as $\mathbb{Z}$-module). Then $Min(M) = \{S_1, S_2\}$, where $S_1 = \{\bar{6}\mathbb{Z}\}$ and $S_2 = \{\bar{9}\mathbb{Z}\}$. It is easy to see that $\frac{M}{S_1}$ is not cocyclic and $\acute{G}(M)$ is not empty.

Theorem 2.7. *The following assertions are equivalent.*

(i) $\acute{G}(M)$ is not connected.

(ii) $|Min(M)| = 2$

(iii) $\acute{G}(M) = \acute{G_1} \cup \acute{G_2}$, where $\acute{G_1}$ and $\acute{G_2}$ are complete and disjoint subgraphs.

Proof. $(i) \Rightarrow (ii)$ Assume to the contrary that $|Min(M)| > 2$. Since $\acute{G}(M)$ is not connected, we can consider two components $\acute{G_1}, \acute{G_2}$ and N, K two submodules of M such that $N \in \acute{G_1}$ and $K \in \acute{G_2}$. Choose $S_1, S_2 \in Min(M)$ such that $S_1 \subseteq N$ and $S_2 \subseteq K$. If $S_1 = S_2$, then $N - S_1 - K$ is a path, a contradiction. So we can assume that $S_1 \neq S_2$. Since $Min(M) > 2$, $S_1 + S_2$ is a non-large submodule of M by Proposition 2.4 (a). Thus $N - S_1 - S_2 - K$ is a path between $\acute{G_1}$ and $\acute{G_2}$, a contradiction. Therefore, $|Min(M)| = 2$.
$(ii) \Rightarrow (iii)$ Let $Min(M) = \{S_1, S_2\}$. Set $\acute{G_j} := \{N \leq M \mid N \supseteq S_j \, and \, N \ntrianglelefteq M\}$, where $j = 1, 2$. Assume that $N, K \in \acute{G_1}$. We claim that N and K are adjacent. Otherwise, if $N + K \trianglelefteq M$, then $S_1 + S_2 = Soc(M) \subseteq N + K$ by Lemma 2.3 (b). So we have $S_2 \subseteq S_1 + S_2 \subseteq N + K$. Thus $S_2 \subseteq N$ or $S_2 \subseteq K$ by Lemma 2.3 (d), a contradiction. By using similar arguments for $\acute{G_2}$, we can conclude that $\acute{G_1}, \acute{G_2}$ are complete subgraphs of $\acute{G}(M)$. We claim that these two subgraphs are disjoint. Assume to the contrary that $N_1 \in \acute{G_1}$ and $N_2 \in \acute{G_2}$ are adjacent. Then $Soc(M) = S_1 + S_2 \subseteq N_1 + N_2$ which implies that $N_1 + N_2$ is a large submodule of M by Lemma 2.3 (b), a contradiction.
$(iii) \Rightarrow (i)$ This is obvious. $\square$

Remark 2.8. The condition that "M is a comultiplication module" can not be omitted in Theorem 2.7. For example, let $M = \mathbb{Z}_2 \oplus \mathbb{Z}_4$ be a $\mathbb{Z}$-module and let $N_1 := (\bar{0}, \bar{1})\mathbb{Z}$, $N_2 := (\bar{0}, \bar{2})\mathbb{Z}$, $N_3 := (\bar{1}, \bar{0})\mathbb{Z}$, $N_4 := (\bar{1}, \bar{1})\mathbb{Z}$, $and \, N_5 := (\bar{1}, \bar{2})\mathbb{Z}$. Then $V(\acute{G}(M)) = \{N_1, N_2, N_3, N_4, N_5\}$ and $Min(M) = \{N_2, N_3, N_5\}$. Thus $|Min(M)| > 2$ but $\acute{G}(M)$ is not a connected graph.

Theorem 2.9. *Let $\acute{G}(M)$ be a connected graph. Then $diam(\acute{G}(M)) \leqslant 2$.*

Proof. Let N and K be two vertices of $\acute{G}(M)$ such that they are not adjacent. By Lemma 2.3 (a), there exist two minimal submodules S_1, S_2 of M such that $S_1 \subseteq N$ and $S_2 \subseteq K$. If $N + S_2 \ntrianglelefteq M$, then $N - S_2 - K$ is a path. So $d(N, K) = 2$. Similarly, if $K + S_1 \ntrianglelefteq M$, then $d(N, K) = 2$. Now assume that $N + S_2 \trianglelefteq M$ and $K + S_1 \trianglelefteq M$. By Theorem 2.7, $|Min(M)| \geq 3$. Let S_3 be a minimal submodule of M such that $S_3 \neq S_1, S_2$. Thus by Lemma 2.3 (b), we have $S_3 \subseteq Soc(M) \subseteq N + S_2$ which implies that $S_3 \subseteq N$ by Lemma 2.3 (d). Also we have $S_3 \subseteq Soc(M) \subseteq K + S_1$ which follows that $S_3 \subseteq K$ by Lemma 2.3 (d). Thus $S_3 \subseteq N$ and $S_3 \subseteq K$. Hence we have $N - S_3 - K$. Therefore, $d(N, K) = 2$. $\square$

Theorem 2.10. *Suppose that $\acute{G}(M)$ contains a cycle. Then $g(\acute{G}(M)) = 3$.*

Proof. If $|Min(M)| = 2$, then $\acute{G}(M) = \acute{G_1} \cup \acute{G_2}$, where $\acute{G_1}$ and $\acute{G_2}$ are complete disjoint subgraphs by Theorem 2.7. Since $\acute{G}(M)$ contains a cycle and $\acute{G_1}, \acute{G_2}$ are disjoint complete subgraphs, $g(\acute{G}(M)) = 3$. Now assume that $|Min(M)| \geq 3$ and choose $S_1, S_2, \, and \, S_3 \in Min(M)$. By Proposition 2.4 (a), $S_1 - S_2 - S_3 - S_1$ is a cycle. Hence $g(\acute{G}(M)) = 3$. $\square$

A vertex a in a connected graph G is a *cut vertex* if $G-\{a\}$ is disconnected.

Theorem 2.11. *If $\acute{G}(M)$ is a connected graph, then $\acute{G}(M)$ has no cut vertex.*

Proof. Assume on the contrary that there exists a vertex $N \in V(\acute{G}(M))$ such that $\acute{G}(M) \setminus \{N\}$ is not connected. Thus there exist at least two vertices K, L such that N lies in every path between them. By Theorem 2.9, the shortest path between K and L has length two. So we have $K-N-L$. Firstly, we claim that N is a minimal submodule of M. Otherwise, there exists a minimal submodule S of M such that $S \subset N$ by Lemma 2.3 (a). Since $S+K \subseteq N+K$ and $N+K \ntrianglelefteq M$, we have $S+K \ntrianglelefteq M$. By similar arguments, $S+L$ is a non-large submodule of M. Hence $K-S-L$ is a path in $\acute{G}(M) \setminus \{N\}$, a contradiction. Thus N is a minimal submodule of M. Now we claim that there is a minimal submodule $S_1 \neq N$ such that $S_1 \nsubseteq K$. Suppose on the contrary that $S_i \subseteq K$ for each $N \neq S_i \in Min(M)$. So we have $Soc(M) \subseteq K+N$. This implies that $K+N$ is a large submodule of M by Lemma 2.3 (b), a contradiction. Similarly, there exists a minimal submodule $S_2 \neq N$ of M such that $S_2 \nsubseteq L$. Note that for each $S_t \in Min(M)$, we have $S_t \subseteq K+L$ because $K+L$ is a large submodule of M. So $S_t \subseteq K$ or $S_t \subseteq L$ by Lemma 2.3 (d). Now let $N \neq S_1, S_2 \in Min(M)$ such that $S_1 \nsubseteq K$ and $S_2 \nsubseteq L$ (Note that $S_1 \neq S_2$). Hence we have $S_1 \subseteq L$ and $S_2 \subseteq K$. This implies that $K-S_1-S_2-L$ is a path in $\acute{G}(M) \setminus \{N\}$, a contradiction. □

Theorem 2.12. *$\acute{G}(M)$ can not be a complete n-partite graph, where $n \geq 2$.*

Proof. Suppose $\acute{G}(M)$ is a complete n-partite graph for some $n \geq 2$, with parts $U_1, ..., U_n$. In particular, $\acute{G}(M)$ is a connected graph. Hence by Theorem 2.7, $|Min(M)| = t \geq 3$. By Proposition 2.4 (a), for every $S_i, S_j \in Min(M)$, $S_i \neq S_j$, S_i is adjacent to S_j. Hence each part U_i contains at most one minimal submodule so that $n \geq 3$. Now we claim that $t = n$. Suppose on the contrary that $t < n$. Without loss of generality, we suppose that $S_i \in U_i$, for $i = 1, ..., t$. Then U_{t+1} contains no minimal submodule of M. By Proposition 2.4 (a), $\Sigma_{j \neq i} S_j$ is a non-large submodule of M. Clearly, $\Sigma_{j \neq i} S_j$ and S_i are not adjacent. Hence $\Sigma_{j \neq i} S_j \in U_i$. Let N be a vertex in U_{t+1}. Then by Lemma 2.3 (a), there exists $S_k \in Min(M)$ such that $S_k \subseteq N$. So N and S_k are adjacent, where $S_k \in U_k$. Since $\acute{G}(M)$ is a complete n-partite graph, N adjacent to all vertices in U_k. So N and $\Sigma_{j \neq k} S_j$ are adjacent. However, $Soc(M) = S_k + \Sigma_{j \neq k} S_j$ which implies that $N + \Sigma_{j \neq k} S_j \trianglelefteq M$ by Proposition 2.4 (a), a contradiction. Hence $|Min(M)| = n$. Now set $K := \Sigma_{i \geq 3} S_i$. By Proposition 2.4 (a), K is a non-large submodule of M. Since $K + S_1 = \Sigma_{i \neq 2} S_i \ntrianglelefteq M$, K and S_1 are adjacent. Similarly, K is adjacent to S_2. Thus $K \notin U_1, U_2$. Furthermore, $K + S_i = K \ntrianglelefteq M$ for each i $(3 \leq i \leq n)$. Hence K is adjacent to all minimal submodules S_i of M. So for each i $(1 \leq i \leq n)$, $K \notin U_i$, a contradiction. □

Proposition 2.13. *Assume that $|Min(M)| < \infty$. Then we have the following.*

(i) There is no vertex in $\acute{G}(M)$ which is adjacent to every other vertex.

(ii) $\acute{G}(M)$ can not be a complete graph.

Proof. (i) Assume on the contrary that there exists a submodule $N \in V(\acute{G}(M))$ such that N is adjacent to all vertices of $\acute{G}(M)$. By Lemma 2.3 (a), there is a minimal submodule $S_i \in Min(M)$ such that $S_i \subseteq N$. Now set $K := \Sigma_{j \neq i} S_j$, where $S_j, j \neq i$, are all the other minimal submodules of M. Clearly, $K \ntrianglelefteq M$ by Proposition 2.4 (a). Since N is adjacent to all other vertices of $\acute{G}(M)$, $N + K$ is a non-large submodule of M. However, $Soc(M) = \Sigma_{j \neq i} S_j + S_i \subseteq N + K$ which shows that $N + K \trianglelefteq M$ by Lemma 2.3 (b), a contradiction.

(ii) This follows from (i).

□

A vertex of a graph G is said to be *pendent* if its neighbourhood contains exactly one vertex.

Theorem 2.14. *(i) $\acute{G}(M)$ contains a pendent vertex if and only if $|Min(M)| = 2$ and $\acute{G}(M) = \acute{G}_1 \cup \acute{G}_2$, where $\acute{G}_1, \acute{G}_2$ are two disjoint complete subgraphs and $|V(\acute{G}_i)| = 2$ for some $i = 1, 2$.*

(ii) $\acute{G}(M)$ is not a star graph.

Proof. (i) Let N be a pendent vertex of $\acute{G}(M)$. Assume on the contrary that $|Min(M)| \geq 3$. Clearly, for each $S_i \in Min(M)$, S_i is adjacent to every other minimal submodules of M. So $deg(S_i) \geq 2$. Thus N is not a minimal submodule of M. By Lemma 2.3 (a), there exists a minimal submodule of S_1 of M such that $S_1 \subseteq N$. Note that the only vertex which is adjacent to N is S_1 because $deg(N) = 1$. Hence there is no minimal submodule $S_i \neq S_1$ such that $S_i \subseteq N$. Moreover, $N + S_2$ is a large submodule of M. So by Lemma 2.3 (b), $S_j \subseteq Soc(M) \subseteq N + S_2$, for each $S_j \neq S_1, S_2$. This implies that $S_j \subseteq N$ by Lemma 2.3 (d), a contradiction. Hence $|Min(M)| = 2$. By Theorem 2.7, $\acute{G}(M) = \acute{G}_1 \cup \acute{G}_2$, where $\acute{G}_1$ and $\acute{G}_2$ are disjoint complete subgraphs. It is easy to see that $|V(\acute{G}_i)| = 2$. The converse is straightforward.

(ii) Suppose that $\acute{G}(M)$ is a star graph. Then $\acute{G}(M)$ has a pendent vertex. So by part (i), we have $|Min(M)| = 2$. Thus $\acute{G}(M)$ is not a connected graph by Theorem 2.7, a contradiction.

□

Theorem 2.15. *(i) Let N, K be two vertex of $\acute{G}(M)$ such that $N \subseteq K$. Then $deg(N) \geq deg(K)$.*

(ii) Let $\acute{G}(M)$ be an r-regular graph. Then $|Min(M)| = 2$ and $|V(\acute{G}(M))| = 2r+2$.

Proof. (i) Let $N, K \in V(\acute{G}(M))$ be such that $N \subseteq K$. Let L be a vertex of $\acute{G}(M)$ such that L is adjacent to K. Thus $K+L$ is a non-large submodule of M and so that $N+L$ is a non-large submodule of M. So L is adjacent to N. Therefore, $deg(N) \geq deg(K)$.

(ii) Suppose on the contrary that $|Min(M)| \geq 3$. By using Proposition 2.4 (a) and our assumption, $Min(M)$ is a finite set. Next for $S_1, S_2 \in Min(M)$, we have $deg(S_1) \geq deg(S_1+S_2)$ by part (i). We claim that $deg(S_1+S_2) < deg(S_1)$. In fact, $\Sigma_{j \neq 2} S_j$ is adjacent to S_1 by Proposition 2.4 (a), but it is not adjacent to S_1+S_2. So $deg(S_1+S_2) < r$, which is a contradiction. Thus $|Min(M)| \leq 2$. If $|Min(M)| = 1$, then $\acute{G}(M)$ is null graph, a contradiction. Thus $|Min(M)| = 2$ and so by Theorem 2.7, $\acute{G}(M) = \acute{G}_1 \cup \acute{G}_2$, where $\acute{G}_1, \acute{G}_2$ are disjoint complete subgraphs. Set $Min(M) = \{S_1, S_2\}$ and $S_i \in G_i$. Since $\acute{G}(M)$ is r-regular, $|V(\acute{G}_i)| = r+1$ for $i = 1,2$. Hence we have $V(\acute{G}(M)) = 2r+2$.

□

3. clique and dominating number

In this section, we provide some information about the clique and dominating number of $\acute{G}(M)$.

Proposition 3.1. *(i) Let $\acute{G}(M)$ be a non-empty graph. Then $\omega(\acute{G}(M)) \geq |Min(M)|$.*

(ii) If $\omega(\acute{G}(M)) < \infty$, then $\omega(\acute{G}(M)) \geq 2^{|Min(M)|-1} - 1$.

Proof. (i) If $|Min(M)| = 2$, then $\omega(\acute{G}(M)) \geq 2$ by Theorem 2.7. Now let $|Min(M)| \geq 3$. Then by Proposition 2.4 (a), the subgraph of $\acute{G}(M)$ with the vertex set of $\{S_i\}_{S_i \in Min(M)}$ is a complete subgraph of $\acute{G}(M)$. So $\omega(\acute{G}(M)) \geq |Min(M)|$.

(ii) Since $\omega(\acute{G}(M)) < \infty$, we have $|Min(M)| < \infty$ by part (i), (ii). Let $Min(M) = \{S_1, ..., S_t\}$. For each $1 \leq i \leq t$, consider

$$A_i = \{S_1, \dots, S_{i-1}, S_{i+1}, S_t\}.$$

Now let $P(A_i)$ be the power set of A_i and for each $X \in P(A_i)$, set $S_X = \bigcap_{S_j \in X} S_j$ for $1 \leq j \leq t$. The subgraph of $\acute{G}(M)$ with the vertex set $\{S_X\}_{X \in P(A_i) \setminus \{\emptyset\}}$ is a complete subgraph of $\acute{G}(M)$ by Proposition 2.4 (a). It is clear that $|\{S_X\}_{X \in P(A_i) \setminus \{\emptyset\}}| = 2^{|Min(M)|-1} - 1$. Thus $\omega(\acute{G}(M)) \geq 2^{|Min(M)|-1} - 1$.

□

Remark 3.2. Note that the condition "M is a comultiplication module" is necessary in Proposition 3.1. For example, let $M = \mathbb{Z}_2 \oplus \mathbb{Z}_4$ be as a $\mathbb{Z}$-module which is not a comultiplication module. Then $\omega(\acute{G}(M)) = 2$ but $|Min(M)| = 3$.

Theorem 3.3. *We have $\gamma(\acute{G}(M)) \leq 2$. In particular, if $|Min(M)| < \infty$, then $\gamma(\acute{G}(M)) = 2$.*

Proof. Clearly, $|Min(M)| \geq 2$ because $\acute{G}(M)$ is a non-null graph. Consider $S = \{S_1, S_2\}$, where $S_1, S_2 \in Min(M)$. Let N be a vertex of $\acute{G}(M)$. We claim that N is adjacent to S_1 or S_2. If $S_1 \subseteq N$ or $S_2 \subseteq N$, then the claim is true. Now assume that $S_1 \nsubseteq N$ and $S_2 \nsubseteq N$. In this case, we also claim that N is adjacent to S_1 or S_2. Without loss of generality, we can assume that N is not adjacent to S_1. So $S_2 \subseteq Soc(M) \subseteq N$ by Lemma 2.3 (b). This shows that $S_2 \subseteq N$, which is a contradiction. By similar arguments, we can show that N is adjacent to S_2. Thus $\gamma(\acute{G}(M)) \leq 2$. The last assertion follows from Theorem 2.13. □

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