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ON THE PRINCIPALLY POLARIZED ABELIAN VARIETIES WITH *M*-MINIMAL CURVES

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In this paper, we study principally polarized abelian varieties X of dimension g with a curve $\nu : C \to X$ such that the class of C is m times the minimal class. In [11], Welters introduced the formalism of complementary pairs to handle this problem in the case $m = 2$. We generalize the results of Welters and construct families of principally polarized abelian varieties for any m and compute the dimension of the locus of these abelian varieties.

1. Introduction

Let $g > 0$, we will denote by $\mathcal{A}_g$ the *moduli space of complex principally polarized abelian varieties* (ppav). In order to study the geometry of $\mathcal{A}_g$, a classical approach is to construct a stratification using moduli spaces of curves. The first stratum is the locus of Jacobians, which is isomorphic to the moduli space of curves by the Torelli theorem, which is contained in the closure of the Prym locus obtained from the space of unramified coverings of degree two. For example the space $\mathcal{A}_5$ is uniformized by the space of degree two unramified coverings of a genus 6 curve (see [5] and [2]). However, the Jacobian locus has dimension

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$3g-3$ and the Prym locus has dimension $3g$ while the space $\mathcal{A}_g$ has dimension $\frac{g(g+1)}{2}$. Thus, the Jacobian and Prym loci will provide a high codimensions subspaces when g becomes big.

Other constructions using spaces of coverings with more general groups of monodromy have been extensively studied (see [7] or [6]); recently this has lead to the uniformization of $\mathcal{A}_6$ (see [1]).

In order to stratify the moduli space $\mathcal{A}_g$, one can generalize the Jacobian and Prym loci. A Prym-Tjurin variety of exponent m is a principally polarized abelian variety (X,Θ) of dimension g such that X is an abelian subvariety of the Jacobian J_N of some genus g smooth projective curve N, and such that the restriction of the principal polarisation of J_N to X is equivalent to $m\Theta$. In particular, if $\nu : N \to X$ is the attached map, we have

$$\nu_*[N] = m.\frac{\Theta^{g-1}}{(g-1)!}.$$

We will often denote by $z = \frac{\Theta^{g-1}}{(g-1)!}$ the *minimal class*. Note that this class is not divisible.
The important point to remark is that, by results of Welters, the existence of $m-$minimal curves is related with the theory of Prym-Tjurin varieties, and moreover, any ppav is a Prym-Tjurin variety for a certain exponent m.

Complementary pairs. In [11], Welters proposed a slightly more general version of the Prym-Tjurin varieties. A ppav will be called m-minimal if there exists a curve $\nu : C \to X$ such that $\nu_*([C])$ is m times the minimal class; we will say that the curve C is m-minimal. The interesting point is that we can reformulate this property using the theory of *complementary pairs*. If X is ppav, we will denote by $\lambda_X : X \to \hat{X}$ the morphism induced by the polarization. Let N be a smooth curve. Then the following two sets of data are equivalent:

1. A ppav (X,Θ) and a morphism $\nu : N \to X$ such that $\nu_*([N]) = m\cdot z$.
2. An abelian subvariety B of JN and a finite subgroup $H \subset JN$, such that, if λ_B is the induced polarization on B by the principal polarization λ_N of JN then:
   (a) $\ker\lambda_B \subset H \subset B_m$ ($m-$torsion points of B).
   (b) H is maximal totally isotropic subgroup of B_m.

We describe a little bit more the construction from 2 to 1. Let N be a smooth curve and B a subvariety satisfying the conditions of 2. Let $A \subset JN$ be obtained by the dual exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{JN/B} & \longrightarrow & \widehat{JN} = JN & \longrightarrow & \widehat{B} & \longrightarrow & 0 \\ & & \uparrow \simeq & & \lambda_N \uparrow \simeq & & \uparrow \simeq & & \\ 0 & \longrightarrow & A & \longrightarrow & JN & \longrightarrow & JN/A & \longrightarrow & 0. \end{array}$$

Thus we get a couple (A,B) of abelian subvarieties of JN satisfying $A \cap B = \ker \lambda_B = \ker \lambda_A$. This pair is called a *complementary pair*. All the statements that follow will be valid if we exchange the roles of A and B.

We denote by $\mu_B : \hat{B} \to B$ the polarization dual to λ_B. We have $\lambda_B \circ \mu_B = [m]_B$ and the following sequence is exact

$$0 \to \ker \lambda_B \to B_m \to \ker \mu_B \to 0.$$

In particular, in the condition 2, we can replace the datum of H a maximal totally isotropic (m.t.i.) subgroup of B_m containing $\ker(\lambda_B)$ by the choice of a m.t.i. subgroup K of $\ker \mu_B$.

We denote by τ the natural isogeny from $A \times B \to JN$ which maps (a,b) to $a+b$. The kernel of τ is given by $\{(x,-x), x \in A \cap B\}$. We define $j : JN \to JN$ the unique morphism which makes the following diagram commutative

$$\begin{array}{ccc} JN & \xrightarrow{\ j\ } & JN \\ \tau \uparrow & & \uparrow \tau \\ A \times B & \xrightarrow[\begin{pmatrix} 1 & 0 \\ 0 & 1-m \end{pmatrix}]{} & A \times B. \end{array}$$

Then $A = \operatorname{Im}(j+m-1) = \ker(1-j)^0$ and $B = \operatorname{Im}(1-j) = \ker(j+m-1)^0$ (we can permute the roles of A and B by taking $j' = m-2-j$). The morphism j satisfies the classical Prym-Tjurin property $(j-1)(j-m+1) = 0$.

With this set-up, the variety X will be defined as $\widehat{B}/K \simeq (JN/A)/K$ and the map ν from N to X is given by the composition of maps

$$N \to JN \to JN/A \to (JN/A)/K.$$

The abelian variety X is principally polarized because K is totally isotropic. We denote by u the map from JN to X and by $u^t = \lambda_N^{-1} \circ u^* \circ \lambda_X$. Then we have $uu^t = [m]$. The class of N in JN is the minimal class of JN, therefore the class $\nu_*([N])$ is equal to m times the minimal class.

Case $m = 2$. We have three different constructions in the case $m = 2$.

1. Quotients of Jacobians. We take $B = JN$ and H is any maximal isotropic subgroup of JN_2.

2. Quotients of Prym varieties. Let $\pi : N \to N_0$ be a double covering. We denote by $\pi^* : JN_0 \to JN$ and by Nm_π (or simply Nm if the context is clear) the norm map associated to π. Moreover, the covering comes with an involution $\iota : N \to N$. Then by taking $B = \ker(\mathrm{Nm})_0$ (the neutral component) the Prym variety attached to π, $A = \pi^*(JN_0)$ and $j = \iota^*$ we get an m-minimal curve by choosing a m.t.i. subgroup of $\ker(\mu_B)$. In this case the map μ_B is the map $JN/\pi^*J_0 \to B$ induced by $1 - j$

3. Quotient of pull-back of Jacobians. This construction is obtained by exchanging the roles of A and B. In this case $B = \pi^*(JN_0)$ and μ_B is induced by $1 + j$. Now $\ker(\mu_B)$ is non-trivial if and only if π is unramified. In which case $\ker(\mu_B) \simeq (\mathbb{Z}/m\mathbb{Z})^2$ thus the maximal isotropic are isomorphic to $\mathbb{Z}/m\mathbb{Z}$ and X can be explicitely described.

Welters proved that any 2-minimal ppav is of one of the three above types.

Statement of the results. The current paper has two purposes. The first one is to compute the dimension of loci defined by Welters in the case $m = 2$.

Proposition 1.1. *The dimension of the locus of* 2*-minimial ppav arising from cases* (1) *and* (3) *is* $3g - 3$*, and for case* (2)*, the dimension is equal to the dimension of the Hurwitz space classifying double coverings. In particular the dimension of the locus of* 2*-minimal ppav in* $\mathcal{A}_g$ *is* $3g$ *(obtained from the Prym varieties of unramified coverings).*

The second purpose is to study the generalization of these three families of m-minimal ppav for $m \geq 3$. Two important questions about m-minimal ppav arise:

1. Compute the dimension of the locus of m-minimal varieties in $\mathcal{A}_g$.

2. Fixing m and g, what are the bound on the geometric genus of N if we assume that the image of N is irreducible? For $m = 1$, Matsusaka criterion (see [9]) implies that $g(N) = g$. For $m = 2$, the classification of 2-minimal curves (see [11]) implies that $g \leq g(N) \leq 2g+1$ and that these bounds can be realized. For general m, the lowest bound is g (and we will provide examples). However the existence (and the value) of an upper bound is still open even for $m = 3$. We expect this upper bound to take the form $g^+(g,m) = mg + P(m)$ where P is a polynomial.

We will prove the following

Theorem 1. *For any* m*, there exists* $(3g-3)$*-dimensional families of* m*-minimal ppav of dimension* g *such that the geometric genus of the curve is given by* g *(see Section 2) and* $mg - m + 1$ *(see Section 3).*

The dimension of these families should be far from the dimension of the locus of m-minimal ppav. However, these families provide an inequality $g^+(g,m) \geq mg - m + 1$ (which is close to the expected one for big values of g).

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2. Quotient of Jacobians

Let $\mathcal{M}_g$ be the moduli space of smooth curves of genus g. The *Torelli map* associates to an element $[C] \in \mathcal{M}_g$ the class of the Jacobian JC in $\mathcal{A}_g$. The Torelli map is injective. Moreover it is an immersion outside the locus of hyperelliptic curves.

Let C be a smooth curve of genus g. Let $K \subset JC_m$ be a m.t.i subgroup of JC with respect to the Riemann bilinear form on JC_m (the m torsion point of JC). Then the quotient JC/K is a ppav and the composition $C \hookrightarrow JC \to X$ gives an m-minimal curve of X.

Lemma 2.1. *For all $[C] \in \mathcal{M}_g$ the map $f : C \to X$ is generically injective.*

Proof. Assume that $f : C \to X$ is not generically injective. Denote by $\tilde{C}$ the reduced image of f in X. Then f is a covering (possibly ramified) on $\tilde{C}$, hence the genus of the normalisation of $\tilde{C}$ is strictly smaller than $g(C)$. On the other hand, by definition, $JC \longrightarrow X$ is an isogeny, it follows that the curve $\tilde{C}$ generates X (as a group). So the genus of the normalisation of $\tilde{C}$ is at least $dim(X)$, a contradiction. □

Let $g > 0$. Let $\delta = (d_1, \ldots, d_g)$ be a polarization type. We will denote by $\mathcal{A}_g^\delta$ the *moduli of abelian varieties with polarization of type* δ. We define the following moduli space

Definition 2.2. Let $g > 0$ and δ be a polarization type. For $m > 0$, we define $\mathcal{A}_{g,m}^\delta$ to be the moduli space of pairs (X,K) where X is an abelian variety of dimension g with polarization type δ and K is a m.t.i. of the group of m-torsion points.

The forgetful map $\mathcal{A}_{g,m}^\delta \to \mathcal{A}_g^\delta$ which maps (X,K) to X is étale. Thus we have $\dim(\mathcal{A}_{g,m}^\delta) = \frac{g(g+1)}{2}$. We have also

Proposition 2.3. *Let $f : \mathcal{A}^{\delta}_{g,m} \to \mathcal{A}_g$ be the map which sends a pair (X,K) to the quotient X/K. The map f is finite.*

Proof. Since the map is projective, it is sufficient to prove that the fibers are finite. Let (X,K) and (Y,L) be two elements of $\mathcal{A}^{\delta}_{g,m}$ such that $X/K \cong Y/L$. Then the composition $f : X \to X/K \to Y/L$ is an isogeny of degree $|K|$. Composing this with the inverse isogeny $Y/L \to Y$ ([3], Proposition 1.2.6) which has degree equals the $e(L)^{2g}/|L|$, where $e(L)$ is the exponent of the finite group L, we get an isogeny $X \to Y$. Moreover, this isogeny has degree $|K| \cdot e(L)^{2g}/|L|$. Since the cardinals of K and L are bounded above, and because that there are finitely many (modulo isomorphism) étale coverings of a given degree of any fixed abelian variety, it follows that, for a fixed (X,K), there exists a finite number of elements $(Y,L) \in \mathcal{A}^{\delta}_{g,m}$ such that $X/K \cong Y/L$. □

Applied to the principally polarized abelian varieties, the dimension of the locus of m-minimal abelian varieties defined by a pair (JC,K) is the same as the dimension of the Jacobian locus, thus $3g-3$.

3. Quotient of pull-back of Jacobians

Let $\pi\colon N \to N_0$ be a finite map of degree m between smooth projective curves N and N_0. Let $A = (\ker \mathrm{Nm})_0$ and $B = \pi^* JN_0$. Then (A,B) is a complementary pair in JN with $A \cap B \subset JN_m$. By Welters construction [11, Proposition 1.17], given a maximal totally isotropic subgroup K of $\ker \mu_B$, one obtains a ppav $X = B'/K$ (where $B' = JN/A \cong B^{\vee}$) together with a morphism $v\colon N \to X$ such that $v_*[N] = mz$, where

$$z = \Theta^{\dim X - 1}/(\dim X - 1)!$$

is the minimal class. The following proposition describes those ppav X obtained in this way such that the morphism $v\colon N \to X$ is birational onto its image.

Proposition 3.1. *With notations as above, the following statements hold.*

(a) *Suppose that m is prime. If the morphism $v\colon N \to X$ is birational onto its image, then $\pi\colon N \to N_0$ is an unramified cyclic cover. Moreover, let $\eta \in (JN_0)_m$ be the m-torsion line bundle on N_0 attached to π. Then $X \cong JN_0/[m]^{-1}\langle\eta\rangle$, where $\langle\eta\rangle$ is the subgroup generated by η in JN_0, and $[m]^{-1}\langle\eta\rangle$ is its preimage under the morphism $[m]\colon JN_0 \to JN_0$.*

(b) *Conversely, let m be any positive integer, η be any nontrivial m-torsion line bundle on N_0, and let $\pi\colon N = \mathbf{Spec}(\mathcal{O}_{N_0} \oplus \eta^{-1}) \to N_0$ be the unramified cyclic cover associated to η. Then there exists a maximal totally isotropic* *subgroup K of* $\ker \mu_B$ *such that* $v\colon N \to X$ *is birational onto its image and* $X \cong JN_0/[m]^{-1}\langle \eta \rangle$.

Before the proof, we first give an elementary algebra lemma.

Lemma 3.2. *Let V be a real vector space, and let $\Gamma \subset V$ be a lattice. Let ω be a non-degenerate alternating bilinear form on V which is integral on Γ. Let $\Lambda = \{v \in V \mid \omega(v,\gamma) \in \mathbb{Z}$ for all $\gamma \in \Gamma\}$. Assume that the quotient group $\bar{\Lambda} = \Lambda/\Gamma$ is isomorphic to $(\mathbb{Z}/m\mathbb{Z})^2$ for some positive integer m.*

(a) *If $\varphi\colon (\mathbb{Z}/m\mathbb{Z})^2 \to \bar{\Lambda}$ is a group isomorphism, then the subgroup $\langle \varphi(1,0) \rangle$ in $\bar{\Lambda}$ generated by the element $\varphi(1,0)$ is a maximal isotropic subgroup of $\bar{\Lambda}$ with respect to the alternating (multiplicative) bilinear form*

$$e^{2i\pi\omega(\cdot,\cdot)}\colon \bar{\Lambda} \times \bar{\Lambda} \to \mathbb{C}^*.$$

(b) *If m is prime, then the maximal isotropic subgroups of $\bar{\Lambda}$ with respect to $e^{2i\pi\omega(\cdot,\cdot)}$ are precisely all the nonzero cyclic subgroups of $\bar{\Lambda}$.*

Proof. (a) By [4, Proposition 6.1], there exists a basis $\{\gamma_1, \gamma_1', \ldots, \gamma_g, \gamma_g'\}$ for Γ such that $\omega(\gamma_i, \gamma_j) = \omega(\gamma_i, \gamma_j') = 0$ for all $i \neq j$, and if we let $m_i = \omega(\gamma_i, \gamma_i')$, then $m_1 \mid m_2 \mid \cdots \mid m_g$. One sees that $\{\gamma_1/m_1, \gamma_1'/m_1, \ldots, \gamma_g/m_g, \gamma_g'/m_g\}$ is a basis for Λ, and $\bar{\Lambda} \cong (\mathbb{Z}/m_1\mathbb{Z})^2 \times \cdots \times (\mathbb{Z}/m_g\mathbb{Z})^2$. Since $\bar{\Lambda} \cong (\mathbb{Z}/m\mathbb{Z})^2$ by assumption, $m_1 = \cdots = m_{g-1} = 1$ and $m_g = m$. Hence $\{\gamma_g/m, \gamma_g'/m\}$ is a basis for $\bar{\Lambda}$. Since $\{\varphi(1,0), \varphi(0,1)\}$ is also a basis for $\bar{\Lambda}$, there exist integers a,b,c,d such that

$$\varphi(1,0) = a\gamma_g/m + b\gamma_g'/m, \quad \varphi(0,1) = c\gamma_g/m + d\gamma_g'/m,$$

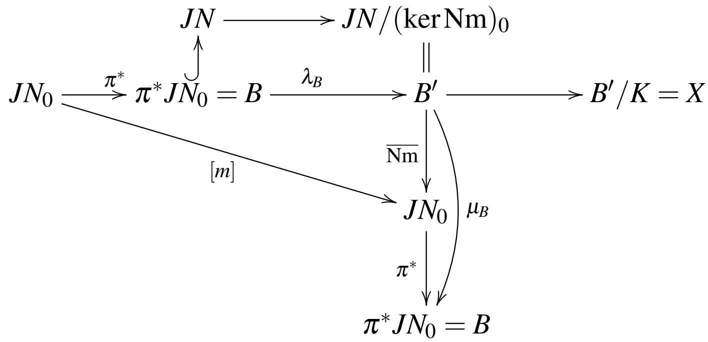
and $ad - bc$ is relatively prime to m. Suppose that there exists a subgroup $\langle \varphi(1,0) \rangle \subsetneq G \subset \bar{\Lambda}$ which is isotropic with respect to $e^{2i\pi\omega(\cdot,\cdot)}$. Since $G \supsetneq \langle \varphi(1,0) \rangle$, G contains $\varphi(0,n)$ for some positive integer $n \mid m$, $n \neq m$. Since G is isotropic with respect to $e^{2i\pi\omega(\cdot,\cdot)}$, $\omega(\varphi(1,0), \varphi(0,n)) \in \mathbb{Z}$. But

$$\omega(\varphi(1,0), \varphi(0,n)) = \omega(a\gamma_g/m + b\gamma_g'/m, nc\gamma_g/m + nd\gamma_g'/m) = n(ad-bc)/m,$$

which is not an integer and thus a contradiction.

(b) If m is prime, then $\bar{\Lambda} \cong (\mathbb{Z}/m\mathbb{Z})^2$ is a two-dimensional vector space over the finite field $\mathbb{Z}/m\mathbb{Z}$, and the subgroups of $\bar{\Lambda}$ coincide with its vector subspaces. Hence proper nonzero subgroups of $\bar{\Lambda}$ are the same as its one-dimensional vector subspaces, which are the same as its nonzero cyclic subgroups, and these are all obviously isotropic. □

Proof of Proposition 3.1. We have the following commutative diagram of morphisms between abelian varieties. All the maps with labels, as well as the quotient map $B' \to B'/K = X$, are isogenies. The central vertical morphism $\overline{\mathrm{Nm}}$ is induced from the norm map $\mathrm{Nm}\colon JN \to JN_0$. From the obvious fact $\mathrm{Nm} \circ \pi^* = m$, we get the commutativity of the central triangle $\overline{\mathrm{Nm}}\lambda_B\pi^* = [m]$. This in turn implies that $\pi^*\overline{\mathrm{Nm}}\lambda_B\pi^* = \pi^*[m] = [m]\pi^*$, and since π^* is an isogeny, $\pi^*\overline{\mathrm{Nm}}\lambda_B = [m]$. Hence $\pi^*\overline{\mathrm{Nm}} = \mu_B$ by the definition of μ_B.



The central horizontal row of the diagram gives $X \cong JN_0/(\lambda_B\pi^*)^{-1}(K)$. Also from the diagram we have $(\lambda_B\pi^*)^{-1}(K) = [m]^{-1}\big(\overline{\mathrm{Nm}}(K)\big)$. Hence

$$X \cong JN_0/[m]^{-1}\big(\overline{\mathrm{Nm}}(K)\big).$$

By [3, Proposition 11.4.3], if m is prime then

$$\ker(\pi^*\colon JN_0 \to JN) = \begin{cases} \langle \eta \rangle, & \text{if } \pi\colon N = \mathbf{Spec}(\mathcal{O}_{N_0} \oplus \eta^{-1}) \to N_0 \text{ is an unramified cyclic cover defined by } \eta \in (JN_0)_m; \\ 0, & \text{otherwise.} \end{cases}$$

If $\ker(\pi^*\colon JN_0 \to JN) = 0$, then $\ker(\mathrm{Nm}\colon JN \to JN_0)$ is connected since $\pi^* = \mathrm{Nm}^\vee$. This means that the two vertical maps $\overline{\mathrm{Nm}}\colon B' \to JN_0$ and $\pi^*\colon JN_0 \to \pi^*JN_0$ in the commutative diagram are isomorphisms, so their composition μ_B is also an isomorphism. Hence $K = \ker \mu_B = 0$. It follows that $X = B' \cong JN_0$, and the map $v\colon N \to X$ is the composition of the Abel-Jacobi map $N \hookrightarrow JN$ followed by the norm map $\mathrm{Nm}\colon JN \to JN_0 = X$. But this v is clearly non birational onto its image since it factorizes through $\pi : N \to N_0$. Hence if m is prime and v is birational onto its image, then $\pi\colon N \to N_0$ must be an unramified cyclic cover.

Let m be any positive integer, η be any nontrivial m-torsion line bundle on N_0, and let $\pi\colon N = \mathbf{Spec}(\mathcal{O}_{N_0} \oplus \eta^{-1}) \to N_0$ be the unramified cyclic cover associated to η. Then there is an automorphism σ of N of order m such that $N_0 = N/\langle \sigma \rangle$. The kernel of the norm map $\mathrm{Nm}\colon JN \to JN_0$ has m connected components $P_0, \ldots, P_{m-1}$, where

$$P_\ell = \{\mathcal{O}_N\big(\textstyle\sum n_i(x_i - \sigma(x_i))\big) \mid x_i \in N,\ n_i \in \mathbb{Z},\ \sum n_i \equiv \ell \pmod{m}\}.$$

Hence $\ker(\overline{Nm}\colon B' = JN/P_0 \to JN_0) = \{P_0, \dots, P_{m-1}\}$ is a cyclic group of order m generated by P_1. Since $\mu_B = \pi^*\overline{\mathrm{Nm}}$, and $\ker \pi^* = \langle \eta \rangle$ is also a cyclic group of order m, $\ker \mu_B \cong (\mathbb{Z}/m\mathbb{Z})^2$. More precisely, pick any $\xi \in JN$ such that $\mathrm{Nm}(\xi) = \eta$, and denote by $\bar{\xi}$ its image in $B' = JN/P_0$. Then an isomorphism $(\mathbb{Z}/m\mathbb{Z})^2 \to \ker \mu_B$ can be defined by sending $(1,0)$ to $\bar{\xi}$ and $(0,1)$ to P_1.

Let K be a maximal totally isotropic subgroup of $\ker \mu_B$. We want to find the conditions such that the morphisms $v\colon N \to X$ is birational onto its image. Recall that $v\colon N \to X$ is the composition

$$v\colon N \hookrightarrow JN \to JN/P_0 = B' \to B'/K = X$$

of the Abel-Jacobi map $N \hookrightarrow JN$ followed by the quotient maps. So for two distinct points $x, y \in N$, since $K \subset \ker \mu_B = \langle \bar{\xi}, P_1 \rangle$, we have

$$\begin{aligned} v(x) = v(y) &\implies \mathcal{O}_N(x-y) \in \langle P_0, \xi, P_1 \rangle \text{ in } JN \\ &\implies \mathcal{O}_{N_0}\big(\pi(x) - \pi(y)\big) = \mathrm{Nm}\big(\mathcal{O}_N(x-y)\big) \in \langle \eta \rangle \\ &\implies \mathcal{O}_N\big(\pi^{-1}(\pi(x)) - \pi^{-1}(\pi(y))\big) = \pi^*\mathcal{O}_{N_0}\big(\pi(x) - \pi(y)\big) = 0. \end{aligned}$$

If v is not birational onto its image, then for all but finitely many points $x \in N$, there exists $y \in N$ different from x such that the divisors $\pi^{-1}(\pi(x))$ and $\pi^{-1}(\pi(y))$ are linear equivalent. If $\pi(x) \neq \pi(y)$ then $\pi^{-1}(\pi(x))$ and $\pi^{-1}(\pi(y))$ generate a g^1_m on N. This can only happen for at most finitely many x, for otherwise the g^1_m consists of fibers of $\pi\colon N \to N_0$ and $N_0 \cong \mathbb{P}^1$. Hence $\pi(x) = \pi(y)$, i.e. $y = \sigma^\ell(x)$ for some $\ell \in \mathbb{Z}/m\mathbb{Z}$. Since

$$\begin{aligned} \mathcal{O}_N(x-y) = \mathcal{O}_N\big(x - \sigma^\ell(x)\big) &= \mathcal{O}_N\Big(\sum_{i=0}^{\ell-1} \sigma^i(x) - \sigma^{i+1}(x)\Big) \\ &= \mathcal{O}_N\Big(\sum_{i=0}^{\ell-1} \sigma^i(x) - \sigma\big(\sigma^i(x)\big)\Big) \in P_\ell, \end{aligned}$$

the image $\overline{\mathcal{O}_N(x-y)}$ of $\mathcal{O}_N(x-y)$ in $B' = JN/P_0$ is equal to $P_\ell = \ell P_1$. It follows that $v(x) = v(y)$ in $X = B'/K$ if and only if $\ell P_1 \in K$ in B'. Hence

$$v \text{ is birational onto its image} \iff \ell P_1 \notin K \text{ for all } \ell \neq 0.$$

To finish the proof of Part (a), if m is prime, maximal isotropic subgroups K of $\ker \mu_B$ are cyclic by Lemma 3.2. So if moreover v is birational onto its image, then $K = \langle a\bar{\xi} + bP_1 \rangle$ for some integers a and b with $m \nmid a$, and hence

$$X \cong JN_0/[m]^{-1}\big(\overline{\mathrm{Nm}}(K)\big) = JN_0/[m]^{-1}\langle \eta \rangle.$$

As for Part (b), given any integer $m > 0$ and any nontrivial m-torsion line bundle η on N_0, pick any $\xi \in JN$ such that $\mathrm{Nm}(\xi) = \eta$, and let $K = \langle \bar{\xi} \rangle$. Then K is a maximal totally isotropic subgroup of $\ker \mu_B$ by Lemma 3.2, and this choice of K makes $v\colon N \to X$ birational onto its image and X isomorphic to $JN_0/[m]^{-1}\langle \eta \rangle$. □

If N_0 is a curve of genus g, and $\pi\colon N \to N_0$ is an unramified cyclic cover of degree m, then the genus of N is $mg - m + 1$. So Proposition 3.1 implies the following

Corollary 3.3. *For any integers $m > 0$ and $g > 1$, the locus of ppav with an m-minimal curve of genus $mg - m + 1$ is at least $3g - 3$.*

Proof. Let $\mathcal{M}_g^{1/m}$ be the moduli space of pairs (N_0, L) where N_0 is a smooth curve of genus g and L is a line bundle such that $L^{\otimes m} \simeq \mathcal{O}_C$. The datum of a point in $\mathcal{M}_g^{1/m}$ is equivalent to the datum of a cyclic étale covering of N_0 of degree d. The space $\mathcal{M}_g^{1/m}$ is of dimension $3g - 3$.

The moduli space has several connected components indexed by the divisors of m: indeed, if $d|m$ we have a natural closed inclusion of $\mathcal{M}_g^{1/d}$ into $\mathcal{M}_g^{1/m}$ corresponding to étale coverings which factors through a covering of degree d.

Let $\tilde{\mathcal{M}}_g^{1/m}$ be the connected component corresponding to pairs (N_0, L) such that: $L^{\otimes d} \neq \mathcal{O}_{N_0}$ for all $d|m$ and $d \neq m$. We recall that the Torelli map is generically an immersion. Therefore the composition of maps from $\tilde{\mathcal{M}}_g^{1/m}$ to $\mathcal{A}_{g,(1,m,\dots,m)}$

$$(N_0, L) \mapsto (JN_0, L) \mapsto JN_0/\langle L \rangle = \pi^* JN_0$$

keeps the dimensions. The dual map $A \to \hat{A}$ from $\mathcal{A}_{g,(1,m,\dots,m)}$ to $\mathcal{A}_{g,(1,\dots,1,m)}$ is étale. Therefore the map which associates JN/P to $(N_0, L) \in \tilde{\mathcal{M}}_g^{1/m}$ keeps the dimension. Finally using Proposition 2.3, the map which associates to (X, K) the ppav X/K where K is a m.t.i. subgroup of X_m is finite.

Therefore the locus of abelian varieties with an m-minimal curve of genus $mg - m + 1$ is at least $3g - 3$. □

4. Quotient of Prym varieties

The last category of examples has been extensively studied. Let r be a positive integer such that $m|r$. We denote $\mathcal{M}_g^{1/m}$ be the moduli space of objects $(C, x_1, \dots, x_r, L)$ where C is smooth curves with markings and L is line bundle satisfying $L^{\otimes m} \simeq \mathcal{O}_C(-x_1 \dots - x_r)$: this is also the moduli space of simply ramified cyclic coverings of a curve of genus g.

Let $(N_0 \to N)$ be a point in $\overline{\mathcal{M}}_{g,r}^{1/m}$, then the Prym variety associated to this map is equal to $(\ker \mathrm{Nm})_0$. Let K be a m.t.i. subgroup of the kernel $\ker(\mu_{\hat{P}})$, where $\mu_{\hat{P}}$ is the induced polarization on the dual of P. The quotient $\hat{P}/K$ is ppav with m-minimal curve. In [8] the authors proved that the map $\overline{\mathcal{M}}_{g,r}^{1/m} \to A_{m(g-1)+1+r/2,\delta}$ is generically finite if (g,m,r) are of the following types:

- $g \geq 2$ and $r \geq 6$ for m even or $r \geq 7$ for m odd,
- $g \geq 3$ and $m = r = 4$ or 5 or $(m,r) = (2,4)$ or $(3,6)$,
- $g \geq 5$ and $m = r = 2$ or 3.

Because the dual map and the quotient by m.t.i. keeps the dimensions we have that the locus obtained by this construction is of the same dimension as $\overline{\mathcal{M}}_{g,r}^{1/m}$ in all above cases.

In particular the dimension of the locus of m-minimal abelian varieties in $\mathcal{A}_g$ that we obtain with this construction is at most $2(g-1+m)-3$. Moreover, with this construction, the genus of an m-minimal curve is at most $2g+1$.

REFERENCES

[1] Valery Alexeev, Ron Donagi, Gavril Farkas, Elham Izadi and Angela Ortega. *The uniformization of the moduli space of principally polarized abelian 6-folds.* 2015, arXiv:1507.05710.

[2] Arnaud Beauville. *Prym varieties and the Schottky problem.* Invent. Math., 41(2):149–196, 1977.

[3] Christina Birkenhake and Herbert Lange. *Complex abelian varieties*. 2nd edition, Springer-Verlag Berlin Heidelberg, 2004.

[4] Olivier Debarre. *Complex tori and abelian varieties*. Volume 11, AMS, 2005.

[5] Ron Donagi. *The unirationality of* $\mathcal{A}_5$. Ann. of Math. 119, 269-307 (1984).

[6] Vassil Kanev. *Principal polarizations of Prym-Tjurin varieties.* Compositio Math., 64(3):243–270, 1987.

[7] Vassil Kanev. *Hurwitz spaces of Galois coverings of* $\mathbb{P}^1$ *whose Galois groups are Weyl groups.* J. Algebra, 305(1):442–456, 2006.

[8] Herbert Lange and Angela Ortega. *Prym varieties of cyclic coverings.* 2010, arXiv:0805.1020.

[9] Teruhisa Matsusaka. *On a characterization of a Jacobian variety.* Mem. College Sci. Univ. Kyoto Ser. A Math. 1–19n, 1959.

[10] David Mumford. *Abelian varieties.* volume 5,Tata Institute of Fundamental Research, Bombay; by Hindustan Book Agency, New Delhi, 2008.

[11] G. E. Welters. *Curves of twice the minimal class on principally polarized abelian varieties.* Nederl. Akad. Wetensch. Indag. Math., 49(1):87–109, 1987.

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