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INFINITELY MANY SOLUTIONS TO THE DIRICHLET PROBLEM FOR QUASILINEAR ELLIPTIC SYSTEMS

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In this paper we deal with the existence of weak solutions for the following Dirichlet problem

$$\begin{cases} -\Delta_p u = f(u, v) \text{ in } \Omega \\ -\Delta_q v = g(u, v) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \\ v = 0 \text{ on } \partial\Omega \end{cases}.$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set. The existence of solutions is proved by applying a critical point variational principle obtained by B. Ricceri as consequence of a more general variational principle.

1. Introduction.

Here and in the sequel:
$\Omega \subset \mathbb{R}^N$ is a bounded open set with boundary of class C^1;
$N \geq 1$; $p > N$; $q > N$;
$f, g \in C^0(\mathbb{R}^2)$ such that the differential form $f(u, v)du + g(u, v)dv$ be exact.

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In this paper we are interested in the following problem:

$$(P) \qquad \begin{cases} -\Delta_p u = f(u,v) \text{ in } \Omega \\ -\Delta_q v = g(u,v) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \\ v = 0 \text{ on } \partial\Omega \end{cases}.$$

More precisely we are interested in the existence of infinitely many weak solutions to such a problem.

Even though the problem (P) has been studied by some other authors (see e.g. [7], [8], [3], [2], [1]) the hypotheses we use in this paper are totally different from those ones and so are our results.

The existence of solutions to Problem (P) is proved by applying the following critical point theorem. The proof of this theorem is very similar to that of Theorem 2.5 of [6] and so it is omitted.

Theorem 1. *Let X be a reflexive real Banach space, and let $\Phi, \Psi : X \to \mathbb{R}$ be two sequentially weakly lower semicontinuous and Gateaux differentiable functionals. Assume also that Ψ is strongly continuous and satisfies $\lim_{\|x\|\to\infty} \Psi(x) = +\infty$. For each $r > inf_X \Psi$, put*

$$\varphi(r) = \inf_{x\in\Psi^{-1}(]-\infty,r[)} \frac{\Phi(x) - \inf_{\overline{(\Psi^{-1}(]-\infty,r[))}_w} \Phi}{r - \Psi(x)},$$

where $\overline{(\Psi^{-1}(]-\infty,r[))}_w$ is the closure of $\Psi^{-1}(]-\infty,r[)$ in the weak topology. Fixed $\lambda > 0$, then

- *(a) if $\{r_n\}_{n\in\mathbb{N}}$ is a real sequence with $\lim_{n\to\infty} r_n = +\infty$ such that $\varphi(r_n) < \lambda$, for each $n \in \mathbb{N}$, the following alternative holds: either $\Phi+\lambda\Psi$ has a global minimum, or there exists a sequence $\{x_n\}$ of critical points of $\Phi+\lambda\Psi$ such that $\lim_{n\to\infty} \Psi(x_n) = +\infty$.*
- *(b) if $\{s_n\}_{n\in\mathbb{N}}$ is a real sequence with $\lim_{n\to\infty} s_n = (\inf_X \Psi)^+$ such that $\varphi(s_n) < \lambda$, for each $n \in \mathbb{N}$, the following alternative holds: either there exists a global minimum of Ψ which is a local minimum of $\Phi+\lambda\Psi$, or there exists a sequence $\{x_n\}$ of pairwise distinct critical points of $\Phi+\lambda\Psi$, with $\lim_{n\to\infty} \Psi(x_n) = \inf_X \Psi$, which weakly converges to a global minimum of Ψ.*

Let $G : \mathbb{R}^2 \to \mathbb{R}$ be the differentiable function such that $G_u(u,v) = f(u,v)$, $G_v(u,v) = g(u,v)$, $G(0,0) = 0$. Then (P) can be written in the form

$$\begin{cases} -\Delta_p u = G_u(u,v) \text{ in } \Omega \\ -\Delta_q v = G_v(u,v) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \\ v = 0 \text{ on } \partial\Omega \end{cases}.$$

and therefore it is a gradient system [4]. We first consider the space $W_0^{1,p}(\Omega)$ with the norm

$$\|u\|_{W_0^{1,p}(\Omega)} = \left(\int_\Omega |\nabla u(x)|^p \, dx \right)^{\frac{1}{p}}$$

and the space $W_0^{1,q}(\Omega)$ with the norm

$$\|v\|_{W_0^{1,q}(\Omega)} = \left(\int_\Omega |\nabla v(x)|^q \, dx \right)^{\frac{1}{q}}.$$

Since by hypotheses $p > N$ and $q > N$, $W^{1,p}(\Omega)$ and $W^{1,q}(\Omega)$ are both compactly embedded in $C^0(\overline{\Omega})$. Then we put

$$c_1 = \sup_{u \in W^{1,p}(\Omega)\setminus\{0\}} \frac{\sup_{x\in\Omega} |u(x)|}{\|u\|}$$

that is finite since $W^{1,p}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$ and

$$c_2 = \sup_{u \in W^{1,q}(\Omega)\setminus\{0\}} \frac{\sup_{x\in\Omega} |u(x)|}{\|u\|}$$

that is finite since $W^{1,q}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$.

In order to apply the former theorem we set

$$\Psi(u, v) = \frac{1}{p}\|u\|^p + \frac{1}{q}\|v\|^q$$

and

$$\Phi(u, v) = -\int_\Omega G(u(x), v(x)) \, dx$$

for all $(u, v) \in X$. Since $X = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega) \subseteq W^{1,p}(\Omega) \times W^{1,q}(\Omega)$, the functionals Φ and Ψ are (well defined and) sequentially weakly lower semicontinuous and Gateaux differentiable in X, the critical points of $\Phi + \Psi$ being precisely the weak solutions to Problem (P). Moreover Ψ is coercive (and strongly continuous as well). For the proofs of the previous statements, which are not difficult but a little bit tedious, the reader is referred to the Author's PHD thesis [5].

If the following definitions are used

$$\alpha = \frac{1}{pc_1^p}$$

$$\beta = \frac{1}{qc_2^q}$$

and for each $r > 0$

$$A(r) = \{(\xi, \eta) \in \mathbb{R}^2 \text{ such that } \alpha|\xi|^p + \beta|\eta|^q \le r\}$$

$$S(r) = \{(\xi, \eta) \in \mathbb{R}^2 \text{ such that } |\xi|^p + |\eta|^q \le r\}$$

then

$$S\left(\frac{r}{\max(\alpha, \beta)}\right) \subseteq A(r) \subseteq S\left(\frac{r}{\min(\alpha, \beta)}\right)$$

Moreover we put $\omega := \frac{\pi^{n/2}}{\frac{n}{2}\Gamma(\frac{n}{2})}$ the measure of the n-dimensional unit ball.

2. Results.

We wish to establish two multiplicity results for Problem (P). Making use of theorem 1, our results guarantee that Problem (P) has infinitely many weak solutions.

Theorem 2. *Assume that* $\inf_{\mathbb{R}^2} G \ge 0$. *Moreover, suppose that there exist two real sequences* $\{a_n\}$ *and* $\{b_n\}$ *in* $]0, +\infty[$ *with* $a_n < b_n$, $\lim_{n\to\infty} b_n = +\infty$, *such that*

$$\lim_{n\to+\infty} \frac{b_n}{a_n} = +\infty$$

$$\max_{S(a_n)} G = \max_{S(b_n)} G > 0$$

$$\max\left\{\frac{2^p(2^N - 1)}{pD^p}, \frac{2^q(2^N - 1)}{qD^q}\right\} < \limsup_{(\xi,\eta)\to\infty} \frac{G(\xi, \eta)}{|\xi|^p + |\eta|^q} < +\infty$$

where $D = \sup_{x\in\Omega} d(x, \partial\Omega)$. *Then Problem* (P) *admits an unbounded sequence of weak solutions.*

Proof. Fix $(\xi_n, \eta_n) \in S(a_n)$ such that

$$\max_{S(b_n)} G = G(\xi_n, \eta_n)$$

Put $\delta = \min\{\alpha, \beta\}$ and $r_n = \delta b_n$ for each $n \in \mathbb{N}$. Then $\{r_n\}_{n\in\mathbb{N}}$ is a real sequence with $r_n > 0$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} r_n = +\infty$ such that

$$\lim_{n\to\infty} \frac{r_n}{|\xi_n|^p + |\eta_n|^q} = +\infty$$

Moreover

$$\max_{A(r_n)} G = G(\xi_n, \eta_n)$$

In our case the function φ of theorem 1 is defined by setting

$$\varphi(r) = \inf_{(u,v)\in\Psi^{-1}(]-\infty,r[)} \frac{\Phi(u,v) - \inf_{\overline{(\Psi^{-1}(]-\infty,r[))}_w} \Phi}{r - \Psi(u,v)} =$$

$$= \inf_{(u,v)\in\Psi^{-1}(]-\infty,r[)} \frac{\Phi(u,v) - \inf_{\Psi^{-1}(]-\infty,r])} \Phi}{r - \Psi(u,v)}$$

for each $r \in]0, +\infty[$. We have

$$\varphi(r_n) = \inf_{(u,v)\in\Psi^{-1}(]-\infty,r_n[)} \frac{\Phi(u,v) - \inf_{\Psi^{-1}(]-\infty,r_n])} \Phi}{r_n - \Psi(u,v)}$$

We wish to prove that $\varphi(r_n) < 1$ provided that $n \in \mathbb{N}$ is large enough; in order to get the previous inequality we show that there exists $(u_n, v_n) \in X$, with $\Psi(u_n, v_n) < r_n$, such that

$$\frac{\Phi(u_n, v_n) - \inf_{\Psi^{-1}(]-\infty,r_n])} \Phi}{r_n - \Psi(u_n, v_n)} < 1$$

From

$$\max\left\{\frac{2^p(2^N-1)}{pD^p}, \frac{2^q(2^N-1)}{qD^q}\right\} < \limsup_{(\xi,\eta)\to\infty} \frac{G(\xi,\eta)}{|\xi|^p + |\eta|^q} < +\infty$$

we can choose a constant h such that

$$\max\left\{\frac{2^p(2^N-1)}{pD^p}, \frac{2^q(2^N-1)}{qD^q}\right\} < h < \limsup_{(\xi,\eta)\to\infty} \frac{G(\xi,\eta)}{|\xi|^p + |\eta|^q} < +\infty$$

and so there exists a $x_0 \in \Omega$ such that

$$\max\left\{\left(\frac{2^p(2^N-1)}{ph}\right)^{\frac{1}{p}}, \left(\frac{2^q(2^N-1)}{qh}\right)^{\frac{1}{q}}\right\} < d(x_0, \partial\Omega) \leq D$$

Therefore we can fix γ satisfying

$$\max\left\{\left(\frac{2^p(2^N-1)}{ph}\right)^{\frac{1}{p}}, \left(\frac{2^q(2^N-1)}{qh}\right)^{\frac{1}{q}}\right\} < \gamma < d(x_0, \partial\Omega) \leq D$$

from which
$$\max\left\{\frac{2^p(2^N-1)}{p\gamma^p}, \frac{2^q(2^N-1)}{q\gamma^q}\right\} < h$$

Now,fix $n \in \mathbb{N}$ and consider the functions $u_n \in W_0^{1,p}(\Omega)$ and $v_n \in W_0^{1,q}(\Omega)$ defined by setting

$$u_n(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0,\gamma) \\ \xi_n & \text{if } x \in B(x_0, \frac{\gamma}{2}) \\ \frac{2\xi_n}{\gamma}(\gamma - |x-x_0|_N) & \text{if } x \in B(x_0,\gamma) \setminus B(x_0, \frac{\gamma}{2}) \end{cases}.$$

$$v_n(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0,\gamma) \\ \eta_n & \text{if } x \in B(x_0, \frac{\gamma}{2}) \\ \frac{2\eta_n}{\gamma}(\gamma - |x-x_0|_N) & \text{if } x \in B(x_0,\gamma) \setminus B(x_0, \frac{\gamma}{2}) \end{cases}.$$

Obviously
$$\Psi(u_n, v_n) = \frac{1}{p}\|u_n\|^p + \frac{1}{q}\|v_n\|^q =$$
$$= \frac{1}{p}\int_\Omega |\nabla u_n(x)|^p\,dx + \frac{1}{q}\int_\Omega |\nabla v_n(x)|^q\,dx =$$
$$= \frac{1}{p}\int_{B(x_0,\gamma)\setminus B(x_0,\frac{\gamma}{2})} |\nabla u_n(x)|^p\,dx + \frac{1}{q}\int_{B(x_0,\gamma)\setminus B(x_0,\frac{\gamma}{2})} |\nabla v_n(x)|^q\,dx =$$
$$= \frac{1}{p}\int_{B(x_0,\gamma)\setminus B(x_0,\frac{\gamma}{2})} \frac{2^p|\xi_n|^p}{\gamma^p}\,dx + \frac{1}{q}\int_{B(x_0,\gamma)\setminus B(x_0,\frac{\gamma}{2})} \frac{2^q|\eta_n|^q}{\gamma^q}\,dx =$$
$$= \Big(\frac{2^p|\xi_n|^p}{p\gamma^p} + \frac{2^q|\eta_n|^q}{q\gamma^q}\Big)\Big|B(x_0,\gamma) \setminus B\Big(x_0, \frac{\gamma}{2}\Big)\Big| =$$
$$= \Big(\frac{2^p|\xi_n|^p}{p\gamma^p} + \frac{2^q|\eta_n|^q}{q\gamma^q}\Big)\omega\gamma^N\frac{2^N-1}{2^N} =$$
$$= \Big(\frac{2^p(2^N-1)}{p\gamma^p}|\xi_n|^p + \frac{2^q(2^N-1)}{q\gamma^q}|\eta_n|^q\Big)\frac{\omega\gamma^N}{2^N} <$$
$$< (|\xi_n|^p + |\eta_n|^q)\frac{h\omega\gamma^N}{2^N}$$

thus $\Psi(u_n, v_n) < r_n$ if $n \in \mathbb{N}$ is large enough.

Moreover we have the inequality:

$$\Big(\frac{2^p|\xi_n|^p}{p\gamma^p} + \frac{2^q|\eta_n|^q}{q\gamma^q}\Big)\omega\gamma^N\frac{2^N-1}{2^N} < (|\xi_n|^p + |\eta_n|^q)\frac{h\omega\gamma^N}{2^N}$$

whence

$$r_n - (|\xi_n|^p + |\eta_n|^q)\frac{h\omega\gamma^N}{2^N} < r_n - \left(\frac{2^p|\xi_n|^p}{p\gamma^p} + \frac{2^q|\eta_n|^q}{q\gamma^q}\right)\omega\gamma^N\frac{2^N - 1}{2^N}$$

Next, since

$$\limsup_{(\xi,\eta)\to\infty} \frac{G(\xi,\eta)}{|\xi|^p + |\eta|^q} < +\infty$$

there exists $L > 0$ such that for all $n \in \mathbb{N}$

$$\frac{G(\xi_n,\eta_n)}{|\xi_n|^p + |\eta_n|^q} < L$$

and since

$$\lim_{n\to\infty} \frac{r_n}{|\xi_n|^p + |\eta_n|^q} = +\infty$$

we have for $n \in \mathbb{N}$ large enough,

$$\frac{r_n}{|\xi_n|^p + |\eta_n|^q} > L\left(|\Omega| - \omega\frac{\gamma^N}{2^N}\right) + \frac{h\omega\gamma^N}{2^N}$$

$$|\Omega| - \omega\frac{\gamma^N}{2^N} < \left[\frac{r_n}{|\xi_n|^p + |\eta_n|^q} - \frac{h\omega\gamma^N}{2^N}\right]\frac{1}{L},$$

hence

$$\Phi(u_n, v_n) - \inf_{\Psi^{-1}(]-\infty,r_n])} \Phi =$$

$$= \sup_{\Psi^{-1}(]-\infty,r_n])} \int_\Omega G(u(x), v(x))\,dx - \int_\Omega G(u_n(x), v_n(x))\,dx \leq$$

$$\leq G(\xi_n,\eta_n)|\Omega| - \int_\Omega G(u_n(x), v_n(x))\,dx \leq$$

$$\leq G(\xi_n,\eta_n)|\Omega| - \int_{B(x_0,\frac{\gamma}{2})} G(u_n(x), v_n(x))\,dx =$$

$$= G(\xi_n,\eta_n)|\Omega| - \int_{B(x_0,\frac{\gamma}{2})} G(\xi_n,\eta_n)\,dx =$$

$$= G(\xi_n,\eta_n)|\Omega| - G(\xi_n,\eta_n)\left|B\left(x_0, \frac{\gamma}{2}\right)\right| =$$

$$= G(\xi_n, \eta_n)\left(|\Omega| - \left|B\left(x_0, \frac{\gamma}{2}\right)\right|\right) =$$

$$= G(\xi_n, \eta_n)\left(|\Omega| - \omega\frac{\gamma^N}{2^N}\right) <$$

$$< G(\xi_n, \eta_n)\left[\frac{r_n}{|\xi_n|^p + |\eta_n|^q} - \frac{h\omega\gamma^N}{2^N}\right]\frac{1}{L} =$$

$$= \frac{1}{L}\left[r_n - \{|\xi_n|^p + |\eta_n|^q\}\frac{h\omega\gamma^N}{2^N}\right]\frac{G(\xi_n, \eta_n)}{|\xi_n|^p + |\eta_n|^q} <$$

$$< \frac{1}{L}\left[r_n - \{|\xi_n|^p + |\eta_n|^q\}\frac{h\omega\gamma^N}{2^N}\right]L =$$

$$= r_n - \{|\xi_n|^p + |\eta_n|^q\}\frac{h\omega\gamma^N}{2^N} <$$

$$< r_n - \left(\frac{2^p|\xi_n|^p}{p\gamma^p} + \frac{2^q|\eta_n|^q}{q\gamma^q}\right)\omega\gamma^N\frac{2^N - 1}{2^N} =$$

$$= r_n - \left(\frac{1}{p}\int_\Omega |\nabla u_n|^p\,dx + \frac{1}{q}\int_\Omega |\nabla v_n|^q\,dx\right) = r_n - \Psi(u_n, v_n)$$

Bearing in mind that $\lim_{n\to\infty} r_n = +\infty$, the previous inequality assures that the conclusion (a) of theorem 1 can be used and either the functional $\Phi + \Psi$ has a global minimum or there exists a sequence $\{(u_n, v_n)\}_{n\in\mathbb{N}}$ of solutions to Problem (P) such that $\lim_{n\to\infty} \|(u_n, v_n)\|_X = +\infty$.

The other step is to verify that the functional $\Phi + \Psi$ has no global minimum. Taking into account

$$h < \limsup_{(\rho,\sigma)\to\infty} \frac{G(\rho, \sigma)}{|\rho|^p + |\sigma|^q} < +\infty$$

one has, for each $n \in \mathbb{N}$

$$h < \inf_{n\in\mathbb{N}}\left(\sup_{\sqrt{\rho^2+\sigma^2}\geq n} \frac{G(\rho, \sigma)}{|\rho|^p + |\sigma|^q}\right)$$

and so there exists $(\rho_n, \sigma_n) \in \mathbb{R}^2$ such that $\sqrt{\rho_n^2 + \sigma_n^2} \geq n$ and

$$\frac{G(\rho_n, \sigma_n)}{|\rho_n|^p + |\sigma_n|^q} > h$$

that is $G(\rho_n, \sigma_n) > (|\rho_n|^p + |\sigma_n|^q)h$.

Now if we consider functions $w_n \in W_0^{1,p}(\Omega)$, $z_n \in W_0^{1,q}(\Omega)$ defined by setting

$$w_n(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0, \gamma) \\ \rho_n & \text{if } x \in B(x_0, \frac{\gamma}{2}) \\ \frac{2\rho_n}{\gamma}(\gamma - |x - x_0|_N) & \text{if } x \in B(x_0, \gamma) \setminus B(x_0, \frac{\gamma}{2}) \end{cases}.$$

$$z_n(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0, \gamma) \\ \sigma_n & \text{if } x \in B(x_0, \frac{\gamma}{2}) \\ \frac{2\sigma_n}{\gamma}(\gamma - |x - x_0|_N) & \text{if } x \in B(x_0, \gamma) \setminus B(x_0, \frac{\gamma}{2}) \end{cases}.$$

one has

$$\begin{aligned}
&(\Phi + \Psi)(w_n, z_n) = \\
&= \frac{1}{p}\int_\Omega |\nabla w_n|^p\,dx + \frac{1}{q}\int_\Omega |\nabla z_n|^q\,dx - \int_\Omega G(w_n(x), z_n(x))\,dx = \\
&= \frac{\omega\gamma^N}{2^N}\left\{\frac{2^p(2^N-1)}{p\gamma^p}|\rho_n|^p + \frac{2^q(2^N-1)}{q\gamma^q}|\sigma_n|^q\right\} - \int_\Omega G(w_n(x), z_n(x))dx \leq \\
&\leq \frac{\omega\gamma^N}{2^N}\left\{\frac{2^p(2^N-1)}{p\gamma^p}|\rho_n|^p + \frac{2^q(2^N-1)}{q\gamma^q}|\sigma_n|^q\right\} - \int_{B(x_0,\frac{\gamma}{2})} G(\rho_n, \sigma_n)dx = \\
&= \frac{\omega\gamma^N}{2^N}\left\{\frac{2^p(2^N-1)}{p\gamma^p}|\rho_n|^p + \frac{2^q(2^N-1)}{q\gamma^q}|\sigma_n|^q\right\} - \omega\frac{\gamma^N}{2^N}G(\rho_n, \sigma_n) < \\
&< \frac{\omega\gamma^N}{2^N}\left\{\frac{2^p(2^N-1)}{p\gamma^p}|\rho_n|^p + \frac{2^q(2^N-1)}{q\gamma^q}|\sigma_n|^q\right\} - \omega\frac{\gamma^N}{2^N}\left(|\rho_n|^p + |\sigma_n|^q\right)h = \\
&= \frac{\omega\gamma^N}{2^N}\left\{\left(\frac{2^p(2^N-1)}{p\gamma^p} - h\right)|\rho_n|^p + \left(\frac{2^q(2^N-1)}{q\gamma^q} - h\right)|\sigma_n|^q\right\}
\end{aligned}$$

The previous inequality shows that the functional $\Phi + \Psi$ is not bounded from below and then it has no global minimum.

Therefore theorem 1 assures that there is a sequence (u_n, v_n) of critical points of $\Phi + \Psi$ such that $\lim_{n\to\infty} \|(u_n, v_n)\|_X = +\infty$. □

We point out that, as a consequence of the former theorem, the following corollary holds:

Theorem 3. *Assume that* $\inf_{\mathbb{R}^2} G \geq 0$. *Moreover, suppose that there exist two real sequences* $\{a_n\}$ *and* $\{b_n\}$ *in* $]0, +\infty[$ *with* $a_n < b_n$, $\lim_{n\to\infty} b_n = +\infty$, *such that*

$$\lim_{n\to+\infty} \frac{b_n}{a_n} = +\infty$$

$$\max_{S(a_n)} G = \max_{S(b_n)} G > 0$$

$$\max\left\{\frac{2^p(2^N-1)}{pD^p}, \frac{2^q(2^N-1)}{qD^q}\right\} < \mu \limsup_{(\xi,\eta)\to\infty} \frac{G(\xi,\eta)}{|\xi|^p + |\eta|^q} < +\infty$$

where $D = \sup_{x\in\Omega} d(x, \partial\Omega)$ *and* $\mu > 0$. *Then the problem*

$$\begin{cases} -\Delta_p u = \mu f(u, v) \text{ in } \Omega \\ -\Delta_q v = \mu g(u, v) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \\ v = 0 \text{ on } \partial\Omega \end{cases}.$$

admits an unbounded sequence of weak solutions.

The proof of the following Theorem is almost the same as that of Theorem 2 and so it is only sketched.

Theorem 4. *Assume that* $\inf_{\mathbb{R}^2} G \geq 0$. *Moreover, suppose that there exist two real sequences* $\{a_n\}$ *and* $\{b_n\}$ *in* $]0, +\infty[$ *with* $a_n < b_n$, $\lim_{n\to\infty} b_n = 0$, *such that*

$$\lim_{n\to+\infty} \frac{b_n}{a_n} = +\infty$$

$$\max_{S(a_n)} G = \max_{S(b_n)} G > 0$$

$$\max\left\{\frac{2^p(2^N-1)}{pD^p}, \frac{2^q(2^N-1)}{qD^q}\right\} < \limsup_{(\xi,\eta)\to(0,0)} \frac{G(\xi,\eta)}{|\xi|^p + |\eta|^q} < +\infty$$

where $D = \sup_{x\in\Omega} d(x, \partial\Omega)$. *Then Problem* (P) *admits a sequence of non-zero weak solutions which strongly converges to* θ_X *in* X.

Proof. Fix $(\xi_n, \eta_n) \in S(a_n)$ such that

$$\max_{S(b_n)} G = G(\xi_n, \eta_n)$$

Put $\delta = \min\{\alpha, \beta\}$ and $s_n = \delta b_n$ for each $n \in \mathbb{N}$. Then $\{s_n\}_{n\in\mathbb{N}}$ is a real sequence with $s_n > 0$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} s_n = 0$ such that

$$\lim_{n\to\infty} \frac{s_n}{|\xi_n|^p + |\eta_n|^q} = +\infty$$

Moreover

$$\max_{A(s_n)} G = G(\xi_n, \eta_n)$$

In our case the function φ of theorem 1 is defined by setting

$$\varphi(r) = \inf_{(u,v)\in\Psi^{-1}(]-\infty,r[)} \frac{\Phi(u,v) - \inf_{\Psi^{-1}(]-\infty,r])} \Phi}{r - \Psi(u,v)}$$

for each $r \in]0, +\infty[$. We have

$$\varphi(s_n) = \inf_{(u,v)\in\Psi^{-1}(]-\infty,s_n[)} \frac{\Phi(u,v) - \inf_{\Psi^{-1}(]-\infty,s_n]} \Phi}{s_n - \Psi(u,v)}$$

We wish to prove that $\varphi(s_n) < 1$ provided that $n \in \mathbb{N}$ is large enough; in order to get the previous inequality we show that there exists $(u_n, v_n) \in X$, with $\Psi(u_n, v_n) < s_n$, such that

$$\frac{\Phi(u_n, v_n) - \inf_{\Psi^{-1}(]-\infty,s_n])} \Phi}{s_n - \Psi(u_n, v_n)} < 1$$

From

$$\max\left\{\frac{2^p(2^N-1)}{pD^p}, \frac{2^q(2^N-1)}{qD^q}\right\} < \limsup_{(\xi,\eta)\to(0,0)} \frac{G(\xi,\eta)}{|\xi|^p + |\eta|^q} < +\infty$$

we can choose a constant h such that

$$\max\left\{\frac{2^p(2^N-1)}{pD^p}, \frac{2^q(2^N-1)}{qD^q}\right\} < h < \limsup_{(\xi,\eta)\to(0,0)} \frac{G(\xi,\eta)}{|\xi|^p + |\eta|^q} < +\infty$$

and so there exists a $x_0 \in \Omega$ such that

$$\max\left\{\left(\frac{2^p(2^N-1)}{ph}\right)^{\frac{1}{p}}, \left(\frac{2^q(2^N-1)}{qh}\right)^{\frac{1}{q}}\right\} < d(x_0, \partial\Omega) \le D$$

Therefore we can fix γ satisfying

$$\max\left\{\left(\frac{2^p(2^N-1)}{ph}\right)^{\frac{1}{p}}, \left(\frac{2^q(2^N-1)}{qh}\right)^{\frac{1}{q}}\right\} < \gamma < d(x_0, \partial\Omega) \le D$$

from which

$$\max\left\{\frac{2^p(2^N-1)}{p\gamma^p}, \frac{2^q(2^N-1)}{q\gamma^q}\right\} < h$$

Now, fix $n \in \mathbb{N}$ and consider the functions $u_n \in W_0^{1,p}(\Omega)$ and $v_n \in W_0^{1,q}(\Omega)$ defined by setting

$$u_n(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0, \gamma) \\ \xi_n & \text{if } x \in B(x_0, \frac{\gamma}{2}) \\ \frac{2\xi_n}{\gamma}(\gamma - |x - x_0|_N) & \text{if } x \in B(x_0, \gamma) \setminus B(x_0, \frac{\gamma}{2}) \end{cases}.$$

$$v_n(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0, \gamma) \\ \eta_n & \text{if } x \in B(x_0, \frac{\gamma}{2}) \\ \frac{2\eta_n}{\gamma}(\gamma - |x - x_0|_N) & \text{if } x \in B(x_0, \gamma) \setminus B(x_0, \frac{\gamma}{2}) \end{cases}.$$

Obviously

$$\Psi(u_n, v_n) < (|\xi_n|^p + |\eta_n|^q)\frac{h\omega\gamma^N}{2^N}$$

thus $\Psi(u_n, v_n) < s_n$ if $n \in \mathbb{N}$ is large enough.

Moreover we have, for $n \in \mathbb{N}$ large enough,

$$\Phi(u_n, v_n) - \inf_{\Psi^{-1}(]-\infty, s_n])} \Phi < s_n - \Psi(u_n, v_n)$$

Bearing in mind that $\lim_{n\to\infty} s_n = 0$, the previous inequality assures that the conclusion (b) of theorem 1 can be used and either there exists a global minimum of Ψ which is a local minimum of $\Phi + \Psi$ or there exists a sequence $\{(u_n, v_n)\}_{n\in\mathbb{N}}$ of pairwise distinct weak solutions of Problem (P) such that $\lim_{n\to\infty} \|(u_n, v_n)\|_X = 0$.

The other step is to verify that $(0, 0)$ is not a local minimum of $\Phi + \Psi$. Taking into account

$$h < \limsup_{(\rho,\sigma)\to(0,0)} \frac{G(\rho, \sigma)}{|\rho|^p + |\sigma|^q} < +\infty$$

one has, for each $n \in \mathbb{N}$

$$h < \inf_{n\in\mathbb{N}} \left(\sup_{\sqrt{\rho^2+\sigma^2} \le \frac{1}{n}} \frac{G(\rho, \sigma)}{|\rho|^p + |\sigma|^q} \right)$$

and so there exists $(\rho_n, \sigma_n) \in \mathbb{R}^2$ such that $\sqrt{\rho_n^2 + \sigma_n^2} \le \frac{1}{n}$ and

$$\frac{G(\rho_n, \sigma_n)}{|\rho_n|^p + |\sigma_n|^q} > h$$

that is $G(\rho_n, \sigma_n) > (|\rho_n|^p + |\sigma_n|^q)h$.

Now if we consider functions $w_n \in W_0^{1,p}(\Omega)$, $z_n \in W_0^{1,q}(\Omega)$ defined by setting

$$w_n(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0, \gamma) \\ \rho_n & \text{if } x \in B(x_0, \frac{\gamma}{2}) \\ \frac{2\rho_n}{\gamma}(\gamma - |x - x_0|_N) & \text{if } x \in B(x_0, \gamma) \setminus B(x_0, \frac{\gamma}{2}) \end{cases} .$$

$$z_n(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0, \gamma) \\ \sigma_n & \text{if } x \in B(x_0, \frac{\gamma}{2}) \\ \frac{2\sigma_n}{\gamma}(\gamma - |x - x_0|_N) & \text{if } x \in B(x_0, \gamma) \setminus B(x_0, \frac{\gamma}{2}) \end{cases} .$$

one has

$$(\Phi + \Psi)(w_n, z_n) < \frac{\omega\gamma^N}{2^N}\Big\{\Big(\frac{2^p(2^N - 1)}{p\gamma^p} - h\Big)|\rho_n|^p +$$

$$+\Big(\frac{2^q(2^N - 1)}{q\gamma^q} - h\Big)|\sigma_n|^q\Big\} < 0$$

The sequence (w_n, z_n) strongly converges to θ_X in X and $\Phi(w_n, z_n) + \Psi(w_n, z_n) < 0$ for all $n \in \mathbb{N}$. Since $\Phi(\theta_X) + \Psi(\theta_X) = 0$, this means that θ_X is not a local minimum of $\Phi + \Psi$. Then, since θ_X is the only global minimum of Ψ, the part (b) of theorem 1 ensures that there is a sequence $\{(u_n, v_n)\}_{n\in\mathbb{N}} \subseteq X$ of critical points of $\Phi+\Psi$ such that $\lim_{n\to\infty} \|(u_n, v_n)\|_X = 0$ and this completes the proof. □

We point out that, as a consequence of the former theorem, the following corollary holds:

Theorem 5. *Assume that* $\inf_{\mathbb{R}^2} G \geq 0$. *Moreover, suppose that there exist two real sequences* $\{a_n\}$ *and* $\{b_n\}$ *in* $]0, +\infty[$ *with* $a_n < b_n$, $\lim_{n\to\infty} b_n = 0$, *such that*

$$\lim_{n\to+\infty} \frac{b_n}{a_n} = +\infty$$

$$\max_{S(a_n)} G = \max_{S(b_n)} G > 0$$

$$\max\Big\{\frac{2^p(2^N - 1)}{pD^p}, \frac{2^q(2^N - 1)}{qD^q}\Big\} < \mu \limsup_{(\xi,\eta)\to(0,0)} \frac{G(\xi, \eta)}{|\xi|^p + |\eta|^q} < +\infty$$

where $D = \sup_{x\in\Omega} d(x, \partial\Omega)$ *and* $\mu > 0$. *Then the problem*

$$\begin{cases} -\Delta_p u = \mu f(u, v) \text{ in } \Omega \\ -\Delta_q v = \mu g(u, v) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \\ v = 0 \text{ on } \partial\Omega \end{cases} .$$

admits a sequence of non-zero weak solutions which strongly converges to θ_X in X.

3. Examples.

Let A be a positive number such that

$$A > \max\left\{\frac{2^p(2^N-1)}{pD^p}, \frac{2^q(2^N-1)}{qD^q}\right\}$$

Let $b_0 = 0$. The sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ with $a_n = (n+1)!$ and $b_n = (n+1)(n+1)!$ satisfy the hypotheses of Theorem 2 and besides $b_{n-1} < a_n$ for all $n \in \mathbb{N}$.

Moreover the sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ with $a_n = \frac{1}{(n+1)(n+1)!}$ and $b_n = \frac{1}{(n+1)!}$ satisfy the hypotheses of Theorem 4 and besides $b_{n+1} < a_n$ for all $n \in \mathbb{N}$.

Here is an example of application of Theorem 2 :

Example 1. Let $b_0 = 0$ and let $\{a_n\}$ and $\{b_n\}$ be two sequences satisfying the hypotheses of Theorem 2 and such that $b_{n-1} < a_n$ for all $n \in \mathbb{N}$. Let us consider the countable family of pairwise disjoint closed bounded intervals

$$\{[b_{n-1}, a_n]\}_{n\in\mathbb{N}}$$

Then for each $n \in \mathbb{N}$ the function

$$t \to \frac{2\pi t - (a_n + b_{n-1})\pi}{a_n - b_{n-1}}$$

is an homeomorphism between the interval $[b_{n-1}, a_n]$ and the interval $[-\pi, \pi]$. For each $n \in \mathbb{N}$ the function

$$f_n(t) = \frac{1}{2}\left\{1 + \cos\left(\frac{2\pi t - (a_n + b_{n-1})\pi}{a_n - b_{n-1}}\right)\right\}$$

satisfies

- $f_n \in C^1([b_{n-1}, a_n])$
- $0 \le f_n(t) \le 1$
- $f_n(b_{n-1}) = f_n(a_n) = 0$
- $f_n'(b_{n-1}) = f_n'(a_n) = 0$

For each $n \in \mathbb{N}$ let

$$\alpha_n(t) = \begin{cases} f_n(t) & \text{if } t \in [b_{n-1}, a_n] \\ 0 & \text{if } t \notin [b_{n-1}, a_n] \end{cases} .$$

and let

$$\alpha(t) = \sum_{n=1}^{\infty} \alpha_n(t) \text{ for each } t \in \mathbb{R}$$

Let $t_0 \in \mathbb{R}$. If there exist $n \in \mathbb{N}$ such that $t_0 \in [b_{n-1}, a_n]$ then $\alpha(t_0) = \alpha_n(t_0) = f_n(t_0)$. Else, if $t_0 \in \mathbb{R} \setminus \cup_{n=1}^{\infty}[b_{n-1}, a_n]$ then $\alpha_n(t_0) = 0$ for each $n \in \mathbb{N}$ so $\alpha(t_0) = 0$.

The function $G : \mathbb{R}^2 \to \mathbb{R}$

$$G(\xi, \eta) = A(|\xi|^p + |\eta|^q)\alpha(|\xi|^p + |\eta|^q)$$

satisfies the hypotheses of Theorem 2. Infact $G(\xi, \eta) \geq 0$ for each $(\xi, \eta) \in \mathbb{R}^2$ and $G(0, 0) = 0$. Moreover $A > 0$, $|\xi|^p + |\eta|^q \geq 0$ and $\alpha(t) \geq 0$ for each $t \in \mathbb{R}$ therefore

$$\inf_{\mathbb{R}^2} G \geq 0$$

From $a_n \leq t \leq b_n$ it follows that $\alpha(t) = 0$, so if $a_n \leq |\xi|^p + |\eta|^q \leq b_n$ then $G(\xi, \eta) = 0$, whence

$$\max_{S(a_n)} G = \max_{S(b_n)} G$$

Finally

$$\limsup_{(\xi,\eta)\to\infty} \frac{G(\xi, \eta)}{|\xi|^p + |\eta|^q} = A$$

Here is an example of application of Theorem 4 :

Example 2. Let $\{a_n\}$ and $\{b_n\}$ be two sequences satisfying the hypotheses of Theorem 4 and such that $b_{n+1} < a_n$ for all $n \in \mathbb{N}$. Let us consider the countable family of pairwise disjoint closed bounded intervals

$$\{[b_{n+1}, a_n]\}_{n\in\mathbb{N}}$$

Then for each $n \in \mathbb{N}$ the function

$$t \to \frac{2\pi t - (a_n + b_{n+1})\pi}{a_n - b_{n+1}}$$

is an homeomorphism between the interval $[b_{n+1}, a_n]$ and the interval $[-\pi, \pi]$.
For each $n \in \mathbb{N}$ the function

$$f_n(t) = \frac{1}{2}\Big\{1 + \cos\Big(\frac{2\pi t - (a_n + b_{n+1})\pi}{a_n - b_{n+1}}\Big)\Big\}$$

satisfies

- $f_n \in C^1([b_{n+1}, a_n])$
- $0 \le f_n(t) \le 1$
- $f_n(b_{n+1}) = f_n(a_n) = 0$
- $f_n'(b_{n+1}) = f_n'(a_n) = 0$

For each $n \in \mathbb{N}$ let

$$\alpha_n(t) = \begin{cases} f_n(t) & \text{if } t \in [b_{n+1}, a_n] \\ 0 & \text{if } t \notin [b_{n+1}, a_n] \end{cases}.$$

and let

$$\alpha(t) = \sum_{n=1}^{\infty} \alpha_n(t) \text{ for each } t \in \mathbb{R}$$

Let $t_0 \in \mathbb{R}$. If there exist $n \in \mathbb{N}$ such that $t_0 \in [b_{n+1}, a_n]$ then $\alpha(t_0) = \alpha_n(t_0) = f_n(t_0)$. Else, if $t_0 \in \mathbb{R} \setminus \cup_{n=1}^{\infty}[b_{n-1}, a_n]$ then $\alpha_n(t_0) = 0$ for each $n \in \mathbb{N}$ so $\alpha(t_0) = 0$.

It is easy to see that the function $G : \mathbb{R}^2 \to \mathbb{R}$

$$G(\xi, \eta) = A(|\xi|^p + |\eta|^q)\alpha(|\xi|^p + |\eta|^q)$$

satisfies the hypotheses of Theorem 4.

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