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DIRICHLET PROBLEM WITH L^p-BOUNDARY DATA FOR REAL SUB-LAPLACIANS

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Let $\mathcal{L}$ be a real sub-Laplacian on a stratified Lie group G. In this note we present some results on the Dirichlet problem for $\mathcal{L}$ with L^p-boundary data, on domains Ω which are contractible with respect to the natural dilations of G. One of the main difficulties we overcome is the presence of non-regular boundary points for the usual Dirichlet problem for $\mathcal{L}$. A potential theoretical approach is followed.

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1. Introduction.

In a paper dated 1937 G. Cimmino introduced a method to study the Dirichlet problem with L^2 boundary data for the Laplace equation [3]. Cimmino method, which is reminiscent the one used in the theory of Hardy spaces of holomorphic functions, naturally extends to the more general setting of the real sub-Laplacians on stratified Lie groups.

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In recent years these operators have received a considerable attention due to they role in the theory of second order partial differential equations with non-negative characteristic form. Sub-Laplacian operators appear in many differet settings, both theoretical and applied, including geometric theory of several complex variables, Cauchy-Riemann and conformal geometry, Weyl formalization of Quantum Mechanics, mathematical models of crystal materials.

The main ideas of Cimmino approach can be described as follows. Let Ω be a bounded open subset of $\mathbb{R}^N$ with sufficiently smooth boundary. Assume Ω is starlike with respect to the origin. More precisely assume that

$$\lambda(\partial\Omega) \subset \Omega, \quad \text{for } 0 < \lambda < 1.$$

Given a function $u : \Omega \to \mathbb{R}$, define

$$u_\lambda : \partial\Omega \to \mathbb{R}, \quad u_\lambda(x) = u(\lambda x)$$

If u is harmonic in Ω and, for a suitable $\varphi \in L^2(\partial\Omega, d\sigma)$, it satisfies

$$u_\lambda \longrightarrow \varphi. \quad \text{as } \lambda \to 1$$

in $L^2(\partial\Omega, d\sigma)$, then Cimmino says that u solves the Dirichlet problem

$$(D) \qquad \begin{cases} \Delta u = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = \varphi & \text{in } L^2 \end{cases}$$

Cimmino proves that this problem is well posed: it has *one and only one* solution for every $\varphi \in L^2(\partial\Omega, d\sigma)$. The *uniqueness* is proved by Cimmino as a consequence of the following noteworthy monotonicity Lemma: the function

$$\lambda \to |u_\lambda|^2_{L^2(\partial\Omega)} = \int_{\partial\Omega} |u(\lambda x)|^2 \, d\sigma(x)$$

is *monotone increasing*. Then, if u solves (D) with $\varphi = 0$, one has

$$0 \le |u_\lambda|^2_{L^2(\partial\Omega)} \le 0, \quad \text{for } 0 < \lambda < 1,$$

which obviously implies $u \equiv 0$ in Ω.

To prove the existence, Cimmino uses what Caccioppoli called the *completeness* method. Define

$$\mathcal{R}(\partial\Omega) := \{\varphi \in L^2(\partial\Omega) : (D) \text{ has a solution}\}.$$

It easy to see that $\mathcal{R}(\partial\Omega)$ contains the space $C(\partial\Omega)$ of the continuous functions on the boundary of Ω. Then, since the closure of $C(\partial\Omega)$ in the $L^2(\partial\Omega, d\sigma)$ norm is the whole $L^2(\partial\Omega, d\sigma)$ one has

$$\overline{\mathcal{R}}(\partial\Omega) = L^2(\partial\Omega, d\sigma)$$

Cimmino proves that $\mathcal{R}$ is *closed* with respect to the $L^2(\partial\Omega, d\sigma)$-norm, obtaining

$$\mathcal{R}(\partial\Omega) = L^2(\partial\Omega, d\sigma),$$

that is the existence of a solution to (D) for every $\varphi \in L^2(\partial\Omega, d\sigma)$.

The full strength of Cimmino method clearly appears by looking at the Dirichlet problem from a potential theoretical point of wiew. Any sub-Laplacian $\mathcal{L}$ endows $\mathbb{R}^N$ with a structure of β-harmonic space. This allows to "solve" the Dirichlet problem, with very general boundary data, by using the Perron-Wiener method in the setting of the abstract harmonic spaces. Our main results shows that the Cimmino solutions actually are the Perron-Wiener solutions.
The *monotonicity lemma*, needed by Cimmino method to get uniqueness , in our paper is proved by using the Poisson-Jensen formula for the $\mathcal{L}$-subharmonic function contained in [1]. This formula suggests to replace the surfaces measure $d\sigma$ used by Cimmino, with the $\mathcal{L}$-harmonic measure.

2. The sub-laplacians and their fundamental solutions.

A stratified Lie group is a connected and simply connected Lie group G whose Lie algebra $\mathfrak{g}$ admits a stratification, i.e. a direct sum decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ with

$$[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1} \text{ for } i \leq r-1, \quad [\mathfrak{g}_1, \mathfrak{g}_r] = \{0\}. \tag{2.1}$$

If $\{X_1, \dots, X_m\}$ is a basis of $\mathfrak{g}_1$, the operator

$$\mathcal{L} = \sum_{i=1}^{m} X_i^2$$

is called a sub-Laplacian on G. Let us denote

$$d_i = \dim(\mathfrak{g}_i) \quad i = 1, \dots, r.$$

By means of the natural identification of G with its Lie algebra via the exponential map, it is non restrictive to suppose that $G = \mathbb{R}^N$ is equipped with a family of *dilations*) $(\delta_\lambda)_{\lambda>0}$, which are automorfisms of G, of the following form

$$\delta_\lambda(x^{(d_1)}, \dots, x^{(d_r)}) = (\lambda x_1^{(d_1)}, \dots, \lambda^r x^{(d_r)}), \tag{2.2}$$

where $x^{(d_i)} \in \mathbb{R}^{d_i}$, $i = 1, \dots, r$. With respect to these dilations the vector fields $X_1, \dots, X_m$ are homogeneous of degree one, so that $\mathcal{L}$ is δ_λ-homogeneous of degree two, i.e.,

$$\mathcal{L}(u \circ \delta_\lambda) = \lambda^2 (\mathcal{L}u) \circ \delta_\lambda \quad \text{for every } u \in C^\infty(G, \mathbb{R}). \tag{2.3}$$

The integer $Q = \sum_{i=1}^{r} i\, d_i$ is called the homogeneous dimension of G. Throughout the note we shall assume $Q \geq 3$ (if $Q = 2$ then $G = (\mathbb{R}^2, +)$ and $\mathcal{L}$ is an elliptic operator with constant coefficients).
The characteristic form of the sub-laplacian $\mathcal{L}$ is non-negative definte, and it is strictly positive definite, if and only if r, the *step* of G, is equal to one. Hence, if $r > 1$, $\mathcal{L}$ is not elliptic at any points. On the other hand, the stratification condition (2.1) ensures that the Lie algebra generated by $X_1, \dots, X_m$ has rank N at any points. Consequently, by a well known theorem of Hörmander [4], $\mathcal{L}$ is hypoelliptic, i.e., any distributional solution to $\mathcal{L}u = f$ is C^∞ whenever f is C^∞. Every smooth function $u : \Omega \to \mathbb{R}$ such that $\mathcal{L}u = 0$ in Ω will be called *$\mathcal{L}$-harmonic* in Ω. We shall denote by $\mathcal{H}(\Omega)$ the space of the $\mathcal{L}$-harmonic functions in Ω.

With respect to the cited logarithmic coordinates on G, $\mathcal{L}$ can be written as

$$\mathcal{L} = \operatorname{div}\big(A(x)\,\nabla\big), \quad \nabla = (\partial_{x_1}, \dots, \partial_{x_N}),$$

where $A(x)$ is a non-negative definite matrix with polynomial entries.

A noteworthy property of $\mathcal{L}$ is the structure of his fundamental solution. Indeed, there exists a homogeneous norm d on G such that

$$\Gamma(x, y) = d^{2-Q}(y^{-1} \circ x), \quad x, y \in G \tag{2.4}$$

is a fundamental solution for $\mathcal{L}$.
We call *homogeneous norm* on G any function $d : G \to [0, \infty)$ such that: $d \in C^\infty(G \setminus \{0\}) \cap C(G)$, $d(\delta_\lambda(x)) = \lambda\, d(x)$, $d(x^{-1}) = d(x)$, $d(x) = 0$ iff $x = 0$.
This striking analogy between $\mathcal{L}$ and the standard Laplace operator allows to develop a Potential Theory that parallels the classical one . A starting point of this theory is the following Mean Value Theorem for $\mathcal{L}$-harmonic functions, that extends to this new setting the classical Gauss-Koebe Theorem.

For every $x \in \mathbb{R}^N$ and $r > 0$ let us define

$$D(x, r) := \{y \in \mathbb{R}^N : d(y^{-1} \circ x) < r\}.$$

Then, for every $\mathcal{L}$-harmonic functions u in an open set $\Omega \subset \mathbb{R}^N$, we have

$$u(x) = M_r(u)(x) \qquad \text{for every } \overline{D(x, r)} \subset \Omega \tag{2.5}$$

where

$$M_r(u)(x) = \frac{C_Q}{r^Q}\int_{D(x,r)} K(x^{-1}\circ y)u(y)\,dy$$

and

$$K = \sum_{j=1}^{m}(X_j d)^2.$$

Viceversa, if u is a *continous* function in Ω satisfying (2.5) then $u \in C^\infty$ and $\mathcal{L}$-harmonic in Ω. The kernel K is δ_λ-homogeneous of degree zero. It is a constant function if and only if G is the Euclidean group and $\mathcal{L}$ is, up to a linear change of coordinates, the standard Laplace operator.

3. Potential Theory for the sub-laplacians .

In this section we still denote by $\mathcal{L}$ a sub-laplacian on a stratified Lie group G. If Ω is an open subset of G , a function $u : \Omega \to [-\infty, \infty[$ will be said *$\mathcal{L}$-subharmonic* if it is upper semicontinuos and satisifies

$$u(x) \leq M_r(u)(x) \qquad \text{for every } \overline{D(x,r)} \in \Omega.$$

The family of all $\mathcal{L}$- subharmonic functions is a cone that will be denoted by $\underline{\mathcal{S}}(\Omega)$. If $-u$ is $\mathcal{L}$- subharmonic we will say that u is $\mathcal{L}$- superharmonic. The cone of all $\mathcal{L}$- superharmonic functions in Ω will be denoted by $\overline{\mathcal{S}}(\Omega)$.

If Ω is a bounded open set and φ is an extended function on the boundary of Ω, i.e.

$$\varphi : \partial\Omega \to [-\infty, \infty],$$

one defines

$$\overline{H}_\varphi^\Omega := \inf\{u \in \overline{\mathcal{S}}(\Omega) : \liminf_{\partial\Omega} u \geq \varphi, \inf u > -\infty\}$$

and

$$\underline{H}_\varphi^\Omega := \sup\{u \in \underline{\mathcal{S}}(\Omega) : \limsup_{\partial\Omega} u \leq \varphi, \sup u < \infty\}.$$

We say that φ is a *risolutive functions* iff the functions $\overline{H}_\varphi^\Omega$ and $\underline{H}_\varphi^\Omega$ are *equal and $\mathcal{L}$-harmonic in* Ω. In this case the function

$$H_\varphi^\Omega := \overline{H}_\varphi^\Omega \equiv \underline{H}_\varphi^\Omega$$

is called the Perron-Wiener solution to the Dirichlet problem

$$(D) \qquad \begin{cases} \mathcal{L}u = 0 & \text{in} \quad \Omega \\ u|_{\partial\Omega} = \varphi. \end{cases}$$

The classical Wiener's Theorem for the standard Laplace operator extends to this general setting. Indeed:

every continuous function is resolutive.

As well as in the classical case, we cannot expect that H_φ^Ω is a *true* solution of (D). However, if (D) is solvable in the classical sense, i.e. if there exists a function $u \in C(\overline{\Omega})$, $\mathcal{L}$-harmonic in Ω and such that $u|_{\partial\Omega} = \varphi$, then $H_\varphi^\Omega = u$. A point $y \in \partial\Omega$ is called $\mathcal{L}$-regular for Ω iff

$$\lim_{x\to y} H_\varphi^\Omega(x) = \varphi(y) \quad \text{for every } \varphi \in C(\partial\Omega).$$

The Dirichlet problem (D) is solvable in the classical sense if and only if every point of $\partial\Omega$ is $\mathcal{L}$-regular for Ω. As we can expect, due to the possible high degeneracy of $\mathcal{L}$, the set

$$\partial_{irr}\Omega := \{y \in \partial\Omega : y \text{ is not } \mathcal{L}\text{-regular for } \Omega\}$$

is in general not empty, even if the boundary of Ω is $C^{1,\alpha}$. Nevertheless, $\partial_{irr}\Omega$ is negligible from a $\mathcal{L}$-potential theoretical point of view. Indeed, for every bounded open set Ω,

$$\partial_{irr}\Omega \quad \text{is} \quad \mathcal{L}\text{-polar}$$

A set $E \subset G$ is called *$\mathcal{L}$-polar* if there exists a $\mathcal{L}$-superharmonic function u such that

$$E \subset \{x : u(x) = \infty\}.$$

For every fixed points $x \in \Omega$ the map

$$C(\partial\Omega) \ni \varphi \longmapsto H_\varphi^\Omega(x) \in \mathbb{R}$$

is linear and non-negative. Then, there exists a unique Radon measure μ_x^Ω such that

$$H_\varphi^\Omega(x) = \int_{\partial\Omega} u(y)\, d\mu_x^\Omega(y)$$

μ_x^Ω is called the *$\mathcal{L}$-harmonic measure* related to Ω at x. From the Harnack inequality for non negative $\mathcal{L}$-harmonic functions, if Ω is connected and $x, x' \in$

Ω, then μ_x^Ω is *absolutely continuous* with respect to $\mu_{x'}^\Omega$ with bounded density function.

The fundamental resolutive theorem states that *a function* $\varphi : \partial\Omega \to [-\infty, \infty]$ *is resoltive if and only if*

$$\varphi \in L^1(\partial\Omega, \mu_x^\Omega)$$

for every $x \in \Omega$. By the previous remark, if Ω is connected, this condition is satisfied if (3.1) holds for just one point $x \in \Omega$.

The set of the boundary points which are not $\mathcal{L}$-regular is negligible also with respect to the harmonic measures. Indeed

$$\mu_x^\Omega(\partial_{irr}\Omega) = 0 \quad \forall x \in \Omega.$$

4. Dirichlet problem with L^p boundary data.

As in the previous sections $\mathcal{L}$ will denote a sub-Laplacian on a stratified Lie group G whose dilations are denoted by δ_λ. A bounded open set $\Omega \subset G$ will be said δ_λ-contractible if

$$\delta_\lambda(\partial\Omega) \subset \Omega \quad \text{for} \quad 0 \le \lambda < 1.$$

In this case, given a function $u : \Omega \to [-\infty, \infty]$, for every $\lambda \in]0, 1[$ we set

$$u_\lambda : \partial\Omega \to [-\infty, \infty], \quad u_\lambda(x) = u(\delta_\lambda(x)).$$

In what follows we shall assume Ω is δ_λ-contractible and denote by μ the $\mathcal{L}$-harmonic measure related to Ω at $x = 0$:

$$\mu := \mu_0^\Omega.$$

Given a function $\varphi \in L^p(\partial\Omega, \mu)$, $1 \le p < \infty$, we shall say that u solves the Dirichlet problema

$$(D_p) \qquad \begin{cases} \mathcal{L}u = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = \varphi, & \text{in } L^p. \end{cases}$$

if u is $\mathcal{L}$- harmonic in Ω and $u_\lambda \to \varphi$ as $\lambda \to 1$ in $L^p(\partial\Omega, \mu)$.
Since $\partial\Omega$ is bounded, $L^p(\partial\Omega, \mu) \subset L^1(\partial\Omega, \mu)$ so that *every* $\varphi \in L^p(\partial\Omega, \mu)$ *is resolutive*. Our main results is the following theorem

Theorem. *For every $\varphi \in L^p(\partial\Omega, \mu)$, $1 \le p < \infty$, the Dirichlet problem (D_p) has a unique solution. It is given by*

$$u := H_\varphi^\Omega$$

An outline of the proof of this theorem is as follows.

Uniqueness. Let u be a $\mathcal{L}$- harmonic funcions in Ω. Then $|u|^p \in \underline{S}(\Omega)$ and there exists a Radon measure ν such that $\mathcal{L}|u|^p = \nu$ in the weak sense of distributions. Let us put $v := |u|^p$. By the Poisson-Jensen formula in [1] we obtain

$$v(0) = \int_{\partial\Omega_\lambda} v(z)\, d\mu_0^{\Omega_\lambda}(z) - \int_{\Omega_\lambda} G_{\Omega_\lambda}(0, z)\, d\nu(z)$$

so that

$$\int_{\partial\Omega} |u(\delta_\lambda(z)|^p\, d\mu(z) = |u(0)|^p + \int_{\Omega_\lambda} G_{\Omega_\lambda}(0, z)\, d\nu(z)\,.$$

Here $\Omega_\lambda := \delta_\lambda(\Omega)$ and G_λ denotes the $\mathcal{L}$-Green function of Ω_λ.

It is quite obvious that this last right hand side is *monotone increasing* with respect to λ. As a consequence, if u is a solution of (D_p) with boundary data $\varphi = 0$, we have

$$0 \le \int_{\partial\Omega} |u(\delta_\lambda(z))|^p\, d\mu \nearrow 0$$

Then, letting $w_\lambda(x) = |u(\delta_\lambda(x))|^p$, $x \in \partial\Omega$, we obtain

$$\int_{\partial\Omega} w_\lambda\, d\mu_0^\Omega = 0\,.$$

This implies

$$0 \le H_{w_\lambda}^\Omega(x) \le C_x H_{w_\lambda}^\Omega(0) = \int_{\partial\Omega} w_\lambda\, d\mu_0^\Omega = 0\,.$$

Hence $H_{w_\lambda}^\Omega \equiv 0$. Then,

$$w_\lambda(z) = \lim_{x \to z} H_{w_\lambda}^\Omega(x) = 0, \quad \forall z \in \Omega \setminus P$$

where $P := \partial_{irr}\Omega$ is the $\mathcal{L}$-polar subset of $\partial\Omega$ of the $\mathcal{L}$-nonregular boundary points. Then $u(\delta x) = 0$ for every $z \in \Omega \setminus P$ and for every $\lambda \in]0, 1[$, that is

$$u = 0 \quad \text{in} \quad \Omega \setminus \cup_{0 \le \lambda \le 1} \delta_\lambda(P)$$

At this point, in order to complete the proof of the uniqueness theorem, we proved the following crucial results: *if P is any $\mathcal{L}$-polar subset of $\partial\Omega$, then $\Omega \setminus \cup_{0\le\lambda\le1}\delta_\lambda(P)$ has no interior points*. As a consquence, since u is continuous in Ω, we get $u \equiv 0$.

Existence. This part of the proof , even if not trivial, does not require particular devices. First of all, one proves that the Perron-Wiener function H_φ^Ω is a solution of (D_p) if φ is continuos. Then, by using a standard approximation argument, one shows that this also hold for every $\varphi \in L^p(\partial\Omega, \mu)$.

REFERENCES

[1] A. Bonfiglioli - C. Cinti, *A Poisson-Jensen type representation formula for subharmonic functions on stratified Lie groups,* Pot. Analysis, 22 (2005), pp. 151–169.

[2] A. Bonfigliolli - E. Lanconelli, *Dirichlet problem with L^p-boundary data and Hardy spaces on Carnot groups,* Preprint.

[3] G. Cimmino, *Nuovo tipo di condizione al contorno e nuovo metodo di trattazione per il problema generalizzato di Dirichlet,* Rend. Circolo Mat. Palermo, 61 (1937), pp. 177–221.

[4] L. Hörmander, *Hypoelliptic second-order differential equations,* Acta Math., 121 (1968), pp. 147–171.

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