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NEF VECTOR BUNDLES ON A PROJECTIVE SPACE WITH FIRST CHERN CLASS 3 AND SECOND CHERN CLASS 8

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We describe nef vector bundles on a projective space with first Chern class three and second Chern class eight over an algebraically closed field of characteristic zero by giving them a minimal resolution in terms of a full strong exceptional collection of line bundles.

1. Introduction

This paper is a continuation of [Ohn16]. Throughout this paper, as in [Ohn16], we work over an algebraically closed field K of characteristic zero. Let $\mathcal{E}$ be a nef vector bundle of rank r on a projective space $\mathbb{P}^n$ with first Chern class c_1 and second Chern class c_2. In [Ohn16, Theorem 1.1], we classified such $\mathcal{E}$'s in case $c_1 = 3$ and $c_2 < 8$, and in [Ohn16, Proposition 1.2], we also gave an example of such $\mathcal{E}$'s on a projective plane with $c_1 = 3$ and $c_2 = 8$. In this paper, we complete the classification of such $\mathcal{E}$'s with $c_1 = 3$ and $c_2 = 8$ by giving them a minimal resolution in terms of a full strong exceptional collection of line bundles. The precise statement is as follows.

Theorem 1.1. *Let $\mathcal{E}$ be as above. Suppose that $c_1 = 3$ and that $c_2 = 8$. Then $n = 2$ and $\mathcal{E}$ fits in an exact sequence*

$$0 \to \mathcal{O}(-2)^{\oplus 2} \to \mathcal{O}^{\oplus r+1} \oplus \mathcal{O}(-1) \to \mathcal{E} \to 0.$$

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This implies that the example given in [Ohn16, Proposition 1.2] is nothing but the unique type of nef vector bundles with $c_1 = 3$ and $c_2 = 8$.

Note that, for a nef vector bundle $\mathcal{E}$ with $c_1 = 3$, the anti-canonical bundle on $\mathbb{P}(\mathcal{E})$ is ample if $n \geq 3$ and nef if $n \geq 2$. Moreover, if $n = 2$, it is big if and only if $c_2 \leq 8$. So we can say that we have classified, except for the case (11) of [Ohn16, Theorem 1.1], weak Fano manifolds of the form $\mathbb{P}(\mathcal{E})$ where $\mathcal{E}$ is a vector bundle on a projective space $\mathbb{P}^n$ under the assumption that $\mathcal{E}$ is nef and $c_1 = 3$. Recall here that a projective manifold M is called weak Fano if its anti-canonical bundle is nef and big, and that a vector bundle $\mathcal{F}$ is called a weak Fano bundle if $\mathbb{P}(\mathcal{F})$ is a weak Fano manifold. We hope that the theorem above together with [Ohn16, Theorem 1.1] would be useful for some part of the classification of weak Fano bundles.

This paper is organized as follows. We first concentrate our attention to the case $n = 2$. In § 2, we recall and summarize results obtained in [Ohn16] by taking into account that we only consider nef vector bundles with $c_1 = 3$ and $c_2 = 8$. In § 3, we show that $\mathcal{E}$ does not contain $\mathcal{O}(1)$ as a subsheaf. In § 4, we first observe that $\mathcal{E}$ must fit in the exact sequence given in [Ohn16, Proposition 1.2] and then show that $\mathcal{E}$ fits in the exact sequence in the theorem above. Finally, in § 5, we show that the case $n \geq 3$ does not happen.

1.1. Notation and conventions

Basically we follow the standard notation and terminology in algebraic geometry. For a vector bundle $\mathcal{E}$, $\mathbb{P}(\mathcal{E})$ denotes $\operatorname{Proj} S(\mathcal{E})$, where $S(\mathcal{E})$ denotes the symmetric algebra of $\mathcal{E}$. For a coherent sheaf $\mathcal{F}$ on a smooth projective variety X, we denote by $c_i(\mathcal{F})$ the i-th Chern class of $\mathcal{F}$. For coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on X, $h^q(\mathcal{F})$ denotes $\dim H^q(\mathcal{F})$. Finally we refer to [Laz04] for the definition and basic properties of nef vector bundles.

2. Set-up for the two-dimensional case

In the following, let $\mathcal{E}$ be a nef vector bundle on a projective space $\mathbb{P}^n$ with $c_1 = 3$ and $c_2 = 8$. In this section, we assume that $n = 2$. It follows from [Ohn16, (3.10), (3.11) and (3.12)] that

$$h^1(\mathcal{E}(-2)) = 5, \tag{1}$$

$$h^0(\mathcal{E}(-1)) - h^1(\mathcal{E}(-1)) = -2, \tag{2}$$

$$h^0(\mathcal{E}) = r + 1. \tag{3}$$

Note here that, for a nef vector bundle $\mathcal{E}'$ in general, unlike the case of globally generated vector bundles, an inequality $h^0(\mathcal{E}') \geq r - 1$ does not necessarily imply that $\mathcal{E}'$ fits in an exact sequence of the form

$$0 \to \mathcal{O}^{\oplus r-1} \to \mathcal{E}' \to \mathcal{I}_Z \otimes \det \mathcal{E}' \to 0$$

for some closed subscheme Z of $\mathbb{P}^2$, where $\mathcal{I}_Z$ denotes the ideal sheaf of Z (see [Ohn16, §13] for some examples). Set

$$e_{0,1} = h^0(\mathcal{E}(-1)).$$

Then

$$h^1(\mathcal{E}(-1)) = e_{0,1} + 2 \geq 2.$$

It follows from [Ohn16, (3.13)] that $5 \geq h^1(\mathcal{E}(-1))$. Therefore

$$0 \leq e_{0,1} \leq 3.$$

We apply to $\mathcal{E}$ the Bondal spectral sequence [OT14, Theorem 1]

$$E_2^{p,q} = \mathcal{T}or^A_{-p}(\mathrm{Ext}^q(G,\mathcal{E}),G) \Rightarrow E^{p+q} = \begin{cases} \mathcal{E} & \text{if} \quad p+q=0 \\ 0 & \text{if} \quad p+q \neq 0. \end{cases} \tag{4}$$

As we have seen in [Ohn16, §3.1 and Lemma 5.1], $E_2^{p,q}$ vanishes unless $(p,q) = (-2,1)$, $(-1,1)$ or $(0,0)$, and $E_2^{-2,1}$ and $E_2^{-1,1}$ fit in an exact sequence of coherent sheaves

$$0 \to E_2^{-2,1} \to \mathcal{O}(-3) \xrightarrow{v_2} \Omega_{\mathbb{P}^2}(1)^{\oplus e_{0,1}} \to E_2^{-1,1} \to k(w) \to 0 \tag{5}$$

for some point w in $\mathbb{P}^2$, where $k(w)$ denotes the residue field of w. Note that this exact sequence is a consequence of the vanishing $H^1(\mathcal{E}) = 0$, and recall that $H^1(\mathcal{E})$ vanishes by the Kawamata-Viehweg vanishing theorem since $c_2 < 9$. Moreover we have the following exact sequences

$$0 \to E_2^{-2,1} \to E_2^{0,0} \to E_3^{0,0} \to 0, \tag{6}$$

$$0 \to E_3^{0,0} \to \mathcal{E} \to E_2^{-1,1} \to 0, \tag{7}$$

$$0 \to \mathcal{O}^{\oplus 3e_{0,1}} \to \mathcal{O}(1)^{\oplus e_{0,1}} \oplus \mathcal{O}^{\oplus r+1} \to E_2^{0,0} \to 0. \tag{8}$$

We shall divide the proof according to the value of $e_{0,1}$.

3. The case $n = 2$ and $e_{0,1} > 0$

Suppose that $n = 2$ and $e_{0,1} > 0$. Since $e_{0,1} > 0$ and $h^0(\mathcal{E}(-2)) = 0$ by the argument in [Ohn16, §3], we have an exact sequence

$$0 \to \mathcal{O}(1) \to \mathcal{E} \to \mathcal{F} \to 0$$

where $\mathcal{F}$ is a torsion-free sheaf with $c_1(\mathcal{F}) = 2$, $c_2(\mathcal{F}) = 6$ and $h^0(\mathcal{F}(-1)) = e_{0,1} - 1$. Denote by $\mathcal{F}^{\vee\vee}$ the double dual of $\mathcal{F}$, and consider the quotient $\mathcal{Q}$ of the inclusion $\mathcal{F} \subset \mathcal{F}^{\vee\vee}$:

$$0 \to \mathcal{F} \to \mathcal{F}^{\vee\vee} \to \mathcal{Q} \to 0.$$

The support of $\mathcal{Q}$ has dimension zero, and its length is equal to $-c_2(\mathcal{Q})$. By [Ohn16, Lemma 12.1], $\mathcal{F}^{\vee\vee}$ is a nef vector bundle of rank $r-1$ with $c_1(\mathcal{F}^{\vee\vee}) = 2$, $c_2(\mathcal{F}^{\vee\vee}) = 6 + c_2(\mathcal{Q})$ and $h^0(\mathcal{F}^{\vee\vee}(-1)) \geq e_{0,1} - 1$.

3.1. The case $e_{0,1} > 1$

Suppose that $e_{0,1} > 1$. Then it follows from [Ohn14, Theorem 6.5] that $\mathcal{F}^{\vee\vee}$ is isomorphic to either $\mathcal{O}(2) \oplus \mathcal{O}^{\oplus r-2}$ or $\mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}^{\oplus r-3}$, or $\mathcal{F}^{\vee\vee}$ fits in an exact sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r-1} \to \mathcal{F}^{\vee\vee} \to 0. \tag{9}$$

Suppose that $\mathcal{F}^{\vee\vee} \cong \mathcal{O}(2) \oplus \mathcal{O}^{\oplus r-2}$. Since $c_2(\mathcal{F}^{\vee\vee}) = 0$, the length of $\mathcal{Q}$ is 6. Let $\mathcal{G}$ be the image of the composite of the inclusion $\mathcal{F} \to \mathcal{O}(2) \oplus \mathcal{O}^{\oplus r-2}$ and the projection $\mathcal{O}(2) \oplus \mathcal{O}^{\oplus r-2} \to \mathcal{O}^{\oplus r-2}$. Note that the kernel of the surjection $\mathcal{F} \to \mathcal{G}$ is a subsheaf of $\mathcal{O}(2)$. Hence it can be written as $\mathcal{I}_Z(2)$ where $\mathcal{I}_Z$ is the ideal sheaf of some closed subscheme Z of $\mathbb{P}^2$. Now we have the following commutative diagram with exact lows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_Z(2) & \longrightarrow & \mathcal{O}(2) & \longrightarrow & \mathcal{O}_Z(2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{O}(2) \oplus \mathcal{O}^{\oplus r-2} & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{O}^{\oplus r-2} & \longrightarrow & \mathcal{Q}_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\mathcal{Q}_1$ is defined by the diagram above. Since $\mathcal{O}_Z(2) \to \mathcal{Q}$ is injective, we see that $\dim Z \leq 0$, and thus $\mathcal{O}_Z(2) \cong \mathcal{O}_Z$. If $\mathcal{Q}_1 \neq 0$, then take a line L intersecting with the support of $\mathcal{Q}_1$. Then the kernel of the surjection $\mathcal{O}_L^{\oplus r-1} \to \mathcal{Q}_1|_L$ has a negative degree line bundle as a direct summand, which implies that some negative degree line bundle is a quotient of $\mathcal{G}|_L$, $\mathcal{F}|_L$ and $\mathcal{E}|_L$. This contradicts that $\mathcal{E}$ is nef. Hence $\mathcal{Q}_1 = 0$. Thus $\mathcal{G} \cong \mathcal{O}^{\oplus r-2}$, $\mathcal{O}_Z \cong \mathcal{Q}$, and $\mathcal{O}_Z$ has length 6. Since $h^0(\mathcal{G}(-1)) = 0$, we infer that $h^0(\mathcal{I}_Z(1)) = e_{0,1} - 1 > 0$. Hence there exists a line L passing through Z. Since length $\mathcal{O}_Z = 6$, this implies that the kernel of the restriction $\mathcal{O}_L(2) \to \mathcal{O}_Z$ to the line L of the surjection $\mathcal{O}(2) \to \mathcal{O}_Z$ is isomorphic to $\mathcal{O}_L(-4)$. By restricting the diagram above to the line L, we see that $\mathcal{F}|_L$ has a negative degree line bundle as a quotient; this is a contradiction. Hence $\mathcal{F}^{\vee\vee}$ cannot be isomorphic to $\mathcal{O}(2) \oplus \mathcal{O}^{\oplus r-2}$.

Suppose that $\mathcal{F}^{\vee\vee} \cong \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}^{\oplus r-2}$. Since $c_2(\mathcal{F}^{\vee\vee}) = 1$, the length of $\mathcal{Q}$ is 5. Let $\mathcal{G}$ be the image of the composite of the inclusion $\mathcal{F} \to \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}^{\oplus r-3}$ and the projection $\mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}^{\oplus r-3} \to \mathcal{O}^{\oplus r-3}$, and $\mathcal{Q}_1$ the cokernel of the inclusion $\mathcal{G} \to \mathcal{O}^{\oplus r-3}$. Then there exists a surjection $\mathcal{Q} \to \mathcal{Q}_1$, and thus the support of $\mathcal{Q}_1$ has dimension ≤ 0. If $\mathcal{Q}_1 \neq 0$, we get a contradiction by the same argument as above. Therefore we may assume that $\mathcal{Q}_1 = 0$; thus $\mathcal{G} \cong \mathcal{O}^{\oplus r-3}$. Let $\mathcal{H}$ be the kernel of the surjection $\mathcal{F} \to \mathcal{O}^{\oplus r-3}$. Then we have the following commutative diagram with exact lows and columns.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{H} & \longrightarrow & \mathcal{O}(1)^{\oplus 2} & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}^{\oplus r-3} & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}^{\oplus r-3} & = & \mathcal{O}^{\oplus r-3} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Since $h^0(\mathcal{O}^{\oplus r-3}(-1)) = 0$, we infer that $h^0(\mathcal{H}(-1)) = e_{0,1} - 1 > 0$. Since $\mathcal{H}(-1)$ is a subsheaf of $\mathcal{O}^{\oplus 2}$, this implies that $\mathcal{H}(-1) \cong \mathcal{I}_Z \oplus \mathcal{O}$ and $\mathcal{Q}(-1) \cong \mathcal{O}_Z$ for some 0-dimensional closed subscheme Z of length 5 in $\mathbb{P}^2$. Now take a line L that intersect with Z in length $l \geq 2$. Then the kernel of $\mathcal{O}_L(1)^{\oplus 2} \to \mathcal{O}_{Z \cap L}(1)$ is of the form $\mathcal{O}_L(1-l) \oplus \mathcal{O}_L(1)$. This implies that $\mathcal{F}|_L$ has a negative degree line bundle as a quotient, which is a contradiction. Hence $\mathcal{F}^{\vee\vee}$ cannot be isomorphic to $\mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}^{\oplus r-3}$ either.

Suppose that $\mathcal{F}^{\vee\vee}$ fits in the exact sequence (9). Since $c_2(\mathcal{F}^{\vee\vee}) = 2$, the length of $\mathcal{Q}$ is 4. Define a torsion-free sheaf $\mathcal{F}_0$ as a quotient of $\mathcal{F}^{\vee\vee}$ by an injection $\mathcal{O}(1) \to \mathcal{F}^{\vee\vee}$. Then $\mathcal{F}_0$ fits in an exact sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O}^{\oplus r-1} \to \mathcal{F}_0 \to 0.$$

Let $\mathcal{G}$ be the image of the composite of the inclusion $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ and the projection $\mathcal{F}^{\vee\vee} \to \mathcal{F}_0$. Since $h^0(\mathcal{F}_0(-1)) = 0$, we see that $h^0(\mathcal{G}(-1)) = 0$. Let $\mathcal{H}$ be the kernel of the surjection $\mathcal{F} \to \mathcal{G}$. Then we have the following commutative diagram with exact lows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{H} & \longrightarrow & \mathcal{O}(1) & \longrightarrow & \mathcal{Q}_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}^{\vee\vee} & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{F}_0 & \longrightarrow & \mathcal{Q}_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are defined by the diagram above. Since $h^0(\mathcal{G}(-1)) = 0$, we see that $h^0(\mathcal{H}(-1)) = e_{0,1} - 1 > 0$. Since $\mathcal{H}(-1)$ is a subsheaf of $\mathcal{O}$, this implies that $\mathcal{H}(-1)$ is $\mathcal{O}$ itself; thus $\mathcal{Q}_2 = 0$, $\mathcal{Q} \cong \mathcal{Q}_1$ and $\mathcal{Q}_1$ has length 4. As we have seen in the proof of [Ohn14, Theorem 6.4], $\mathcal{F}_0$ is locally free outside at most one point, and if $\mathcal{F}_0$ is not locally free at a point z, then $\mathcal{F}_0$ is isomorphic to $\mathfrak{m}_z(1) \oplus \mathcal{O}^{\oplus r-3}$, where $\mathfrak{m}_z$ is the ideal sheaf of z, since $n = 2$. Suppose that $\mathcal{F}_0$ is not locally free. Then take a line L passing through z and meeting the support of $\mathcal{Q}_1$. We see that the surjection $\mathcal{F}_0 \to \mathcal{Q}_1$ induces a surjection $\mathcal{O}_L^{\oplus r-2} \to \mathcal{Q}_1|_L$, whose kernel has a negative degree line bundle as a quotient, and thus so does $\mathcal{G}|_L$, $\mathcal{F}|_L$ and $\mathcal{E}|_L$. This is a contradiction. Suppose that $\mathcal{F}_0$ is locally free. Then take a line L which intersects with $\mathcal{Q}_1$ in length $l \geq 2$. Since $\mathcal{F}_0|_L \cong \mathcal{O}_L(1) \oplus \mathcal{O}^{\oplus r-3}$, we see that $\mathcal{G}|_L$ admits a negative degree line bundle as a quotient; this is a contradiction. Hence $\mathcal{F}^{\vee\vee}$ cannot fit in the exact sequence (9).

Therefore we conclude that the case $e_{0,1} > 1$ does not happen.

3.2. The case $e_{0,1} = 1$

Suppose that $e_{0,1} = 1$. If the morphism v_2 in (5) is zero, then $E_2^{-1,1}|_L \cong \Omega_{\mathbb{P}^2}(1)|_L \cong \mathcal{O}_L(-1) \oplus \mathcal{O}_L$ for a line L not containing w. By (7), this implies that $\mathcal{E}|_L$ has $\mathcal{O}_L(-1)$ as a quotient; this is a contradiction. Hence $v_2 \neq 0$, and thus $E_2^{-2,1} = 0$, $E_2^{0,0} \cong E_3^{0,0}$ by (6), and $E_2^{-1,1}$ fits in an exact sequence

$$0 \to \mathcal{O}(-3) \xrightarrow{v_2} \Omega_{\mathbb{P}^2}(1) \to E_2^{-1,1} \to k(w) \to 0. \tag{10}$$

We see that $E_2^{-1,1}$ is a coherent sheaf of rank one. Since $E_3^{0,0}$ is torsion-free by (7), so is $E_2^{0,0}$, and thus $E_2^{0,0}$ has $\mathcal{O}(1)$ as a subsheaf and consequently is isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}^{\oplus r-2}$ by (8). Hence the exact sequence (7) becomes an exact sequence

$$0 \to \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r-2} \xrightarrow{\varphi} \mathcal{E} \to E_2^{-1,1} \to 0.$$

By taking the dual of φ and $(r-1)$-th wedge product of the dual, we obtain a morphism $\wedge^{r-1}\mathcal{E}^\vee \to \mathcal{O}(-1)$. Let $\mathcal{I}_Z(-1)$ be the image of this morphism, where $\mathcal{I}_Z$ is the ideal sheaf of a closed subscheme Z of $\mathbb{P}^2$ of dimension ≤ 1. Note that Z is the degeneracy locus of φ and that if we denote by ψ the induced surjection $\mathcal{E} \cong \wedge^{r-1}\mathcal{E}^\vee \otimes \det\mathcal{E} \to \mathcal{I}_Z(-1) \otimes \det\mathcal{E} \cong \mathcal{I}_Z(2)$ then $\psi \circ \varphi = 0$.

Suppose that the degeneracy locus Z of φ has codimension ≥ 2. Then $E_2^{-1,1}$ is torsion-free. This implies that $E_2^{-1,1} \cong \mathcal{I}_Z(2)$ and that $\mathcal{E}$ fits in an exact sequence

$$0 \to \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r-2} \xrightarrow{\varphi} \mathcal{E} \to \mathcal{I}_Z(2) \to 0.$$

Note that $\operatorname{length} Z = 6$. Since $\mathcal{E}$ is nef, $\operatorname{length}(Z \cap L) \leq 2$ for any line L in $\mathbb{P}^2$; let us call this the basic property of Z. Let p be any point in Z. We may assume that Z is in an affine open subscheme $\operatorname{Spec} K[x,y]$ and that $p = (0,0)$. The local ring $\mathcal{O}_{Z,p}$ can be written as A/I, where $A = \hat{\mathcal{O}}_{\mathbb{P}^2,p} = K[[x,y]]$ and I the ideal of Z in the local ring A. Observe here that if $\operatorname{length}(A/I) \leq 4$ and thus the support of Z contains another point $q \neq p$, then the basic property of Z implies $I \not\subseteq \mathfrak{m}^2$, where $\mathfrak{m}$ denotes the maximal ideal of A. Based on this observation, we can deduce from the basic property of Z that $I \not\subseteq \mathfrak{m}^2$ without any assumption on $\operatorname{length}(A/I)$. Now that Z is curvilinear, after changing coordinates (x,y) if necessary, we may assume that $I = \langle y - \varphi(x), x^l \rangle$, where $\varphi(x) = a_2x^2 + a_3x^3 + \cdots \in K[[x]]$ ($a_2 \neq 0$) and $l = \operatorname{length}(A/I)$. Local computation then shows that there exists a smooth conic C such that $\operatorname{length}(Z \cap C) \geq 5$; e.g., if $l \geq 3$, we can take a defining equation of C to be $y = a_2x^2 + dxy + ey^2$ for some $d, e \in K$. However this again contradicts that $\mathcal{E}$ is nef. Therefore this case cannot happen.

Suppose that $\dim Z = 1$. Then the ideal sheaf $\mathcal{I}_Z$ of Z is decomposed as $\mathcal{I}_Z \cong \mathcal{I}_{Z_d}(-d)$, where d is the degree of the divisor contained in Z and $\mathcal{I}_{Z_d}$ is the ideal sheaf of a 0-dimensional closed subscheme Z_d of $\mathbb{P}^2$. Consider the following commutative diagram with exact lows and columns

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}(1)\oplus\mathcal{O}^{\oplus r-2} & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{T} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}(1)\oplus\mathcal{O}^{\oplus r-2} & \xrightarrow{\varphi} & \mathcal{E} & \longrightarrow & E_2^{-1,1} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{I}_{Z_d}(2-d) & = & \mathcal{I}_{Z_d}(2-d) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

where $\mathcal{K}$ and $\mathcal{T}$ are defined by the diagram above. We see that $\mathcal{K}$ is a coherent sheaf of rank $r-1$ and thus $\mathcal{T}$ is the torsion subsheaf of $E_2^{-1,1}$, and that $\operatorname{Supp} Z = \operatorname{Supp}\mathcal{T} \cup \operatorname{Supp} Z_d$. Hence $E_2^{-1,1}$ has an associated point of codimension one. Now recall the exact sequence (10) and split this sequence into the following two exact sequences of coherent sheaves

$$0 \to \mathcal{O}(-3) \xrightarrow{\nu_2} \Omega_{\mathbb{P}^2}(1) \to \mathcal{C} \to 0, \tag{11}$$

$$0 \to \mathcal{C} \to E_2^{-1,1} \to k(w) \to 0. \tag{12}$$

Note that $\mathcal{C}$ has an associated point of codimension one since so does $E_2^{-1,1}$. Hence ν_2 passes through $\mathcal{O}(-1)$ or $\mathcal{O}(-2)$.

Suppose that ν_2 passes through $\mathcal{O}(-1)$. Then we have the following commutative diagram with exact lows and columns

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}(-3) & = & \mathcal{O}(-3) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{O}(-1) & \longrightarrow & \Omega_{\mathbb{P}^2}(1) & \longrightarrow & \mathcal{I}_p & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{O}_D(-1) & \longrightarrow & \mathcal{C} & \longrightarrow & \mathcal{I}_p & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

where $\mathcal{I}_p$ is the ideal sheaf of a point p, and D is a conic in $\mathbb{P}^2$. We also have the following commutative diagram with exact lows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_D(-1) & = & \mathcal{O}_D(-1) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{C} & \longrightarrow & E_2^{-1,1} & \longrightarrow & k(w) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{I}_p & \longrightarrow & \mathcal{D} & \longrightarrow & k(w) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where $\mathcal{D}$ is defined by the diagram above. Suppose that $\mathcal{D}$ has an associated point other than the generic point. Then it must be w, and thus $\mathcal{D} \cong \mathcal{I}_w \oplus k(w)$, which also contradicts that $\mathcal{E}$ is nef. Therefore $\mathcal{D}$ is torsion-free. Since $\mathcal{D}$ has rank one, $c_1(\mathcal{D}) = 0$ and $c_2(\mathcal{D}) = 0$, $\mathcal{D}$ is isomorphic to its double dual $\mathcal{O}_{\mathbb{P}^2}$. Moreover we see that $p = w$, that $\mathcal{O}_D(-1)$ is the torsion subsheaf $\mathcal{T}$ of $E_2^{-1,1}$, that $Z_d = \emptyset$, and that $Z = D$. If $h^0(E_2^{-1,1}) \neq 0$, then $E_2^{-1,1} \cong \mathcal{O}_D(-1) \oplus \mathcal{O}_{\mathbb{P}^2}$, which contradicts that $\mathcal{E}$ is nef. Hence $h^0(E_2^{-1,1}) = 0$. Since $h^1(\mathcal{O}_D(-1)) = h^2(\mathcal{O}_{\mathbb{P}^2}(-3)) = 1$, this implies that $H^0(\mathcal{D}) = H^0(\mathcal{O}_{\mathbb{P}^2}) \cong H^1(\mathcal{O}_D(-1))$. Suppose that D is smooth. Consider the pull back $\mathcal{O}_D(-1) \to E_2^{-1,1}|_D \to \mathcal{O}_D \to 0$ of the exact sequence above. Note that $E_2^{-1,1}|_D$ has rank at least two since D is the degeneracy locus of φ. Hence we obtain an exact sequence

$$0 \to \mathcal{O}_D(-1) \to E_2^{-1,1}|_D \to \mathcal{O}_D \to 0.$$

Note that $D \cong \mathbb{P}^1$ and that $\mathcal{O}_D(-1) \cong \mathcal{O}_{\mathbb{P}^1}(-2)$ via this isomorphism. Since the sequence above does not split, this implies that $E_2^{-1,1}|_D \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$, which contradicts that $\mathcal{E}$ is nef. Suppose that D is a double line. Then $D_{\text{red}} \cong \mathbb{P}^1$, and we have a surjection $\mathcal{O}_D(-1) \to \mathcal{O}_{D_{\text{red}}}(-1)$. The similar argument as above shows that there exists an exact sequence

$$0 \to \mathcal{O}_{D_{\text{red}}}(-1) \to E_2^{-1,1}|_{D_{\text{red}}} \to \mathcal{O}_{D_{\text{red}}} \to 0.$$

Hence $E_2^{-1,1}|_{D_{\text{red}}} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$; this contradicts that $\mathcal{E}$ is nef. Suppose that D is a union of two distinct lines: $D = L_1 + L_2$. Then $E_2^{-1,1}|_{L_1} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$ by the similar argument as above, and hence this case does not occur either.

Suppose that v_2 passes through $\mathcal{O}(-2)$ and does not pass through $\mathcal{O}(-1)$. Then we have the following commutative diagram with exact lows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}(-3) & = & \mathcal{O}(-3) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{O}(-2) & \longrightarrow & \Omega_{\mathbb{P}^2}(1) & \longrightarrow & \mathcal{I}_W(1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{O}_L(-2) & \longrightarrow & \mathcal{C} & \longrightarrow & \mathcal{I}_W(1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where $\mathcal{I}_W$ is the ideal sheaf of a 0-dimensional locally complete intersection W of length three, and L is a line in $\mathbb{P}^2$. We also have the following commutative diagram with exact lows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_L(-2) & = & \mathcal{O}_L(-2) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{C} & \longrightarrow & E_2^{-1,1} & \longrightarrow & k(w) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{I}_W(1) & \longrightarrow & \mathcal{D} & \longrightarrow & k(w) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where $\mathcal{D}$ is defined by the diagram above. If $\mathcal{D}$ is not torsion-free, then $\mathcal{D} \cong \mathcal{I}_w \oplus k(w)$, which contradicts that $\mathcal{E}$ is nef. Therefore $\mathcal{D}$ is a torsion-free coherent sheaf of rank one with $c_1(\mathcal{D}) = 1$ and $c_2(\mathcal{D}) = 2$. Hence $\mathcal{O}_L(-2)$ is the torsion subsheaf $\mathcal{T}$ of $E_2^{-1,1}$, and we infer that $\mathcal{D} \cong \mathcal{I}_{Z_1}(1)$ with length $Z_1 = 2$. This also contradicts that $\mathcal{E}$ is nef.

Therefore we conclude that the case $e_{0,1} = 1$ does not happen.

4. The case $n = 2$ and $e_{0,1} = 0$

Suppose that $e_{0,1} = 0$. Then $E_2^{-2,1} \cong \mathcal{O}(-3)$ and $E_2^{-1,1} \cong k(w)$ by (5), and $E_2^{0,0} \cong \mathcal{O}^{\oplus r+1}$ by (8). Thus we have the following two exact sequences by (6) and (7)

$$0 \to \mathcal{O}(-3) \to \mathcal{O}^{\oplus r+1} \to E_3^{0,0} \to 0, \tag{13}$$

$$0 \to E_3^{0,0} \to \mathcal{E} \to k(w) \to 0. \tag{14}$$

These two exact sequences show that $\mathcal{E}$ must fit in the exact sequence given in [Ohn16, Proposition 1.2]. We shall show that $\mathcal{E}$ has a resolution in terms of a full strong exceptional sequence of line bundles as in Theorem 1.1 in accordance with the framework given in [Ohn14].

Since $h^1(E_3^{0,0}(1)) = 0$, we have the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}(-3) & \longrightarrow & \mathcal{J} & \longrightarrow & \mathcal{I}_w(-1) & \longrightarrow & 0 \\
 & & \downarrow & & {\scriptstyle g}\downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}^{\oplus r+1} & \longrightarrow & \mathcal{O}^{\oplus r+1} \oplus \mathcal{O}(-1) & \longrightarrow & \mathcal{O}(-1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_3^{0,0} & \longrightarrow & \mathcal{E} & \longrightarrow & k(w) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\mathcal{I}_w$ is the ideal sheaf of w, and $\mathcal{J}$ and g are defined by the diagram above.

We also have the following commutative diagram with exact rows and columns

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}(-3) & = & \mathcal{O}(-3) & & \\
 & & & & {\scriptstyle f}\downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}(-3) & \longrightarrow & \mathcal{O}(-3)\oplus\mathcal{O}(-2)^{\oplus 2} & \longrightarrow & \mathcal{O}(-2)^{\oplus 2} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}(-3) & \longrightarrow & \mathcal{J} & \longrightarrow & \mathcal{I}_w(-1) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

where f is defined by the diagram above. We claim here that the composite of f and the projection $\mathcal{O}(-3)\oplus\mathcal{O}(-2)^{\oplus 2}\to\mathcal{O}(-3)$ is non-zero. Suppose, to the contrary, that the composite is zero. Then $\mathcal{J}\cong\mathcal{O}(-3)\oplus\mathcal{I}_w(-1)$. By taking the double dual, the composite of the inclusion $\mathcal{I}_w(-1)\to\mathcal{J}$ and g extends to a splitting injection of the projection $\mathcal{O}^{\oplus r+1}\oplus\mathcal{O}(-1)\to\mathcal{O}(-1)$; we obtain the following commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{I}_w(-1) & \longrightarrow & \mathcal{O}(-1) & \longrightarrow & k(w) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{J} & \xrightarrow{g} & \mathcal{O}^{\oplus r+1}\oplus\mathcal{O}(-1) & \longrightarrow & \mathcal{E} & \longrightarrow & 0.
\end{array}
$$

Since the induced morphism $k(w)\to\mathcal{E}$ is a splitting injection of the surjection $\mathcal{E}\to k(w)$, we have an isomorphism $\mathcal{E}\cong E_3^{0,0}\oplus k(w)$, which is absurd. Hence the claim holds; thus $\mathcal{J}\cong\operatorname{Coker}(f)\cong\mathcal{O}(-2)^{\oplus 2}$. Therefore we obtain the desired exact sequence

$$0\to\mathcal{O}(-2)^{\oplus 2}\to\mathcal{O}^{\oplus r+1}\oplus\mathcal{O}(-1)\to\mathcal{E}\to 0.$$

5. The case $n\geq 3$

In this section, we shall show that the case $n\geq 3$ does not happen. By considering the restriction $\mathcal{E}|_{L^3}$ to a 3-dimensional linear subspace $L^3\subseteq\mathbb{P}^n$, we may assume that $n=3$. We have

$$\chi(\mathcal{E}(-1))=\frac{c_3}{2}-10$$

by [Ohn16, (3.20)]. In particular, c_3 is even. We also have

$$c_3 \geq 21$$

by [Ohn16, (3.23)]. Since the equality in $c_3 \geq 21$ does not hold, we infer that $H(\mathcal{E})$ is big, and thus $h^q(\mathcal{E}(-1)) = 0$ for all $q > 0$ by [Ohn16, (3.3)]. Therefore $h^0(\mathcal{E}(-1)) \geq 1$. On the other hand, $H^0(\mathcal{E}(-2)) = 0$ by the argument in [Ohn16, §3], and $h^0(\mathcal{E}|_H(-1)) = 0$ for any plane $H \subset \mathbb{P}^3$ as is shown in § 3. Hence $h^0(\mathcal{E}(-1)) = 0$, which is a contradiction. Therefore the case $n \geq 3$ does not happen.

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