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ON THE CODIMENSION OF SUBALGEBRAS OF THE ALGEBRA OF MATRICES OVER A FIELD

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In this paper we provide an elementary and easy proof that a proper subalgebra of the matrix algebra $\mathbb{K}^{n,n}$, with $n \geq 3$ and $\mathbb{K}$ an arbitrary field, has dimension strictly less than $n^2 - 1$.

1. Introduction

The aim of this paper is to give an elementary proof of the following result:

Theorem 1.1. *If $\mathcal{A}$ is a proper subalgebra of $\mathbb{K}^{n,n}$, the algebra of $n \times n$ matrices with entries in an arbitrary field $\mathbb{K}$, and if $n \geq 3$, then $\dim(\mathcal{A}) \leq n^2 - 2$, where $\dim(\mathcal{A})$ is the dimension of $\mathcal{A}$ as a vector space over $\mathbb{K}$.*

When $\mathbb{K}$ is an algebraically closed field, this result can be deduced and generalized by the following well known Theorem of Burnside (see [4], [5], [6] and [7] for elementary proofs):

Theorem 1.2 (Burnside's Theorem)**.** *If $\mathcal{A}$ is an irreducible subalgebra of $\mathbb{K}^{n,n}$, with $n \geq 1$ and $\mathbb{K}$ is an algebraically closed field, then $\mathcal{A} = \mathbb{K}^{n,n}$.*

Recall that a subalgebra $\mathcal{A} \subseteq \mathbb{K}^{n,n}$ is said to be irreducible if the only linear subspaces $U \subseteq \mathbb{K}^n$ invariant under all the elements of $\mathcal{A}$, i.e. such that $AU \subseteq U$ for all $A \in \mathcal{A}$, are $\{0\}$ and $\mathbb{K}^n$.

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Then, if $\mathcal{A}$ is a proper subalgebra of $\mathbb{K}^{n,n}$, with $\mathbb{K}$ an algebraically closed field, there exists a non trivial subspace U of $\mathbb{K}^n$ that is invariant under all the elements of $\mathcal{A}$ with $0 < \dim(U) = r < n$. This immediately implies $\dim(\mathcal{A}) \leq n^2 - r(n-r) \leq n^2 - n + 1$, refining significantly the bound in Theorem 1.1.

So, for $n \geq 4$, Theorem 1.1 gives less information on the dimension of a proper subalgebra of $\mathbb{K}^{n,n}$ than the consequences of Burnside's Theorem, but it holds over an arbitrary field.

From another perspective the reason for which there do not exist codimension 1 subalgebras of $\mathbb{K}^{n,n}$ resides on Wedderburn's Theorem providing the decomposition of $\mathbb{K}^{n,n}$ into the direct sum of its *semi-simple* and *radical* parts. Over an algebraically closed field the semi simple part is a direct sum of matrix algebras while the dimension of the radical is controlled by Gerstenhaber's Theorem on the dimension of linear subspaces of nilpotent matrices, see [3]. Putting together these not trivial facts one obtains a different proof of the above inequality $\dim(\mathcal{A}) \leq n^2 - n + 1$. This approach is present in [1], where this inequality is also proved to be true over a field of characteristic zero, thus requiring less restrictive hypothesis about $\mathbb{K}$. A, not trivial, proof of Theorem 1.1 with $\mathcal{A}$ over an arbitrary field, can be also found as a consequence of the results contained in [2].

In conclusion we post our novelty only on the simple and elementary proof of Theorem 1.1 but not on its contents which are surely well known to any expert in the field. Some introduction can go here, for example

2. Proof of Theorem 1.1

Suppose that $\dim(\mathcal{A}) = n^2 - 1$. Since $\dim(\mathbb{K}^{n,n}) = n^2$, given a basis for the vector space $\mathbb{K}^{n,n}$, the subspace $\mathcal{A}$ is represented by only one homogeneous equation in the associated coordinates. We consider the standard basis of $\mathbb{K}^{n,n}$ which consists of the matrices E_{ij} such that $(E_{ij})_{k,l} = \delta_{ik} \cdot \delta_{jl}$, where δ is the Kronecker delta.

We have:

$$(E_{ij} \cdot E_{kl})_{m,q} = \sum_{p=1}^{n} (E_{ij})_{m,p} (E_{kl})_{p,q} = \sum_{p=1}^{n} \delta_{im}\delta_{jp}\delta_{kp}\delta_{lq} = \delta_{im}\delta_{kj}\delta_{lq}.$$

If $k \neq j$ then $E_{ij} \cdot E_{kl} = 0$, while, if $k = j$, from

$$\left(E_{ij} \cdot E_{jl}\right)_{m,q} = \delta_{im}\delta_{lq} = (E_{il})_{m,q},$$

it follows that $E_{ij} \cdot E_{jl} = E_{il}$.

We now suppose the subspace $\mathcal{A}$ is given by the following equation:

$$a_{11}x_{11} + \ldots + a_{1n}x_{1n} + a_{21}x_{21} + \ldots + a_{nn}x_{nn} = 0 \tag{1}$$

where $a_{ij} \in \mathbb{K}$ for all $i, j = 1, \ldots, n$.

We want to show that $a_{ij} = 0$ for all i and j. To this aim, for all i, j, k and $l \in \{1, \ldots, n\}$, we define the matrices

$$D_{ij}^{kl} = a_{kl}E_{ij} - a_{ij}E_{kl}.$$

By construction D_{ij}^{kl} satisfies the equation of $\mathcal{A}$, and therefore we have $D_{ij}^{kl} \in \mathcal{A}$ for all i, j, k and $l \in \{1, \ldots, n\}$.

We now consider a triple of pairwise distinct indices i, j and k (we can find them because $n \geq 3$). D_{ik}^{ij} and D_{kj}^{ij} belong to $\mathcal{A}$, which is closed under matrix multiplication. Therefore we deduce:

$$\mathcal{A} \ni D_{ik}^{ij} \cdot D_{kj}^{ij} = \left(a_{ij}E_{ik} - a_{ik}E_{ij}\right) \cdot \left(a_{ij}E_{kj} - a_{kj}E_{ij}\right) =$$
$$= a_{ij}^2 E_{ik}E_{kj} - a_{ij}a_{kj}E_{ik}E_{ij} - a_{ik}a_{ij}E_{ij}E_{kj} + a_{ik}a_{kj}E_{ij}^2 = a_{ij}^2 E_{ij},$$

where the last equality follows from the properties of the matrices E_{ij} and from the choice of i, j e k. The matrix $a_{ij}^2 E_{ij}$ belongs to $\mathcal{A}$, so it must satisfy the equation (1). From this it follows that $a_{ij}^3 = 0 \Rightarrow a_{ij} = 0$.

We have just proved that $a_{ij} = 0$ for all $i \neq j$, so the equation of $\mathcal{A}$ can be written in the form:

$$a_{11}x_{11} + a_{22}x_{22} + \ldots + a_{nn}x_{nn} = 0. \tag{2}$$

From this equation we can deduce that, for all $i = 1, \ldots, n$ and $k \neq i$, the matrices $P_i = a_{ii}E_{ik}$ and $Q_i = a_{ii}E_{ki}$ belong to $\mathcal{A}$.

Then we have:

$$P_i \cdot Q_i = a_{ii}^2 E_{ik}E_{ki} = a_{ii}^2 E_{ii} \in \mathcal{A}.$$

The matrix $a_{ii}^2 E_{ii}$ belongs to $\mathcal{A}$, so it must satisfy the equation (2). From this it follows that $a_{ii}^3 = 0 \Rightarrow a_{ii} = 0$ for all $i = 1, \ldots n$.

We proved that $a_{ij} = 0$ for all i and j, in contradiction with the assumption that $\mathcal{A}$ can be represented by only one equation.

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