NEF VECTOR BUNDLES ON A PROJECTIVE SPACE WITH FIRST CHERN CLASS 3 AND SECOND CHERN CLASS 8

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We describe nef vector bundles on a projective space with first Chern class three and second Chern class eight over an algebraically closed field of characteristic zero by giving them a minimal resolution in terms of a full strong exceptional collection of line bundles.

1. Introduction

This paper is a continuation of [Ohn16]. Throughout this paper, as in [Ohn16], we work over an algebraically closed field K of characteristic zero. Let \mathcal{E} be a nef vector bundle of rank r on a projective space \mathbb{P}^n with first Chern class c_1 and second Chern class c_2 . In [Ohn16, Theorem 1.1], we classified such \mathcal{E} 's in case $c_1=3$ and $c_2<8$, and in [Ohn16, Proposition 1.2], we also gave an example of such \mathcal{E} 's on a projective plane with $c_1=3$ and $c_2=8$. In this paper, we complete the classification of such \mathcal{E} 's with $c_1=3$ and $c_2=8$ by giving them a minimal resolution in terms of a full strong exceptional collection of line bundles. The precise statement is as follows.

Theorem 1.1. Let \mathcal{E} be as above. Suppose that $c_1 = 3$ and that $c_2 = 8$. Then n = 2 and \mathcal{E} fits in an exact sequence

$$0 \to \mathcal{O}(-2)^{\oplus 2} \to \mathcal{O}^{\oplus r+1} \oplus \mathcal{O}(-1) \to \mathcal{E} \to 0.$$

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This implies that the example given in [Ohn16, Proposition 1.2] is nothing but the unique type of nef vector bundles with $c_1 = 3$ and $c_2 = 8$.

Note that, for a nef vector bundle \mathcal{E} with $c_1=3$, the anti-canonical bundle on $\mathbb{P}(\mathcal{E})$ is ample if $n\geq 3$ and nef if $n\geq 2$. Moreover, if n=2, it is big if and only if $c_2\leq 8$. So we can say that we have classified, except for the case (11) of [Ohn16, Theorem 1.1], weak Fano manifolds of the form $\mathbb{P}(\mathcal{E})$ where \mathcal{E} is a vector bundle on a projective space \mathbb{P}^n under the assumption that \mathcal{E} is nef and $c_1=3$. Recall here that a projective manifold M is called weak Fano if its anti-canonical bundle is nef and big, and that a vector bundle \mathcal{F} is called a weak Fano bundle if $\mathbb{P}(\mathcal{F})$ is a weak Fano manifold. We hope that the theorem above together with [Ohn16, Theorem 1.1] would be useful for some part of the classification of weak Fano bundles.

This paper is organized as follows. We first concentrate our attention to the case n=2. In § 2, we recall and summarize results obtained in [Ohn16] by taking into account that we only consider nef vector bundles with $c_1=3$ and $c_2=8$. In § 3, we show that $\mathcal E$ does not contain $\mathcal O(1)$ as a subsheaf. In § 4, we first observe that $\mathcal E$ must fit in the exact sequence given in [Ohn16, Proposition 1.2] and then show that $\mathcal E$ fits in the exact sequence in the theorem above. Finally, in § 5, we show that the case $n \geq 3$ does not happen.

1.1. Notation and conventions

Basically we follow the standard notation and terminology in algebraic geometry. For a vector bundle \mathcal{E} , $\mathbb{P}(\mathcal{E})$ denotes $\operatorname{Proj} S(\mathcal{E})$, where $S(\mathcal{E})$ denotes the symmetric algebra of \mathcal{E} . For a coherent sheaf \mathcal{F} on a smooth projective variety X, we denote by $c_i(\mathcal{F})$ the i-th Chern class of \mathcal{F} . For coherent sheaves \mathcal{F} and \mathcal{G} on X, $h^q(\mathcal{F})$ denotes $\dim H^q(\mathcal{F})$. Finally we refer to [Laz04] for the definition and basic properties of nef vector bundles.

2. Set-up for the two-dimensional case

In the following, let \mathcal{E} be a nef vector bundle on a projective space \mathbb{P}^n with $c_1 = 3$ and $c_2 = 8$. In this section, we assume that n = 2. It follows from [Ohn16, (3.10), (3.11) and (3.12)] that

$$h^1(\mathcal{E}(-2)) = 5, (1)$$

$$h^0(\mathcal{E}(-1)) - h^1(\mathcal{E}(-1)) = -2,$$
 (2)

$$h^0(\mathcal{E}) = r + 1. \tag{3}$$

Note here that, for a nef vector bundle \mathcal{E}' in general, unlike the case of globally generated vector bundles, an inequality $h^0(\mathcal{E}') \ge r - 1$ does not necessarily

imply that \mathcal{E}' fits in an exact sequence of the form

$$0 \to \mathcal{O}^{\oplus r-1} \to \mathcal{E}' \to \mathcal{I}_Z \otimes \det \mathcal{E}' \to 0$$

for some closed subscheme Z of \mathbb{P}^2 , where \mathcal{I}_Z denotes the ideal sheaf of Z (see [Ohn16, §13] for some examples). Set

$$e_{0,1} = h^0(\mathcal{E}(-1)).$$

Then

$$h^1(\mathcal{E}(-1)) = e_{0,1} + 2 \ge 2.$$

It follows from [Ohn16, (3.13)] that $5 \ge h^1(\mathcal{E}(-1))$. Therefore

$$0 \le e_{0,1} \le 3$$
.

We apply to \mathcal{E} the Bondal spectral sequence [OT14, Theorem 1]

$$E_2^{p,q} = \mathcal{T}or_{-p}^A(\operatorname{Ext}^q(G,\mathcal{E}),G) \Rightarrow E^{p+q} = \begin{cases} \mathcal{E} & \text{if} \quad p+q=0\\ 0 & \text{if} \quad p+q\neq 0. \end{cases} \tag{4}$$

As we have seen in [Ohn16, §3.1 and Lemma 5.1], $E_2^{p,q}$ vanishes unless (p,q)=(-2,1), (-1,1) or (0,0), and $E_2^{-2,1}$ and $E_2^{-1,1}$ fit in an exact sequence of coherent sheaves

$$0 \to E_2^{-2,1} \to \mathcal{O}(-3) \xrightarrow{\nu_2} \Omega_{\mathbb{P}^2}(1)^{\oplus e_{0,1}} \to E_2^{-1,1} \to k(w) \to 0 \tag{5}$$

for some point w in \mathbb{P}^2 , where k(w) denotes the residue field of w. Note that this exact sequence is a consequence of the vanishing $H^1(\mathcal{E}) = 0$, and recall that $H^1(\mathcal{E})$ vanishes by the Kawamata-Viehweg vanishing theorem since $c_2 < 9$. Moreover we have the following exact sequences

$$0 \to E_2^{-2,1} \to E_2^{0,0} \to E_3^{0,0} \to 0, \tag{6}$$

$$0 \to E_3^{0,0} \to \mathcal{E} \to E_2^{-1,1} \to 0,$$
 (7)

$$0 \to \mathcal{O}^{\oplus 3e_{0,1}} \to \mathcal{O}(1)^{\oplus e_{0,1}} \oplus \mathcal{O}^{\oplus r+1} \to E_2^{0,0} \to 0. \tag{8}$$

We shall divide the proof according to the value of $e_{0,1}$.

3. The case n = 2 and $e_{0.1} > 0$

Suppose that n=2 and $e_{0,1}>0$. Since $e_{0,1}>0$ and $h^0(\mathcal{E}(-2))=0$ by the argument in [Ohn16, §3], we have an exact sequence

$$0 \to \mathcal{O}(1) \to \mathcal{E} \to \mathcal{F} \to 0$$

where \mathcal{F} is a torsion-free sheaf with $c_1(\mathcal{F})=2$, $c_2(\mathcal{F})=6$ and $h^0(\mathcal{F}(-1))=e_{0,1}-1$. Denote by $\mathcal{F}^{\vee\vee}$ the double dual of \mathcal{F} , and consider the quotient \mathcal{Q} of the inclusion $\mathcal{F}\subset\mathcal{F}^{\vee\vee}$:

$$0 \to \mathcal{F} \to \mathcal{F}^{\vee\vee} \to \mathcal{Q} \to 0.$$

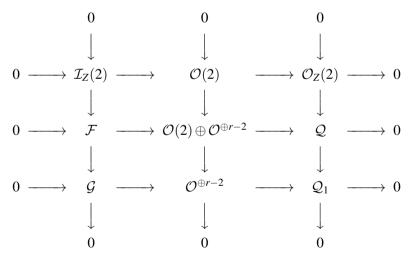
The support of \mathcal{Q} has dimension zero, and its length is equal to $-c_2(\mathcal{Q})$. By [Ohn16, Lemma 12.1], $\mathcal{F}^{\vee\vee}$ is a nef vector bundle of rank r-1 with $c_1(\mathcal{F}^{\vee\vee})=2$, $c_2(\mathcal{F}^{\vee\vee})=6+c_2(\mathcal{Q})$ and $h^0(\mathcal{F}^{\vee\vee}(-1))\geq e_{0,1}-1$.

3.1. The case $e_{0,1} > 1$

Suppose that $e_{0,1} > 1$. Then it follows from [Ohn14, Theorem 6.5] that $\mathcal{F}^{\vee\vee}$ is isomorphic to either $\mathcal{O}(2) \oplus \mathcal{O}^{\oplus r-2}$ or $\mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}^{\oplus r-3}$, or $\mathcal{F}^{\vee\vee}$ fits in an exact sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r-1} \to \mathcal{F}^{\vee\vee} \to 0. \tag{9}$$

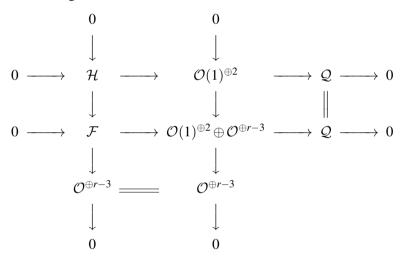
Suppose that $\mathcal{F}^{\vee\vee}\cong\mathcal{O}(2)\oplus\mathcal{O}^{\oplus r-2}$. Since $c_2(\mathcal{F}^{\vee\vee})=0$, the length of \mathcal{Q} is 6. Let \mathcal{G} be the image of the composite of the inclusion $\mathcal{F}\to\mathcal{O}(2)\oplus\mathcal{O}^{\oplus r-2}$ and the projection $\mathcal{O}(2)\oplus\mathcal{O}^{\oplus r-2}\to\mathcal{O}^{\oplus r-2}$. Note that the kernel of the surjection $\mathcal{F}\to\mathcal{G}$ is a subsheaf of $\mathcal{O}(2)$. Hence it can be written as $\mathcal{I}_Z(2)$ where \mathcal{I}_Z is the ideal sheaf of some closed subscheme Z of \mathbb{P}^2 . Now we have the following commutative diagram with exact lows and columns



where \mathcal{Q}_1 is defined by the diagram above. Since $\mathcal{O}_Z(2) \to \mathcal{Q}$ is injective, we see that $\dim Z \leq 0$, and thus $\mathcal{O}_Z(2) \cong \mathcal{O}_Z$. If $\mathcal{Q}_1 \neq 0$, then take a line L intersecting with the support of \mathcal{Q}_1 . Then the kernel of the surjection $\mathcal{O}_L^{\oplus r-1} \to \mathcal{Q}_1|_L$ has a negative degree line bundle as a direct summand, which implies that some negative degree line bundle is a quotient of $\mathcal{G}|_L$, $\mathcal{F}|_L$ and $\mathcal{E}|_L$. This contradicts

that \mathcal{E} is nef. Hence $\mathcal{Q}_1=0$. Thus $\mathcal{G}\cong\mathcal{O}^{\oplus r-2}$, $\mathcal{O}_Z\cong\mathcal{Q}$, and \mathcal{O}_Z has length 6. Since $h^0(\mathcal{G}(-1))=0$, we infer that $h^0(\mathcal{I}_Z(1))=e_{0,1}-1>0$. Hence there exists a line L passing through Z. Since length $\mathcal{O}_Z=6$, this implies that the kernel of the restriction $\mathcal{O}_L(2)\to\mathcal{O}_Z$ to the line L of the surjection $\mathcal{O}(2)\to\mathcal{O}_Z$ is isomorphic to $\mathcal{O}_L(-4)$. By restricting the diagram above to the line L, we see that $\mathcal{F}|_L$ has a negative degree line bundle as a quotient; this is a contradiction. Hence $\mathcal{F}^{\vee\vee}$ cannot be isomorphic to $\mathcal{O}(2)\oplus\mathcal{O}^{\oplus r-2}$.

Suppose that $\mathcal{F}^{\vee\vee}\cong\mathcal{O}(1)^{\oplus 2}\oplus\mathcal{O}^{\oplus r-2}$. Since $c_2(\mathcal{F}^{\vee\vee})=1$, the length of \mathcal{Q} is 5. Let \mathcal{G} be the image of the composite of the inclusion $\mathcal{F}\to\mathcal{O}(1)^{\oplus 2}\oplus\mathcal{O}^{\oplus r-3}$ and the projection $\mathcal{O}(1)^{\oplus 2}\oplus\mathcal{O}^{\oplus r-3}\to\mathcal{O}^{\oplus r-3}$, and \mathcal{Q}_1 the cokernel of the inclusion $\mathcal{G}\to\mathcal{O}^{\oplus r-3}$. Then there exists a surjection $\mathcal{Q}\to\mathcal{Q}_1$, and thus the support of \mathcal{Q}_1 has dimension ≤ 0 . If $\mathcal{Q}_1\neq 0$, we get a contradiction by the same argument as above. Therefore we may assume that $\mathcal{Q}_1=0$; thus $\mathcal{G}\cong\mathcal{O}^{\oplus r-3}$. Let \mathcal{H} be the kernel of the surjection $\mathcal{F}\to\mathcal{O}^{\oplus r-3}$. Then we have the following commutative diagram with exact lows and columns.

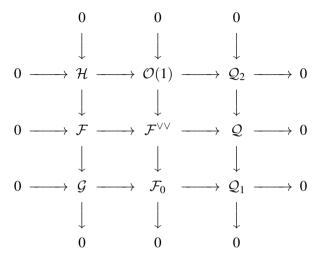


Since $h^0(\mathcal{O}^{\oplus r-3}(-1))=0$, we infer that $h^0(\mathcal{H}(-1))=e_{0,1}-1>0$. Since $\mathcal{H}(-1)$ is a subsheaf of $\mathcal{O}^{\oplus 2}$, this implies that $\mathcal{H}(-1)\cong\mathcal{I}_Z\oplus\mathcal{O}$ and $\mathcal{Q}(-1)\cong\mathcal{O}_Z$ for some 0-dimensional closed subscheme Z of length S in \mathbb{P}^2 . Now take a line S that intersect with S in length S in the kernel of S in S in the form S in length S in the kernel of S in the form S in

Suppose that $\mathcal{F}^{\vee\vee}$ fits in the exact sequence (9). Since $c_2(\mathcal{F}^{\vee\vee})=2$, the length of \mathcal{Q} is 4. Define a torsion-free sheaf \mathcal{F}_0 as a quotient of $\mathcal{F}^{\vee\vee}$ by an injection $\mathcal{O}(1) \to \mathcal{F}^{\vee\vee}$. Then \mathcal{F}_0 fits in an exact sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O}^{\oplus r-1} \to \mathcal{F}_0 \to 0.$$

Let $\mathcal G$ be the image of the composite of the inclusion $\mathcal F\to\mathcal F^{\vee\vee}$ and the projection $\mathcal F^{\vee\vee}\to\mathcal F_0$. Since $h^0(\mathcal F_0(-1))=0$, we see that $h^0(\mathcal G(-1))=0$. Let $\mathcal H$ be the kernel of the surjection $\mathcal F\to\mathcal G$. Then we have the following commutative diagram with exact lows and columns



where \mathcal{Q}_1 and \mathcal{Q}_2 are defined by the diagram above. Since $h^0(\mathcal{G}(-1))=0$, we see that $h^0(\mathcal{H}(-1))=e_{0,1}-1>0$. Since $\mathcal{H}(-1)$ is a subsheaf of \mathcal{O} , this implies that $\mathcal{H}(-1)$ is \mathcal{O} itself; thus $\mathcal{Q}_2=0$, $\mathcal{Q}\cong\mathcal{Q}_1$ and \mathcal{Q}_1 has length 4. As we have seen in the proof of [Ohn14, Theorem 6.4], \mathcal{F}_0 is locally free outside at most one point, and if \mathcal{F}_0 is not locally free at a point z, then \mathcal{F}_0 is isomorphic to $\mathfrak{m}_z(1)\oplus\mathcal{O}^{\oplus r-3}$, where \mathfrak{m}_z is the ideal sheaf of z, since n=2. Suppose that \mathcal{F}_0 is not locally free. Then take a line L passing through z and meeting the support of \mathcal{Q}_1 . We see that the surjection $\mathcal{F}_0\to\mathcal{Q}_1$ induces a surjection $\mathcal{O}_L^{\oplus r-2}\to\mathcal{Q}_1|_L$, whose kernel has a negative degree line bundle as a quotient, and thus so does $\mathcal{G}|_L$, $\mathcal{F}|_L$ and $\mathcal{E}|_L$. This is a contradiction. Suppose that \mathcal{F}_0 is locally free. Then take a line L which intersects with \mathcal{Q}_1 in length $l\geq 2$. Since $\mathcal{F}_0|_L\cong\mathcal{O}_L(1)\oplus\mathcal{O}^{\oplus r-3}$, we see that $\mathcal{G}|_L$ admits a negative degree line bundle as a quotient; this is a contradiction. Hence $\mathcal{F}^{\vee\vee}$ cannot fit in the exact sequence (9).

Therefore we conclude that the case $e_{0,1} > 1$ does not happen.

3.2. The case $e_{0,1} = 1$

Suppose that $e_{0,1}=1$. If the morphism v_2 in (5) is zero, then $E_2^{-1,1}|_L\cong\Omega_{\mathbb{P}^2}(1)|_L\cong\mathcal{O}_L(-1)\oplus\mathcal{O}_L$ for a line L not containing w. By (7), this implies that $\mathcal{E}|_L$ has $\mathcal{O}_L(-1)$ as a quotient; this is a contradiction. Hence $v_2\neq 0$, and thus $E_2^{-2,1}=0$, $E_2^{0,0}\cong E_3^{0,0}$ by (6), and $E_2^{-1,1}$ fits in an exact sequence

$$0 \to \mathcal{O}(-3) \xrightarrow{\nu_2} \Omega_{\mathbb{P}^2}(1) \to E_2^{-1,1} \to k(w) \to 0. \tag{10}$$

We see that $E_2^{-1,1}$ is a coherent sheaf of rank one. Since $E_3^{0,0}$ is torsion-free by (7), so is $E_2^{0,0}$, and thus $E_2^{0,0}$ has $\mathcal{O}(1)$ as a subsheaf and consequently is isomorphic to $\mathcal{O}(1)\oplus\mathcal{O}^{\oplus r-2}$ by (8). Hence the exact sequence (7) becomes an exact sequence

$$0 \to \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r-2} \xrightarrow{\varphi} \mathcal{E} \to E_2^{-1,1} \to 0.$$

By taking the dual of φ and (r-1)-th wedge product of the dual, we obtain a morphism $\wedge^{r-1}\mathcal{E}^\vee \to \mathcal{O}(-1)$. Let $\mathcal{I}_Z(-1)$ be the image of this morphism, where \mathcal{I}_Z is the ideal sheaf of a closed subscheme Z of \mathbb{P}^2 of dimension ≤ 1 . Note that Z is the degeneracy locus of φ and that if we denote by ψ the induced surjection $\mathcal{E} \cong \wedge^{r-1}\mathcal{E}^\vee \otimes \det \mathcal{E} \to \mathcal{I}_Z(-1) \otimes \det \mathcal{E} \cong \mathcal{I}_Z(2)$ then $\psi \circ \varphi = 0$.

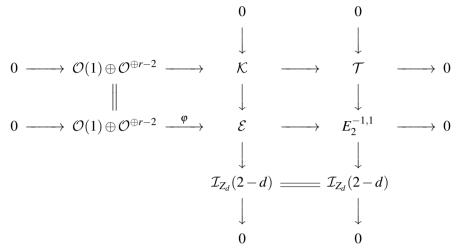
Suppose that the degeneracy locus Z of φ has codimension ≥ 2 . Then $E_2^{-1,1}$ is torsion-free. This implies that $E_2^{-1,1} \cong \mathcal{I}_Z(2)$ and that \mathcal{E} fits in an exact sequence

$$0 \to \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r-2} \xrightarrow{\varphi} \mathcal{E} \to \mathcal{I}_Z(2) \to 0.$$

Note that length Z=6. Since $\mathcal E$ is nef, length $(Z\cap L)\leq 2$ for any line L in $\mathbb P^2$; let us call this the basic property of Z. Let p be any point in Z. We may assume that Z is in an affine open subscheme $\operatorname{Spec} K[x,y]$ and that p=(0,0). The local ring $\mathcal O_{Z,p}$ can be written as A/I, where $A=\hat{\mathcal O}_{\mathbb P^2,p}=K[[x,y]]$ and I the ideal of Z in the local ring A. Observe here that if $\operatorname{length}(A/I)\leq 4$ and thus the support of Z contains another point $q\neq p$, then the basic property of Z implies $I\not\subseteq \mathfrak m^2$, where $\mathfrak m$ denotes the maximal ideal of A. Based on this observation, we can deduce from the basic property of Z that $I\not\subseteq \mathfrak m^2$ without any assumption on $\operatorname{length}(A/I)$. Now that Z is curvilinear, after changing coordinates (x,y) if necessary, we may assume that $I=\langle y-\varphi(x),x^l\rangle$, where $\varphi(x)=a_2x^2+a_3x^3+\cdots\in K[[x]]$ $(a_2\neq 0)$ and $l=\operatorname{length}(A/I)$. Local computation then shows that there exists a smooth conic C such that $\operatorname{length}(Z\cap C)\geq 5$; e.g., if $l\geq 3$, we can take a defining equation of C to be $y=a_2x^2+dxy+ey^2$ for some $d,e\in K$. However this again contradicts that $\mathcal E$ is nef. Therefore this case cannot happen.

Suppose that $\dim Z = 1$. Then the ideal sheaf \mathcal{I}_Z of Z is decomposed as $\mathcal{I}_Z \cong \mathcal{I}_{Z_d}(-d)$, where d is the degree of the divisor contained in Z and \mathcal{I}_{Z_d} is the ideal sheaf of a 0-dimensional closed subscheme Z_d of \mathbb{P}^2 . Consider the

following commutative diagram with exact lows and columns



where \mathcal{K} and \mathcal{T} are defined by the diagram above. We see that \mathcal{K} is a coherent sheaf of rank r-1 and thus \mathcal{T} is the torsion subsheaf of $E_2^{-1,1}$, and that $\operatorname{Supp} Z = \operatorname{Supp} \mathcal{T} \cup \operatorname{Supp} Z_d$. Hence $E_2^{-1,1}$ has an associated point of codimension one. Now recall the exact sequence (10) and split this sequence into the following two exact sequences of coherent sheaves

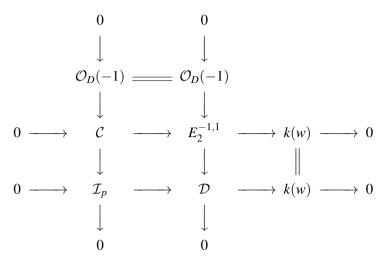
$$0 \to \mathcal{O}(-3) \xrightarrow{\nu_2} \Omega_{\mathbb{P}^2}(1) \to \mathcal{C} \to 0, \tag{11}$$

$$0 \to \mathcal{C} \to E_2^{-1,1} \to k(w) \to 0. \tag{12}$$

Note that C has an associated point of codimension one since so does $E_2^{-1,1}$. Hence v_2 passes through O(-1) or O(-2).

Suppose that v_2 passes through $\mathcal{O}(-1)$. Then we have the following commutative diagram with exact lows and columns

where \mathcal{I}_p is the ideal sheaf of a point p, and D is a conic in \mathbb{P}^2 . We also have the following commutative diagram with exact lows and columns



where \mathcal{D} is defined by the diagram above. Suppose that \mathcal{D} has an associated point other than the generic point. Then it must be w, and thus $\mathcal{D} \cong \mathcal{I}_w \oplus k(w)$, which also contradicts that \mathcal{E} is nef. Therefore \mathcal{D} is torsion-free. Since \mathcal{D} has rank one, $c_1(\mathcal{D})=0$ and $c_2(\mathcal{D})=0$, \mathcal{D} is isomorphic to its double dual $\mathcal{O}_{\mathbb{P}^2}$. Moreover we see that p=w, that $\mathcal{O}_D(-1)$ is the torsion subsheaf \mathcal{T} of $E_2^{-1,1}$, that $Z_d=\emptyset$, and that Z=D. If $h^0(E_2^{-1,1})\neq 0$, then $E_2^{-1,1}\cong \mathcal{O}_D(-1)\oplus \mathcal{O}_{\mathbb{P}^2}$, which contradicts that \mathcal{E} is nef. Hence $h^0(E_2^{-1,1})=0$. Since $h^1(\mathcal{O}_D(-1))=h^2(\mathcal{O}_{\mathbb{P}^2}(-3))=1$, this implies that $H^0(\mathcal{D})=H^0(\mathcal{O}_{\mathbb{P}^2})\cong H^1(\mathcal{O}_D(-1))$. Suppose that D is smooth. Consider the pull back $\mathcal{O}_D(-1)\to E_2^{-1,1}|_D\to \mathcal{O}_D\to 0$ of the exact sequence above. Note that $E_2^{-1,1}|_D$ has rank at least two since D is the degeneracy locus of φ . Hence we obtain an exact sequence

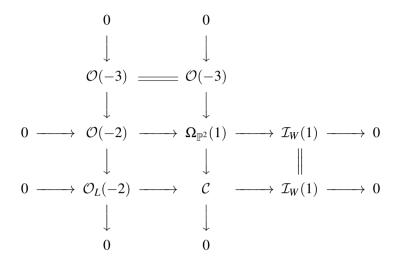
$$0 \to \mathcal{O}_D(-1) \to E_2^{-1,1}|_D \to \mathcal{O}_D \to 0.$$

Note that $D\cong \mathbb{P}^1$ and that $\mathcal{O}_D(-1)\cong \mathcal{O}_{\mathbb{P}^1}(-2)$ via this isomorphism. Since the sequence above does not split, this implies that $E_2^{-1,1}|_D\cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$, which contradicts that \mathcal{E} is nef. Suppose that D is a double line. Then $D_{\mathrm{red}}\cong \mathbb{P}^1$, and we have a surjection $\mathcal{O}_D(-1)\to \mathcal{O}_{D_{\mathrm{red}}}(-1)$. The similar argument as above shows that there exists an exact sequence

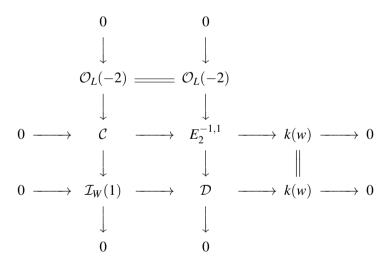
$$0 \to \mathcal{O}_{D_{\mathrm{red}}}(-1) \to E_2^{-1,1}|_{D_{\mathrm{red}}} \to \mathcal{O}_{D_{\mathrm{red}}} \to 0.$$

Hence $E_2^{-1,1}|_{D_{\text{red}}} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$; this contradicts that \mathcal{E} is nef. Suppose that D is a union of two distinct lines: $D = L_1 + L_2$. Then $E_2^{-1,1}|_{L_1} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$ by the similar argument as above, and hence this case does not occur either.

Suppose that v_2 passes through $\mathcal{O}(-2)$ and does not pass through $\mathcal{O}(-1)$. Then we have the following commutative diagram with exact lows and columns



where \mathcal{I}_W is the ideal sheaf of a 0-dimensional locally complete intersection W of length three, and L is a line in \mathbb{P}^2 . We also have the following commutative diagram with exact lows and columns



where \mathcal{D} is defined by the diagram above. If \mathcal{D} is not torsion-free, then $\mathcal{D} \cong \mathcal{I}_w \oplus k(w)$, which contradicts that \mathcal{E} is nef. Therefore \mathcal{D} is a torsion-free coherent sheaf of rank one with $c_1(\mathcal{D}) = 1$ and $c_2(\mathcal{D}) = 2$. Hence $\mathcal{O}_L(-2)$ is the torsion subsheaf \mathcal{T} of $E_2^{-1,1}$, and we infer that $\mathcal{D} \cong \mathcal{I}_{Z_1}(1)$ with length $Z_1 = 2$. This also contradicts that \mathcal{E} is nef.

Therefore we conclude that the case $e_{0,1} = 1$ does not happen.

4. The case n = 2 and $e_{0,1} = 0$

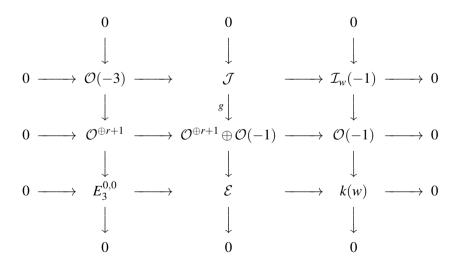
Suppose that $e_{0,1}=0$. Then $E_2^{-2,1}\cong \mathcal{O}(-3)$ and $E_2^{-1,1}\cong k(w)$ by (5), and $E_2^{0,0}\cong \mathcal{O}^{\oplus r+1}$ by (8). Thus we have the following two exact sequences by (6) and (7)

$$0 \to \mathcal{O}(-3) \to \mathcal{O}^{\oplus r+1} \to E_3^{0,0} \to 0, \tag{13}$$

$$0 \to E_3^{0,0} \to \mathcal{E} \to k(w) \to 0. \tag{14}$$

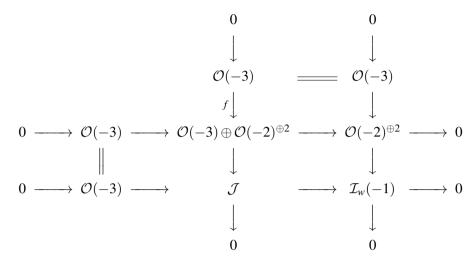
These two exact sequences show that \mathcal{E} must fit in the exact sequence given in [Ohn16, Proposition 1.2]. We shall show that \mathcal{E} has a resolution in terms of a full strong exceptional sequence of line bundles as in Theorem 1.1 in accordance with the framework given in [Ohn14].

Since $h^1(E_3^{0,0}(1)) = 0$, we have the following commutative diagram with exact rows and columns



where \mathcal{I}_w is the ideal sheaf of w, and \mathcal{J} and g are defined by the diagram above.

We also have the following commutative diagram with exact rows and columns



where f is defined by the diagram above. We claim here that the composite of f and the projection $\mathcal{O}(-3)\oplus\mathcal{O}(-2)^{\oplus 2}\to\mathcal{O}(-3)$ is non-zero. Suppose, to the contrary, that the composite is zero. Then $\mathcal{J}\cong\mathcal{O}(-3)\oplus\mathcal{I}_w(-1)$. By taking the double dual, the composite of the inclusion $\mathcal{I}_w(-1)\to\mathcal{J}$ and g extends to a splitting injection of the projection $\mathcal{O}^{\oplus r+1}\oplus\mathcal{O}(-1)\to\mathcal{O}(-1)$; we obtain the following commutative diagram with exact rows

Since the induced morphism $k(w) \to \mathcal{E}$ is a splitting injection of the surjection $\mathcal{E} \to k(w)$, we have an isomorphism $\mathcal{E} \cong E_3^{0,0} \oplus k(w)$, which is absurd. Hence the claim holds; thus $\mathcal{J} \cong \operatorname{Coker}(f) \cong \mathcal{O}(-2)^{\oplus 2}$. Therefore we obtain the desired exact sequence

$$0 \to \mathcal{O}(-2)^{\oplus 2} \to \mathcal{O}^{\oplus r+1} \oplus \mathcal{O}(-1) \to \mathcal{E} \to 0.$$

5. The case $n \ge 3$

In this section, we shall show that the case $n \ge 3$ does not happen. By considering the restriction $\mathcal{E}|_{L^3}$ to a 3-dimensional linear subspace $L^3 \subseteq \mathbb{P}^n$, we may assume that n = 3. We have

$$\chi(\mathcal{E}(-1)) = \frac{c_3}{2} - 10$$

by [Ohn16, (3.20)]. In particular, c_3 is even. We also have

$$c_3 > 21$$

by [Ohn16, (3.23)]. Since the equality in $c_3 \ge 21$ does not hold, we infer that $H(\mathcal{E})$ is big, and thus $h^q(\mathcal{E}(-1)) = 0$ for all q > 0 by [Ohn16, (3.3)]. Therefore $h^0(\mathcal{E}(-1)) \ge 1$. On the other hand, $H^0(\mathcal{E}(-2)) = 0$ by the argument in $[Ohn16, \S 3]$, and $h^0(\mathcal{E}|_H(-1)) = 0$ for any plane $H \subset \mathbb{P}^3$ as is shown in $\S 3$. Hence $h^0(\mathcal{E}(-1)) = 0$, which is a contradiction. Therefore the case $n \ge 3$ does not happen.

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