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NO GEOMETRIC APPROACH FOR GENERAL OVERDETERMINED ELLIPTIC PROBLEMS WITH NONCOSTANT SOURCE

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We discuss the geometric approach developed in [8] for the study of overdetermined boundary value problems for general elliptic operators. We show that this approach does not apply when nonconstant sources are involved.

1. Introduction.

The purpose of the present note is to discuss the geometric approach recently developed in [8] for the study of the elliptic problem

$$\begin{cases} -\mathrm{div}(v(|\nabla u|^2)\nabla u) = g(x, u, |\nabla u|) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u_\nu = -c & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where u_ν denotes the outward normal derivative of u and $c > 0$. Throughout the paper we assume that $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is an open bounded domain with

$$\partial\Omega \in C^{2,\alpha} \ . \tag{2}$$

The operator v is required to satisfy the (possibly degenerate) ellipticity conditions

$$v \in C^2(0, +\infty) \ , \ \lim_{q\to 0^+} \sqrt{q}\, v(q) = 0 \ , \ v(q) + 2qv'(q) > 0 \ \text{ for } q > 0 \ , \tag{3}$$

and the source term g is assumed to be smooth (at least $g \in C^1$). Under these assumptions, we consider C^1 distributional solutions of (1):

Definition 1.1. We say that u is a *solution* of (1) if $u \in C^1_0(\overline{\Omega})$, $u_\nu = -c$ on $\partial\Omega$ and

$$\int_\Omega v(|\nabla u|^2)\nabla u \nabla\varphi = \int_\Omega g(x, u, |\nabla u|)\varphi \qquad \text{for all } \varphi \in C^\infty_c(\Omega)\ .$$

Problem (1) is *overdetermined* since its solution is required to satisfy both a Dirichlet and a Neumann boundary condition. Therefore, in general, it admits no solution. Starting from the celebrated paper by Serrin [15], several authors have studied problem (1), see [3], [4], [6], [7], [9], [14], [17], [18]. We refer to the introduction in [8] for a survey of these results. Under quite different assumptions and using fairly different techniques, all these papers establish that if (1) admits a solution, then Ω is a ball.

The approach developed in [8] is geometric and considers the case of constant source, $g \equiv 1$. On the other hand, it allows to treat very general (possibly degenerate) elliptic operators. We postpone the discussion of this approach until Section 2. Here, we just recall the results.

According to [10], [11] we say that Ω is a *Cheeger set* if

$$\frac{|\partial\Omega|}{|\Omega|} = \min_D \frac{|\partial D|}{|D|} := h(\Omega)$$

where the minimum is taken over all open, nonempty, simply connected subdomains D of $\overline{\Omega}$. The constant $h(\Omega)$ is named after [5] and called the Cheeger constant of Ω. Then, we have

Theorem 1.2. ([8]) *Assume* (2)-(3) *and* $g \equiv 1$. *If problem* (1) *admits a solution* u, *then* $|\nabla u(x)| \le c$ *for all* $x \in \overline{\Omega}$, *and* Ω *is a Cheeger set.*

Since there are several examples of Cheeger sets, Theorem 2.1 does not allow to conclude that Ω is a ball. In order to obtain such result we need a further assumption on the domain. We say that Ω is star-shaped if there exists $x_0 \in \Omega$ such that $(x - x_0) \cdot \nu \ge 0$ on $\partial\Omega$. Then, we have

Theorem 1.3. ([8]) *Assume* (2)-(3), *let* $g \equiv 1$ *and assume that* Ω *is star-shaped. If problem* (1) *admits a solution, then* Ω *is a ball of radius* $R = ncv(c^2)$.

Several geometric tools are used in [8] for the proofs of the above results. In next section we briefly recall them and we discuss the main steps of the proofs of Theorems 1.2 and 1.3 .

The purpose of the present note is to show that the very same approach cannot be used to extend Theorems 1.2 and 1.3 to the case of nonconstant sources g. A crucial tool in the proofs is a suitable P-function which attains its maximum on $\partial\Omega$. In Section 3 we construct a P-function for problem (1) in the cases $g = g(u)$, $g = g(|\nabla u|)$ and $g = g(x)$. In Section 4 we show that in presence of nonconstant g some tools used in the proof "play against each other". Therefore, using the same geometric approach, a generalization of Theorems 1.2 and 1.3 to the case of nonconstant sources g seems out of reach.

2. Sketch of the proofs of Theorems 1.2 and 1.3.

In this section, we briefly recall the main steps in the proofs of Theorems 1.2 and 1.3. For the details, we refer to [8]. Without further mention we assume (2)-(3).

First, we remark that if u solves (1) (with $g \equiv 1$) in the sense of Definition 1.1, by elliptic regularity it satisfies

$$u \in C^{2,\alpha}(\overline{\Omega} \setminus \{x \ : \ \nabla u(x) \neq 0\}) . \tag{4}$$

Then, we put

$$\Phi(t) := \int_0^{t^2} [v(s) + 2sv'(s)]\, ds . \tag{5}$$

We assume that u solves (1) according to Definition 1.1 and we consider the function defined by

$$P(x) := \Phi(|\nabla u(x)|) + \frac{2}{n} u(x) \qquad (x \in \overline{\Omega}) . \tag{6}$$

Clearly, P is continuous in $\overline{\Omega}$ and, by (4), it is of class C^1 in a neighborhood of $\partial\Omega$. The next lemma extends a result in [13] (see also [16]) to the case of possibly degenerate elliptic operators:

Lemma 2.1. ([8]) *Let* $g \equiv 1$. *If* u *solves* (1) *in the sense of Definition 1.1, then the* P*-function defined by* (6) *is either constant in* $\overline{\Omega}$ *or it satisfies* $P_\nu > 0$ *on* $\partial\Omega$.

Proof. Assume that P is not constant in Ω.

If the operator is assumed to be *uniformly elliptic* (i.e. $v \in C^2[0, +\infty)$ and the inequality in (3) holds for $q \geq 0$) then one arrives at (2.39) in [13], namely

$$\Delta P + 2\frac{v'(|\nabla u|^2)}{v(|\nabla u|^2)} \nabla^2 P \nabla u \cdot \nabla u + L(u) \cdot \nabla P \geq 0 \qquad \text{in } \Omega , \tag{7}$$

where $L(u)$ is a smooth vector in Ω. The elliptic inequality (7) tells us that P attains its maximum only on $\partial\Omega$ (where it is constant). The boundary point Lemma gives $P_\nu > 0$ on $\partial\Omega$.

If the operator merely satisfies (3), then the vector $L(u)$ in (7) becomes unbounded at critical points of u. The statement may then be obtained by a suitable approximation procedure, see [8]. We point out that in this case P may attain its maximum also at critical points of u. □

As a consequence of Lemma 2.1, an upper bound for the mean curvature of $\partial\Omega$ is obtained:

Lemma 2.2. ([8]) *Let* $g \equiv 1$. *If problem* (1) *admits a solution, then the mean curvature* $H(x)$ *of* $\partial\Omega$ *satisfies*

$$\text{either} \quad H(x) < \frac{1}{n\,c\,v(c^2)} \quad \text{for all } x \in \partial\Omega \qquad \text{or} \quad H(x) \equiv \frac{1}{n\,c\,v(c^2)}\,.$$

Proof. Since $c > 0$, the equation in (1) is nondegenerate in a neighborhood of $\partial\Omega$ so that it also holds on $\partial\Omega$. Therefore, (recall $g \equiv 1$)

$$\left[v(c^2) + 2c^2 v'(c^2)\right] u_{\nu\nu} - (n-1)cv(c^2)H(x) = -1\,. \tag{8}$$

Distinguishing the two cases which arise from Lemma 2.1, the alternative in the statement follows. □

Proof of Theorem 1.2. Note first that integrating the differential equation in (1) (recalling $g \equiv 1$) and using the divergence Theorem yields

$$|\Omega| = -\int_\Omega \operatorname{div}(v(|\nabla u|^2)\nabla u) = cv(c^2)|\partial\Omega|\,. \tag{9}$$

By [8], Lemma 3.7, we know that $u(x) \geq 0$ for all $x \in \overline{\Omega}$. This, together with Lemma 2.1, shows that

$$\Phi(|\nabla u(x)|) \leq \Phi(|\nabla u(x)|) + \frac{2}{n}u(x) \leq \Phi(c) \qquad \text{for all } x \in \overline{\Omega}\,.$$

Since $t \mapsto \Phi(t)$ is strictly increasing in view of (3), we infer that $|\nabla u(x)| \leq c$ for all $x \in \overline{\Omega}$. Hence, for any subdomain $D \subseteq \Omega$ an integration of the differential equation (1) over D and an integration by parts yield

$$\begin{aligned} |D| &= -\int_D \operatorname{div}(v(|\nabla u|^2)\nabla u) = -\int_{\partial D} v(|\nabla u|^2)u_\nu \\ &\leq \int_{\partial D} v(|\nabla u|^2)|\nabla u| \leq cv(c^2)|\partial D|\,. \end{aligned} \tag{10}$$

This, combined with (9), shows that

$$\frac{|\partial\Omega|}{|\Omega|} = \frac{1}{cv(c^2)} \leq \frac{|\partial D|}{|D|} \qquad \text{for all } D \subseteq \Omega$$

and completes the proof of Theorem 1.2. $\square$

Proof of Theorem 1.3. Since the problem is autonomous, we may assume that Ω is star-shaped with respect to the origin. Recalling the Minkowski formula (see for instance Section 2A in [12]) and using the divergence Theorem we obtain the following identities:

$$\text{(11)} \qquad \int_{\partial\Omega} H(x)\, x \cdot \nu = |\partial\Omega| \,, \qquad \int_{\partial\Omega} x \cdot \nu = n|\Omega| \,.$$

The claim of Theorem 1.3 follows from Alexandrov's characterization of spheres [1], [2] if we show that

$$\text{(13)} \qquad H(x) \equiv \frac{1}{n\, c\, v(c^2)} \qquad \text{on } \partial\Omega \,.$$

Assume for contradiction that (12) is false. In view of Lemma 2.2, this means that

$$\text{(13)} \qquad H(x) < \frac{1}{n\, c\, v(c^2)} \quad \text{on } \partial\Omega \,.$$

But (11) and starshapedness with respect to the origin tell us that $x \cdot \nu \geq 0$ on $\partial\Omega$ with $x \cdot \nu > 0$ on a subset of positive $(n-1)$ measure. Therefore, multiplying inequality (13) by $x \cdot \nu$ and integrating over $\partial\Omega$ yields

$$\text{(14)} \qquad \int_{\partial\Omega} H(x)\, x \cdot \nu < \int_{\partial\Omega} \frac{x \cdot \nu}{n\, c\, v(c^2)} \,.$$

By (11) and (14) we get $cv(c^2)|\partial\Omega| < |\Omega|$. This contradicts (9) and completes the proof of Theorem 1.3. $\square$

3. Some P-functions for nonconstant sources.

In this section we construct suitable P-functions for problem (1) in the case where g is not constant. The results are partly known [13]. We extend the results in [13] to the case of possibly degenerate elliptic operators. Moreover, although Proposition 3.3 holds in a quite restrictive situation, it is somehow the result of most interest in this section because P-functions for nonautonomous problems are a quite delicate matter and are not considered in the standard references [13], [16].

We consider separately the three cases where $g = g(u)$, $g = g(|\nabla u|)$ and $g = g(x)$. We first prove

Proposition 3.1. *Assume* (2)-(3) *and let* $c > 0$. *Let* $g \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ *be such that* $g'(s) \leq 0$ *for all* $s \geq 0$. *Let* u *be a solution (according to Definition 1.1) of the problem*

$$\begin{cases} -\mathrm{div}(v(|\nabla u|^2)\nabla u) = g(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u_\nu = -c & \text{on } \partial\Omega. \end{cases}$$

Let Φ *be as in* (5) *and let*

$$G(s) = \int_0^s g(t)\,dt\,.$$

Then the P*-function defined by*

$$P(x) := \Phi(|\nabla u(x)|) + \frac{2}{n}\,G[u(x)] \qquad (x \in \overline{\Omega})\,,$$

is either constant in $\overline{\Omega}$ *or it satisfies* $P_\nu > 0$ *on* $\partial\Omega$. *Moreover,* $|\nabla u(x)| \leq c$ *for all* $x \in \overline{\Omega}$.

Proof. Elliptic regularity ensures (4). Condition (2.35) in [13] reads $g'(s) \leq 0$ which is precisely our assumption. Therefore, if the operator is *nondegenerate* then [13], Theorem 4, applies and shows that P attains its maximum on $\partial\Omega$. The boundary point Lemma gives $P_\nu > 0$ on $\partial\Omega$ unless P is constant in $\overline{\Omega}$. The approximation argument used in [8], Lemma 3.2, enables us to obtain the same result for degenerate operators. In any case, one obtains that P achieves its maximum on $\partial\Omega$. Therefore, the upper bound for $|\nabla u|$ follows. □

In the case where g only depends on the gradient we prove:

Proposition 3.2. *Assume* (2)-(3) *and let* $c > 0$. *Let* $g \in C^1(\mathbb{R}_+, (0, +\infty))$ *be such that* $g'(s) \geq 0$ *for all* $s \geq 0$. *Let* $u \in C^{1,\alpha}(\overline{\Omega})$ *be a solution (according to Definition 1.1) of the problem*

$$\begin{cases} -\mathrm{div}(v(|\nabla u|^2)\nabla u) = g(|\nabla u|) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u_\nu = -c & \text{on } \partial\Omega. \end{cases}$$

Then the P*-function defined by*

$$P(x) := \int_0^{|\nabla u(x)|^2} \frac{v(s) + 2sv'(s)}{g(\sqrt{s})}\, ds + \frac{2}{n} u(x) \qquad (x \in \overline{\Omega}) ,$$

is either constant in $\overline{\Omega}$ *or it satisfies* $P_\nu > 0$ *on* $\partial\Omega$. *Moreover,* $|\nabla u(x)| \leq c$ *for all* $x \in \overline{\Omega}$.

Proof. The further regularity assumption on u entails $g(|\nabla u|) \in C^{0,\alpha}(\overline{\Omega})$ so that (4) still holds. Condition (2.35) in [13] now reads $g'(s) \geq 0$ which is precisely our assumption. Therefore, if the operator is *nondegenerate* then [13], Theorem 4, applies and shows that P attains its maximum on $\partial\Omega$. The boundary point Lemma gives $P_\nu > 0$ on $\partial\Omega$ unless P is constant. The approximation argument used in [8], Lemma 3.2, yields the same result for degenerate operators. In any case, one obtains that P achieves its maximum on $\partial\Omega$ and the upper bound for $|\nabla u|$ follows. □

Finally, we turn to the case where g only depends on x:

Proposition 3.3. *Assume that* $\Omega \subset \mathbb{R}^2$ *is a bounded domain satisfying* (2). *Let* $g \in C^2(\overline{\Omega}, \mathbb{R}_+)$ *be such that* $\Delta g \geq 0$ *in* Ω. *Let* u *be a (classical) solution of the problem*

$$\begin{cases} -\Delta u = g(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then the P*-function defined by*

$$P(x) := |\nabla u(x)|^2 + u(x)g(x) \qquad (x \in \overline{\Omega}) ,$$

is either constant in $\overline{\Omega}$ *or it satisfies* $P_\nu > 0$ *on* $\partial\Omega$.

Proof. Using the fact that $-\Delta u = g$, some computations lead to

$$\Delta P = (u_{11} - u_{22})^2 + 4u_{12}^2 + u\Delta g ,$$

where $u_{ij} = \partial^2 u/\partial x_i \partial x_j$. Since $u \geq 0$ and g is subharmonic, this yields $\Delta P \geq 0$ in Ω. Therefore, P assumes it maximum on $\partial\Omega$. The boundary point Lemma gives $P_\nu > 0$ on $\partial\Omega$ unless P is constant. □

4. No extensions of Theorems 1.2 and 1.3 with the same proof.

In this section we show that the same arguments used in the proofs of Theorems 1.2 and 1.3 *cannot* be used to extend the results to the case of a nonconstant source g.

In the proof of Theorem 1.2 one may relax the equality in (9) to the inequality

$$|\Omega| \leq cv(c^2)|\partial\Omega| \ . \tag{15}$$

And inequality (15) is ensured (by the same argument leading to (9) if we assume that

$$g(x, u, |\nabla u|) \geq 1 \qquad \text{in } \Omega \ . \tag{16}$$

Once (16) is obtained, one reaches a contradiction using (10). But in order to obtain (10) one needs to assume that

$$g(x, u, |\nabla u|) \leq 1 \qquad \text{in } \Omega \ . \tag{17}$$

Clearly, (16) and (17) are possible only if $g \equiv 1$.

Also in the proof of Theorem 1.3 one needs (15) so that (16) seems unavoidable. The proof of Theorem 1.3 also uses Lemma 2.1, namely that a suitable P-function assumes its maximum on $\partial\Omega$. The final ingredient of the proof of Theorem 1.3 is Alexandrov's Theorem and therefore the alternative stated in Lemma 2.2 is needed in order to argue by contradiction assuming (13). With (13) one then arrives at $cv(c^2)|\partial\Omega| < |\Omega|$ which contradicts (15). But in order to prove Lemma 2.2, one needs (8) so that one also has to assume that $g \equiv 1$ on $\partial\Omega$, namely

$$g(x, 0, c) = 1 \qquad \text{for all } x \in \partial\Omega \ . \tag{18}$$

Let us discuss separately the three different cases considered in Propositions 3.1–3.3.

<u>The case $g = g(u)$.</u> In this case, (18) reads $g(0) = 1$ whereas (16) reads $g(s) \geq 1$ for all $s \geq 0$. In Proposition 3.1 it is assumed that $g'(s) \leq 0$ for all $s \geq 0$. These three assumptions necessarily yield $g \equiv 1$.

<u>The case $g = g(|\nabla u|)$.</u> Since Proposition 3.2 states that $|\nabla u(x)| \leq c$ for all x, only the assumptions for g over $[0, c]$ should be considered. In this case, (18) reads $g(c) = 1$ whereas (16) reads $g(s) \geq 1$ for all $s \geq 0$. In Proposition 3.2

it is assumed that $g'(s) \geq 0$ for all $s \geq 0$. These three assumptions necessarily yield $g \equiv 1$ on $[0, c]$.
The case $g = g(x)$. In this case, (18) reads $g(x) = 1$ on $\partial\Omega$ whereas (16) reads $g \geq 1$ in Ω. In Proposition 3.3 it is assumed that $\Delta g \geq 0$ in Ω. Once more, these three assumptions yield $g \equiv 1$.

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