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ISOGENIES OF PRYM VARIETIES

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We prove an extension of the Babbage-Enriques-Petri theorem for semi-canonical curves. We apply this to show that the Prym variety of a generic element of a codimension k subvariety of $\mathcal{R}_g$ is not isogenous to another distinct Prym variety, under some mild assumption on k.

1. Introduction

Let $\mathcal{R}_g$ denote the moduli space of unramified irreducible double covers of complex smooth curves of genus g. Given an element $\pi : D \to C$ in $\mathcal{R}_g$, we can lift this morphism to the corresponding Jacobians via the norm map

$$\mathrm{Nm}_\pi : J(D) \to J(C).$$

By taking the neutral connected component of its kernel, we obtain an abelian variety of dimension $g-1$ called the *Prym variety* attached to π.

In this note, we study the isogeny locus in $\mathcal{A}_{g-1}$ of Prym varieties attached to generic elements in $\mathcal{R}_g$; that is, principally polarized abelian varieties of dimension $g-1$ which are isogenous to such Prym varieties. More concretely, given a subvariety $\mathcal{Z}$ of $\mathcal{R}_g$ of codimension k and a generic element $\pi : D \to C$ in $\mathcal{Z}$, we prove that the Prym variety attached to π is not isogenous to a distinct Prym variety, whenever $g \geq \max\{7, 3k+5\}$, see Theorem 3.2.

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This result can be seen as an extension of the analogue statements for Jacobians of generic curves proven by Bardelli and Pirola [1] for the case $k = 0$, and Marcucci, Naranjo and Pirola [2] for $k > 0$, $g \geq 3k+5$ or $k = 1$ and $g \geq 5$. In the latter, to prove the case $g \geq 3k+5$, they use an argument on infinitesimal variation of Hodge structure proposed by Voisin in [1, Remark (4.2.5)] which allows them to translate the question to a geometric problem of intersection of quadrics. In doing so, they give a generalization of Babbage-Enriques-Petri's theorem which allows them to recover a canonical curve from the intersection of a system of quadrics in $\mathbb{P}^{g-1}$ of codimension k. The strategy we follow to prove Theorem 3.2 is an adaptation of these techniques to the setting of Prym varieties. We are also able to give an extension of Babbage-Enriques-Petri's theorem for semicanonical curves in a similar fashion as in [2], see Proposition 2.2. Our result generalises the one by Lange and Sernesi [3] for curves of genus $g \geq 9$, since it recovers a semicanonical curve of genus $g \geq 7$ from a system of quadrics in $\mathbb{P}^{g-2}$ of codimension k, $g \geq 3k+5$.

2. Intersection of quadrics

Let C be a smooth curve. Given a globally generated line bundle $L \in \mathrm{Pic}(C)$, we denote by $\varphi_L : C \to \mathbb{P}\mathrm{H}^0(C,L)^*$ its induced morphism. If L is very ample, we say that $\varphi_L(C)$ is *projectively normal* if its homogeneous coordinate ring is integrally closed; or equivalently, if for all $k \geq 0$, the homomorphism

$$\mathrm{Sym}^k \mathrm{H}^0(L) \longrightarrow \mathrm{H}^0(L^{\otimes k})$$

is surjective.

We also recall that the *Clifford index* of C is defined as

$$\min\{\deg(L) - 2\mathrm{h}^0(C,L) + 2\},$$

where the minimum ranges over the line bundles $L \in \mathrm{Pic}(C)$ such that $\mathrm{h}^0(C,L) \geq 2$ and $\mathrm{h}^0(C, \omega_C \otimes L^{-1}) \geq 2$. Its value is an integer between 0 and $\lfloor \frac{g-1}{2} \rfloor$, where g is the genus of the curve.

Let C be of genus g and with Clifford index c. For any non-trivial 2-torsion point η in the Jacobian of C, we call $\omega_C \otimes \eta$ a *semicanonical line bundle* of C whenever it is globally generated, and we denote by $\varphi_{\omega_C \otimes \eta} : C \to \mathbb{P}^{g-2}$ its associated morphism. In that case, we call its image $C_\eta := \varphi_{\omega_C \otimes \eta}(C)$ a *semicanonical curve*. The following is a result of Lange and Sernesi [3], and Lazarsfeld [4]:

Lemma 2.1. *If $g \geq 7$ and $c \geq 3$, then $\omega_C \otimes \eta$ is very ample and the semicanonical curve C_η is projectively normal.*

Furthermore, Lange and Sernesi prove that C_η is the only non-degenerate curve in the intersection of all quadrics in $\mathbb{P}^{g-2}$ containing C_η if $c > 3$, or $c = 3$ and $g \geq 9$, see [3]. The following proposition generalises this result for a smaller family of quadrics.

Proposition 2.2. *Let C be a curve of genus g and Clifford index c, and η be a non-trivial 2-torsion point in $J(C)$. Let $I_2(C_\eta) \subset \operatorname{Sym}^2 \mathrm{H}^0(C, \omega_C \otimes \eta)$ be the vector space of equations of the quadrics containing C, and $K \subset I_2(C_\eta)$ be a linear subspace of codimension k. If $g \geq \max\{7, 2k+6\}$ and $c \geq \max\{3, k+2\}$, then C_η is the only irreducible non-degenerate curve in the intersection of the quadrics of K.*

Notice that for $k = 0$, this proposition extends the result of Lange and Sernesi [3] to the cases when $c = 3$ and $g = 7$ and 8. We refer to Remark 2.3 for a brief discussion on a simplified version of the following proof in this case.

Proof. We start by assuming that there exists an irreducible non-degenerate curve C_0 in the intersection of quadrics $\bigcap_{Q \in K} Q \subset \mathbb{P}\mathrm{H}^0(C, \omega_C \otimes \eta)^*$, which is different from C_η. In particular, we can choose $k+1$ linearly independent points in $\bigcap_{Q \in K} Q$ such that $x_i \notin C_\eta$ for all i. By abuse of notation, we denote also as x_i the representatives in $\mathrm{H}^0(C, \omega_C \otimes \eta)^*$. We define $L \subset \operatorname{Sym}^2 \mathrm{H}^0(C, \omega_C \otimes \eta)^*$ as the linear subspace spanned by $x_i \otimes x_i$.

Let $R = I_2(C_\eta)/K$ and $R' = \operatorname{Sym}^2 \mathrm{H}^0(C, \omega_C \otimes \eta)/K$. By Lemma 2.1 and the fact that $g \geq 7$ and $c \geq 3$, we have that C_η is projectively normal. Hence, we can build the following diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K & \longrightarrow & I_2(C_\eta) & \longrightarrow & R & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \operatorname{Sym}^2 \mathrm{H}^0(C, \omega_C \otimes \eta) & = & \operatorname{Sym}^2 \mathrm{H}^0(C, \omega_C \otimes \eta) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 \longrightarrow R & \longrightarrow & R' & \longrightarrow & \mathrm{H}^0(C, \omega_C^{\otimes 2}) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

where the last row is obtained by applying the snake lemma to the first two rows.

By dualizing this diagram, we get

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathrm{H}^0(C,\omega_C^{\otimes 2})^* = \mathrm{H}^1(C,T_C) & \longrightarrow & R'^* & \longrightarrow & R^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathrm{Sym}^2\,\mathrm{H}^0(C,\omega_C\otimes\eta)^* & = & \mathrm{Sym}^2\,\mathrm{H}^0(C,\omega_C\otimes\eta)^* & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 \longrightarrow R^* & \longrightarrow & I_2(C_\eta)^* & \longrightarrow & K^* & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Notice that $Q(\alpha)=0$ for every $\alpha\in L$ and every $Q\in K$. Therefore, $L\subset R'^*$ Since $\dim(L)=k+1$ and $\dim(R)=k$, there is a non-trivial element $\alpha\in L\cap \mathrm{H}^1(C,T_C)$. By the isomorphism $\mathrm{H}^1(C,T_C)\simeq \mathrm{Ext}^1(\omega_C,\mathcal{O}_C)$, there is a 2 vector bundle E_α associated to α satisfying the following exact sequence:

$$0\longrightarrow \mathcal{O}_C\longrightarrow E_\alpha\longrightarrow \omega_C\longrightarrow 0. \tag{1}$$

The cup product with α is the coboundary map $\mathrm{H}^0(C,\omega_C)\to \mathrm{H}^1(C,\mathcal{O}_C)$. By writing the element $\alpha=\sum_{i=1}^{k+1} a_i x_i\otimes x_i$, we have

$$\mathrm{Ker}(\cdot\cup\alpha)=\bigcap_{i\,|\,a_i\neq 0} H_i,$$

where $H_i=\mathrm{Ker}(x_i)$. After reordering, we may assume that $x_1,\dots,x_{k'}$ are the points such that $a_i\neq 0$, for some $k'\leq k+1$. This means that there are $g-k'$ linearly independent sections in $\mathrm{H}^0(C,\omega_C)$ lifting to $\mathrm{H}^0(C,E_\alpha)$. Denote by $W\subset \mathrm{H}^0(C,E_\alpha)$ the vector space generated by these sections, and consider the morphism $\psi:\wedge^2 W\to \mathrm{H}^0(C,\omega_C)$ obtained by the following composition:

$$\wedge^2 W\longrightarrow \wedge^2\mathrm{H}^0(C,E_\alpha)\longrightarrow \mathrm{H}^0(C,\det E_\alpha)=\mathrm{H}^0(C,\omega_C).$$

The kernel of ψ has codimension at most g, and the Grassmannian of the decomposable elements in $\mathbb{P}(\wedge^2 W)$ has dimension $2(g-k'-2)$. Since $g>2k+5$ by hypothesis, we have that their intersection is not trivial. Thus, take $s_1,s_2\in \mathrm{H}^0(C,E_\alpha)$ such that $\psi(s_1\wedge s_2)=0$. They generate a line bundle $M_\alpha\subset E_\alpha$ and $h^0(C,M_\alpha)\geq 2$. Take Q_α the neutral component of the quotient E_α/M_α, and L_α the kernel of $E_\alpha\to Q_\alpha$, then we obtain the following exact sequence:

$$0\longrightarrow L_\alpha\longrightarrow E_\alpha\longrightarrow Q_\alpha\longrightarrow 0. \tag{2}$$

Notice that $M_\alpha \subset L_\alpha$, hence $h^0(C, L_\alpha) \geq 2$. Moreover from (1) and (2), we obtain $\omega_C \simeq \det E_\alpha \simeq L_\alpha \otimes Q_\alpha$, which implies that $Q_\alpha \simeq \omega_C \otimes L_\alpha^{-1}$. We have the following diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & L_\alpha & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{O}_C & \longrightarrow & E_\alpha & \longrightarrow & \omega_C & \longrightarrow & 0 \\
 & & & \searrow & \downarrow & & & & \\
 & & & & \omega_C \otimes L_\alpha^{-1} & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

Assume that $\mathcal{O}_C \to \omega_C \otimes L_\alpha^{-1}$ is 0. Then the section of E_α that represents $\mathcal{O}_C \to E_\alpha$ would be a section of L_α, in particular, a section in W. Since the sections in W map to sections of ω_C, this contradicts the exactness of the horizontal sequence. So $\mathcal{O}_C \to \omega_C \otimes L_\alpha^{-1}$ is not 0 and the $h^0(C, \omega_C \otimes L_\alpha^{-1}) > 0$.

If $h^0(C, \omega_C \otimes L_\alpha^{-1}) \geq 2$, we have that

$$c \leq \deg(L_\alpha) - 2h^0(C, L_\alpha) + 2. \tag{3}$$

Moreover, $h^0(C, L_\alpha) + h^0(C, \omega_C \otimes L_\alpha^{-1}) \geq h^0(C, E_\alpha) > \dim(W) = g - k'$ and, using Riemann-Roch we obtain that $2h^0(C, L_\alpha) \geq \deg(L_\alpha) + 2 - k'$. Combining this with (3), we obtain that $c \leq k' \leq k+1$ which contradicts our hypothesis on c ($c \geq k+2$). Hence, $h^0(C, \omega_C \otimes L_\alpha^{-1}) = 1$.

Write $\omega_C \otimes L_\alpha^{-1} \simeq \mathcal{O}_C(p_1 + \cdots + p_e)$, where $e = \deg(\omega_C \otimes L_\alpha^{-1})$. Notice that $h^0(C, L_\alpha) \geq g - k'$ and $\deg(L_\alpha) = 2g - 2 - e$. Using Riemann-Roch, we get

$$g - k' \leq h^0(C, L_\alpha) = h^0(C, \omega_C \otimes L_\alpha^{-1}) + 2g - 2 - e - (g-1) = g - e.$$

So $e \leq k'$.

By (2), we have that $L_\alpha \simeq \omega_C(-p_1 - \cdots - p_e)$. Moreover, the sections of L_α lie in W, and by construction of W we have that $\mathrm{H}^0(\omega_C(-p_1 - \cdots - p_e)) \subset \operatorname{Ker}(\cdot \cup \alpha) = \cap_{i|a_i \neq 0} H_i$. Therefore, by dualizing this inclusion, we obtain that

$$\langle x_1, \ldots, x_{k'} \rangle_{\mathbb{C}} \subset \langle p_1, \ldots, p_e \rangle_{\mathbb{C}}. \tag{4}$$

Let $\gamma : N_0 \to C_0$ be a normalization. For any generic choice of $k+1$ points $x_i \in N_0$, we can repeat the construction above for $\gamma(x_1), \ldots, \gamma(x_{k+1})$, and we can assume that k' and e are constant. We define the correspondence

$$\Gamma = \Big\{ (x_1 + \cdots + x_{k'}, p_1 + \cdots + p_e) \in N_0^{(k')} \times C_\eta^{(e)}, \\ \text{such that } \langle \gamma(x_1), \ldots, \gamma(x_{k'}) \rangle_{\mathbb{C}} \subset \langle p_1, \ldots, p_e \rangle_{\mathbb{C}} \Big\}.$$

Observe that Γ dominates $N_0^{(k')}$, so $e \leq k' \leq \dim \Gamma$. In addition, the second projection $\Gamma \to C_\eta^{(e)}$ has finite fibers, since both curves are non-degenerate. This implies that $\dim \Gamma \leq e$, and so we have $k' = e$. Since $k' \leq k+1 \leq g-3$, by the uniform position theorem we have that the rational maps

$$C^{(k')} \dashrightarrow Sec^{(k')}(C_\eta) \subset \mathbb{G}(e-1, \mathbb{P}^{g-2}),$$
$$N_0^{(k')} \dashrightarrow Sec^{(k')}(N_0) \subset \mathbb{G}(e-1, \mathbb{P}^{g-2}),$$

are generically injective. This gives a birational map between $C_\eta^{(k')}$ and $N_0^{(k')}$. In particular, it induces dominant morphisms $JC_\eta \to JN_0$ and $JN_0 \to JC_\eta$. Therefore, $g(C_\eta) = g(N_0)$ and by a theorem of Ran [5], the birational map $C_\eta^{(k')} \dashrightarrow N_0^{(k')}$ is defined by a birational map between C_η and N_0. By composing it with the normalization map γ, we obtain a birational map

$$\varphi : C_\eta \dashrightarrow C_0,$$

that defines the correspondence Γ; that is $\langle \varphi(x_1), \ldots, \varphi(x_{k'}) \rangle = \langle x_1, \ldots, x_{k'} \rangle$ for generic elements $x_1 + \ldots + x_{k'} \in C_\eta^{(k')}$. This implies that φ is generically the identity map over C_η. Thus $C_\eta = C_0$, which is a contradiction and ends the proof. □

Remark 2.3. The proof of Corollary 3.1 can be simplified for the case $K = I_2(C_\eta)$, that is $k = 0$. Under this assumption, we only consider one point $x \notin C_\eta$, and $k' = e = 1$. Therefore, the inclusion (4) already implies the equality $C_\eta = C_0$.

3. Main theorem

An element in $\mathcal{R}_g$ can be identified with a pair (C, η), where C is a complex smooth curve of genus g, and η is a non-trivial 2-torsion element in the Jacobian of C. This allows us to consider $\mathcal{R}_g$ as a covering of the moduli space $\mathcal{M}_g$ of complex smooth curves of genus g. It is given by the morphism

$$\mathcal{R}_g \longrightarrow \mathcal{M}_g, \quad (C, \eta) \longmapsto C,$$

which has degree $2^{2g}-1$. Thus, a generic choice of an element in a subvariety $\mathcal{Z}\subset\mathcal{R}_g$ is equivalent to a generic choice of a curve C in the image of $\mathcal{Z}$ in $\mathcal{M}_g$, and any non-trivial element $\eta\in J(C)[2]$.

The following result is a direct consequence of Proposition 2.2 and it is the version of Babbage-Enriques-Petri's theorem that we use in the proof of the main result in this article.

Corollary 3.1. *Let (C,η) be a generic point in a subvariety $\mathcal{Z}$ of $\mathcal{R}_g$ of codimension k. Let $I_2(C_\eta)\subset \operatorname{Sym}^2 \mathrm{H}^0(C,\omega_C\otimes\eta)$ be the vector space of the equations of quadrics in $\mathbb{P}^{g-2}$ containing C_η. Let $K\subset I_2(C_\eta)$ be a linear subspace of codimension k. If $g\geq \max\{7,3k+5\}$, then C_η is the only irreducible non-degenerate curve in the intersection of the quadrics of K.*

Proof. Let $\mathcal{M}_g^c$ be the locus in $\mathcal{M}_g$ corresponding to curves with Clifford index c. Then $\mathcal{M}_g^c$ is a finite union of subvarieties of $\mathcal{M}_g$, where the one of higher dimension corresponds to the curves whose Clifford index is realized by a g^1_{c+2} linear series, see [7]. By Riemann-Hurwitz, the codimension in $\mathcal{M}_g$ of the component of the curves with a g^1_{c+2} linear series is

$$3g-3-(2g-2c+2-3)=g-2c-2.$$

If $k=0$, a generic curve in $\mathcal{M}_g$ has Clifford index $c\geq 3$, because $g\geq 7$. As when $k>0$, since $g\geq 3k+5$, we obtain

$$k\geq g-2c-2\geq 3k+5-2c-2=3k-2c+3,$$

and thus $c\geq k+2$. The corollary follows by applying Proposition 2.2. □

Let $\widetilde{\mathcal{A}_g}^m$ be the space of isogenies of principally polarized Abelian varieties of degree m (up to isomorphism); that is the space of classes of isogenies $\chi: A\longrightarrow A'$ such that $\chi^*L_{A'}\cong L_A^{\otimes m}$, where L_A (respectively $L_{A'}$) is a principal polarization on A (respectively A'). There are two forgetful maps to the moduli space $\mathcal{A}_g$ of p.p.a.v. of dimension g

$$\begin{array}{ccc} & \widetilde{\mathcal{A}_g}^m & \\ \varphi\swarrow & & \searrow\psi \\ \mathcal{A}_g & & \mathcal{A}_g, \end{array} \tag{5}$$

such that $\varphi(\chi)=(A,L_A)$ and $\psi(\chi)=(A',L_{A'})$. These maps yield the following commutative diagram,

$$\begin{array}{ccc} & T_{[\chi]}\widetilde{\mathcal{A}_{g-1}}^{m} & \\ {}^{d\varphi}\swarrow & & \searrow^{d\psi} \\ T_{[A]}\mathcal{A}_{g-1} & \xrightarrow{\quad\lambda\quad} & T_{[A']}\mathcal{A}_{g-1}. \end{array} \tag{6}$$

where all maps are isomorphisms.

Theorem 3.2. *Let $\mathcal{Z} \subset \mathcal{R}_g$ be a (possibly reducible) codimension k subvariety. Assume that $g \geq \max\{7, 3k+5\}$, and let (C, η) be a generic element in $\mathcal{Z}$. If there is a pair $(C', \eta') \in \mathcal{R}_g$ such that there exists an isogeny $\chi : P(C, \eta) \longrightarrow P(C', \eta')$, then $(C, \eta) \cong (C', \eta')$ and $\chi = [n]$, for some $n \in \mathbb{Z}$.*

Proof. Suppose that (C, η) is generic in $\mathcal{Z}$. By the assumption on g, the Clifford index of a generic element of $\mathcal{Z}$ is at least three (as shown in the proof of Corollary 3.1). However, by [8], if the Clifford index of a curve C is $c \geq 3$, then the corresponding fiber of the Prym map is 0-dimensional, i.e. $\dim P^{-1}(P(C, \eta)) = 0$. Therefore, the restriction of the Prym map to $\mathcal{Z}$,

$$P_{|\mathcal{Z}} : \mathcal{Z} \longrightarrow \mathcal{R}_g \longrightarrow \mathcal{A}_{g-1},$$

has generically fixed degree d onto its image, for some $d \in \mathbb{N}$. So, by the genericity of the pair (C, η), we can assume that (C, η) lies in the locus of $\mathcal{Z}$ where $P_{|\mathcal{Z}}$ is étale. This gives the isomorphisms of the tangent spaces

$$T_{P[(C,\eta)]}P(\mathcal{Z}) \cong T_{[C,\eta]}\mathcal{Z} \quad \text{and} \quad T_{P[(C,\eta)]}P(\mathcal{R}_g) \cong T_{[C,\eta]}\mathcal{R}_g. \tag{7}$$

Let us assume that the locus of curves in $\mathcal{R}_g$ whose corresponding Prym variety is isogenous to the Prym variety of an element in $\mathcal{Z}$ has an irreducible component $\mathcal{Z}'$ of codimension k. By [6], since $k < g-2$, we have $\mathrm{End}(P(C, \eta)) \cong \mathbb{Z}$. Suppose that we are given an isogeny $\chi : P(C, \eta) \longrightarrow P(C', \eta')$; then, it must have the property that the pull-back of the principal polarization Ξ' is a multiple of the principal polarization Ξ on $P(C, \eta)$, say $\chi^*\Xi' \cong \Xi^{\otimes m}$, for some $m \in \mathbb{Z}$.

For such m, we have the diagram of forgetful maps as in (5) with $g-1$ in place of g. We can find an irreducible subvariety $\mathcal{V} \subset \widetilde{\mathcal{A}_{g-1}}^{m}$ which dominates both $P(\mathcal{Z})$ and $P(\mathcal{Z}')$ through φ and ψ respectively. Setting $\mathcal{R} := \varphi^{-1}(P(\mathcal{R}_g))$ and $\mathcal{R}' := \psi^{-1}(P(\mathcal{R}_g))$, we have the inclusion $\mathcal{V} \subset \mathcal{R} \cap \mathcal{R}'$.

For a generic element $\chi : P(C, \eta) \longrightarrow P(C', \eta')$ in $\mathcal{V}$, the diagram (6) becomes

$$\begin{array}{ccc} & T_{[\chi]}\widetilde{\mathcal{A}_{g-1}}^{m} & \\ {}^{d\varphi}\swarrow & & \searrow^{d\psi} \\ T_{[P(C,\eta)]}\mathcal{A}_{g-1} & \underset{\cong}{\xrightarrow{\quad\lambda\quad}} & T_{[P(C',\eta')]}\mathcal{A}_{g-1}. \end{array}$$

In addition, $T_{[P(C,\eta)]}\mathcal{A}_{g-1} \cong \mathrm{Sym}^2 \mathrm{H}^0(P(C,\eta), T_{P(C,\eta)}) \cong \mathrm{Sym}^2 \mathrm{H}^0(\omega_C \otimes \eta)^*$. By looking at $d\varphi$, and the isomorphisms in (7), we see that we have the following diagram of tangents spaces and identifications:

$$\begin{array}{ccccccc}
T_{[\chi]}\mathcal{V} & \hookrightarrow & T_{[\chi]}\mathcal{R} & \hookrightarrow & T_{[\chi]}\mathcal{R} + T_{[\chi]}\mathcal{R}' & \hookrightarrow & T_{[\chi]}\widetilde{\mathcal{A}_{g-1}}^{m} \\
\cong\downarrow & & \cong\downarrow & & \| & & \cong\downarrow \\
T_{[C,\eta]}\mathcal{Z} & \hookrightarrow & T_{C,\eta]}\mathcal{R}_g & \hookrightarrow & \bar{T} & \hookrightarrow & \mathrm{Sym}^2 \mathrm{H}^0(\omega_C \otimes \eta)^*
\end{array}$$

where the vertical arrows are $d\varphi$.

By the Grassmann formula, $\dim \bar{T} \leq 3g-3+k$. Set

$$K(C_\eta) := \ker\Big(\mathrm{Sym}^2 \mathrm{H}^0(\omega_C \otimes \eta) \longrightarrow \bar{T}^*\Big).$$

It is a subspace of the space of quadrics containing the semicanonical curve C_η. Notice that $\mathrm{codim}_{I_2(C_\eta)} K(C_\eta) \leq k$. By repeating the above argument with ψ in place of φ, we get the corresponding inclusion of vector spaces $K(C'_{\eta'}) \subset I_2(C'_{\eta'})$, and by using the (canonical) isomorphism λ above, we get a (canonical) isomorphism $K(C_\eta) \cong K(C'_{\eta'})$.

A closer look at $\lambda : T_{[P(C,\eta)]}\mathcal{A}_{g-1} \longrightarrow T_{[P(C',\eta')]}\mathcal{A}_{g-1}$ reveals that this map is induced by the isogeny $\chi : P(C,\eta) \longrightarrow P(C',\eta')$. In fact, one has that $d_0\chi : \mathrm{H}^0(\omega_C \otimes \eta) \longrightarrow \mathrm{H}^0(\omega_{C'} \otimes \eta')$ is an isomorphism, and λ is induced by it. This means that $d_0\chi$ induces an isomorphism of projective spaces $\mathbb{P}\mathrm{H}^0(\omega_C \otimes \eta)^* \longrightarrow \mathbb{P}\mathrm{H}^0(\omega_{C'} \otimes \eta')^*$, which sends quadrics containing $C'_{\eta'}$ to quadrics containing C_η, by means of λ. By using Lemma 3.1, we get that $C_\eta \cong C'_{\eta'}$, and thus $C \cong C'$. This gives us the following commutative diagram

$$\begin{array}{ccccc}
C & \overset{\varphi_{\omega_C \otimes \eta}}{\hookrightarrow} & C_\eta & \hookrightarrow & \mathbb{P}\mathrm{H}^0(\omega_C \otimes \eta)^* \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
C' & \overset{\varphi_{\omega_{C'} \otimes \eta'}}{\hookrightarrow} & C'_{\eta'} & \hookrightarrow & \mathbb{P}\mathrm{H}^0(\omega_{C'} \otimes \eta')^*
\end{array}$$

from which we deduce that $(C,\eta) \cong (C',\eta')$. Indeed, pulling back hyperplanes to C and C', yields an isomorphism $\omega_{C'} \otimes \eta' \cong \omega_C \otimes \eta$, from which it follows that $\eta \cong \eta'$. The isogeny is necessarily of the form $[n]$, for some $n \in \mathbb{Z}$, because $\mathrm{End}(P(C,\eta)) \cong \mathbb{Z}$. □

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REFERENCES

[1] F. Bardelli and G. P. Pirola, *Curves of genus g lying on a g-dimensional Jacobian variety*, Inventiones Mathematicae 95 (2) (1989), 263–276.

[2] V. Marcucci, J. C. Naranjo and G. P. Pirola, *Isogenies of Jacobians*, To appear in *Algebraic Geometry*.

[3] H. Lange and E. Sernesi, *Quadrics containing a Prym-canonical curve*, Journal of Algebraic Geometry 5 (2) (1996), 387–399.

[4] R. Lazarsfeld, *A sampling of vector bundle techniques in the study of linear series*, Lectures on Riemann surfaces (Trieste, 1987), 500–559, World Sci. Publ., Teaneck, NJ (1989).

[5] Z. Ran, *On a theorem of Martens*, Rend. Sem. Mat. Univ. Politec. Torino 44 (2) (1986), 287–291.

[6] G. P. Pirola, *Base number theorem for abelian varieties. An infinitesimal approach*, Math. Ann. 282 (3) (1988), 361–368.

[7] M. Coppens, G. Martens, *Secant spaces and Clifford's theorem*, Compositio Math. 78 (2) (1991), 193–212.

[8] J. C. Naranjo, *The positive dimensional fibres of the Prym map*, Pac. J. Math. 172 (1) (1996), 223–226.

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