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THE FORMAL ANALOGY BETWEEN THE STATIONARY AXISYMMETRIC EINSTEIN-MAXWELL EQUATIONS AND THE EQUATIONS OF ELECTRICAL HEATING OF CONDUCTORS

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Two problems of the general theory of relativity and a problem in the electrical heating of conductors (the so-called thermistor problem), lead to the same set of partial differential equations. This permits a unified treatment of these different problems. The related boundary value problems is studied using a suitable transformation.

1. Introduction

In this paper we study the system of PDE

$$\psi_{\rho\rho} + \frac{1}{\rho}\psi_\rho + \psi_{zz} = e^{-2\psi}\left(\phi_\rho^2 + \phi_z^2\right) \tag{1.1}$$

$$\left(\rho e^{-2\psi}\phi_\rho\right)_\rho + \left(\rho e^{-2\psi}\phi_z\right)_z = 0 \tag{1.2}$$

which is relevant in three different contests: (I) in the axially symmetric problem of the general theory of relativity, (II) in the extension of the Weyl's metric to the case of steadily spinning sources and (III) last, but not least, to the completely

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different problem of the electrical heating of conductors. We start with problem (I). The axially symmetric static Weyl metric reads (see [18], [1], [2], [4] and reference therein)

$$ds^2 = e^{2\psi}dt^2 - e^{-2\psi}\left[e^{2\gamma}\left(dz^2 + d\rho^2\right) + \rho^2 d\varphi^2\right], \tag{1.3}$$

where ψ and γ are functions of ρ and z only. In terms of the antisymmetric electromagnetic fields tensor F_{ik} the Maxwell equations [10], [17] are

$$F_{ij;k} + F_{jk;i} + F_{ki;j} = 0, \quad F^{ik}_{\ ;k} = 0.$$

If ϕ is the electric potential, also depending on ρ and z only, and magnetic effects are neglected, the only non-vanishing components of F_{ik} are $F_{21} = -F_{12} = \phi_\rho$, $F_{31} = -F_{13} = \phi_z$. Under these assumptions the system of the Einstein-Maxwell equations reduces to (1.1) and (1.2), see [7], [8]. If ψ and ϕ are known from (1.1) and (1.2), γ can be determined by integration up to an additive constant [1], [11]. Problem (II) is closely related to (I). For, the metric corresponding to axially symmetric rotating matter reads, [11], [1], [16]

$$ds^2 = e^{2\mu}\left(dt + \omega d\varphi\right)^2 - e^{-2\mu}\left[e^{2\nu}\left(dz^2 + d\rho^2\right) + \rho^2 d\varphi^2\right]. \tag{1.4}$$

The vacuum field equations reduce to two for μ and ω, which are

$$\mu_{\rho\rho} + \frac{1}{\rho}\mu_\rho + \mu_{zz} = -\frac{1}{2}\frac{e^{4\mu}}{\rho^2}\left(\omega_\rho^2 + \omega_z^2\right) \tag{1.5}$$

$$\left(\frac{e^{4\mu}}{\rho}\omega_\rho\right)_\rho + \left(\frac{e^{4\mu}}{\rho}\omega_z\right)_z = 0, \tag{1.6}$$

with ν determined by quadrature as in the previous case. The relationship between the solutions of (1.1), (1.2) and the solutions of (1.5), (1.6) is quite simple and is given in Section 2. To prove that problem III can, in certain cases, be modeled with (1.1), (1.2), let us consider an axially symmetric conductor of electricity and heat under steady conditions. Suppose that the boundary conditions are such that the temperature and the electric potential in cylindrical coordinates do not depend on the angular variable. Let $u(\rho, z)$ denote the temperature. Assume the electrical conductivity σ to depend on the temperature according to the law $\sigma(u) = e^{2u}$ [1] and take the thermal conductivity $\kappa = 1$. The heat flux density is given by the Fourier's law $\mathbf{q} = -\nabla u$ and the current density reads $\mathbf{J} = -\mathrm{e}^{2u}\nabla\phi$. Thus the conservation of energy and charge, i.e. $\nabla \cdot \mathbf{J} = 0$, $\nabla \cdot \mathbf{q} = \mathbf{E} \cdot \mathbf{J}$, implies

$$\nabla \cdot \left(\mathrm{e}^{2u}\nabla\phi\right) = 0, \quad -\Delta u = \mathrm{e}^{2u}|\nabla\phi|^2. \tag{1.7}$$

[1] This happens in certain technical devices called thermistors [9].

On the other hand, the laplacian in cylindrical coordinates reads $\Delta u = u_{\rho\rho} + \frac{1}{\rho}u_\rho + u_{zz} + \frac{1}{\rho^2}u_{\theta\theta}$. If u and ϕ do not depend on the angular variable and we set $\psi = -u$, we obtain precisely (1.1), (1.2). The results of this paper apply to each of these three physical situations.

Many papers have been devoted to the study of the system of (1.1), (1.2). We quote in particular [12] where a simplified form of the system is studied in which the electric field is so weak that its influence on the metric can be neglected. Thus the right hand side in equation (1.1) is put equal to zero. An orthogonal electrostatic conforming coordinates system is used in [3] to find properties of the system (1.1), (1.2).

In Section 3 we introduce a transformation which permits to rewrite the system (1.1), (1.2) in a more symmetric form. To single out a specific solution of this system of PDE we need to prescribe boundary conditions. This is made according to the specific nature of the problem. We prove that the solution of this non linear boundary value problem depends only on the solution of an auxiliary linear Dirichlet's problem for a single equation.

2. Equivalence of the systems (1.1), (1.2) and (1.5), (1.6)

If

$$\phi = \omega, \quad \psi = \log\rho - 2\mu \tag{2.1}$$

the system

$$\psi_{\rho\rho} + \frac{1}{\rho}\psi_\rho + \psi_{zz} = e^{-2\psi}\left(\phi_\rho^2 + \phi_z^2\right) \tag{2.2}$$

$$\left(\rho e^{-2\psi}\phi_\rho\right)_\rho + \left(\rho e^{-2\psi}\phi_z\right)_z = 0 \tag{2.3}$$

becomes

$$\mu_{\rho\rho} + \frac{1}{\rho}\mu_\rho + \mu_{zz} = -\frac{1}{2}\frac{e^{4\mu}}{\rho^2}\left(\omega_\rho^2 + \omega_z^2\right) \tag{2.4}$$

$$\left(\frac{e^{4\mu}}{\rho}\omega_\rho\right)_\rho + \left(\frac{e^{4\mu}}{\rho}\omega_z\right)_z = 0 \tag{2.5}$$

and vice-versa. This is seen immediately, in fact from (2.1) we have

$$e^{-2\psi} = \frac{e^{4\mu}}{\rho^2}. \tag{2.6}$$

Substituting (2.6) in (2.2) we obtain (2.5). Moreover, from (2.1) we obtain

$$\psi_{\rho\rho}+\frac{1}{\rho}\psi_\rho+\psi_{zz}=-2\big(\mu_{\rho\rho}+\frac{1}{\rho}\mu_\rho+\mu_{zz}\big)+\frac{1}{\rho}\Big(\rho\frac{\partial\log\rho}{\partial\rho}\Big)_\rho= \tag{2.7}$$

$$-2\big(\mu_{\rho\rho}+\frac{1}{\rho}\mu_\rho+\mu_{zz}\big).$$

Putting (2.6) and (2.7) into (2.2) we have (2.4). Hence every result for the system (2.2), (2.3) translates, via (2.1), into a result for the system (2.4), (2.5).

3. The main transformation

It is useful to apply to (2.2), (2.3) the transformation

$$\theta(\rho,z)=\frac{\phi^2(\rho,z)}{2}+\frac{1}{2}\big(1-e^{2\psi(\rho,z)}\big), \tag{3.1}$$

where $(\phi(\rho,z),\psi(\rho,z))$ is a solution of (2.2), (2.3). We have

$$\theta_\rho=\phi\phi_\rho-e^{2\psi}\psi_\rho,\quad \theta_z=\phi\phi_z-e^{2\psi}\psi_z. \tag{3.2}$$

From (2.3), (2.2) we have, using (3.2),

$$\frac{1}{\rho}\big(\rho e^{-2\psi}\theta_\rho\big)_\rho+\Big(e^{-2\psi}\theta_z\Big)_z=0. \tag{3.3}$$

Therefore, the system (1.1), (1.2) can be reformulated in the following more symmetric form

$$\frac{1}{\rho}\big(\rho e^{-2\psi}\phi_\rho\big)_\rho+\Big(e^{-2\psi}\phi_z\Big)_z=0 \tag{3.4}$$

$$\frac{1}{\rho}\big(\rho e^{-2\psi}\theta_\rho\big)_\rho+\Big(e^{-2\psi}\theta_z\Big)_z=0, \tag{3.5}$$

where ϕ, ψ and θ are related by the functional relation (3.1). However, with this approach (3.1) needs to be solved with respect to ψ if we want to return to the physical quantities ϕ and ψ. This is possible only if

$$\phi^2+1-2\theta>0. \tag{3.6}$$

To decide whether or not the condition (3.6) is satisfied we need to specify a particular set of boundary conditions for the system (3.4), (3.5). On the other hand, both systems are of little use without this particularization which only permits to determine a definite solution out of the infinite set of solutions of the indefinite systems. The problem of finding suitable boundary conditions for the Einstein-Maxwell equations is an old one. We refer in this respect to [5], [11] and [15]. The boundary conditions in which we state our problem are quite special, however, they permit a near complete solution of the corresponding problem. Let D be an open and bounded subset of $\mathbf{R}^2$ homeomorphic to an annulus, laying entirely in the half plane $\rho > 0$ and with a positive distance from the z-axis. Let Γ_1 and Γ_2 be the two disjoint parts of the boundary of D. Let Ω be the subset of $\mathbf{R}^3$ obtained rotating D around the z-axis and let T be the toroidal surface obtained rotating Γ_1 around the z-axis. Referring to problem (I), we assume that all the matter and electric charges which give rise to the potentials ψ and ϕ are contained inside the solid torus of boundary T and determine the values of ϕ and ψ on Γ_1 as two constants $\bar{\psi}$ and $\bar{\phi}$ such that $\psi = \bar{\psi}$, $\phi = \bar{\phi}$ on Γ_1. The torus obtained rotating Γ_2 around the z-axis will be so-to-speak our "horizon". Thus we assume on Γ_2 the values pertaining to the flat space solution i.e. $\psi = 0$, $\phi = 0$. For problem (III) with the present choice of boundary conditions, we prescribe two different constant values of temperature on the two part of the boundary. We arrive for the determination of ϕ and ψ to the following boundary value problem

$$\frac{1}{\rho}\left(\rho\psi_\rho\right)_\rho + \psi_{zz} = e^{-2\psi}\left(\phi_\rho^2 + \phi_z^2\right) \quad \text{in} \quad D \tag{3.7}$$

$$\psi = \bar{\psi} \quad \text{on} \quad \Gamma_1, \qquad \psi = 0 \quad \text{on} \quad \Gamma_2 \tag{3.8}$$

$$\frac{1}{\rho}\left(\rho e^{-2\psi}\phi_\rho\right)_\rho + \left(e^{-2\psi}\phi_z\right)_z = 0 \quad \text{in} \quad D \tag{3.9}$$

$$\phi = \bar{\phi} \quad \text{on} \quad \Gamma_1, \qquad \phi = 0 \quad \text{on} \quad \Gamma_2. \tag{3.10}$$

The case $\bar{\phi} = 0$ is immediately dealt with, since we have $\phi(\rho, z) = 0$ from (3.9) and (3.10), and we obtain $\psi(\rho, z)$ from (3.7), (3.8) which becomes a simple linear Dirichlet's problem. Thus there is no loss in generality if we assume $\bar{\phi} \neq 0$. In terms of ϕ and θ, see (3.1), the problem (3.7)-(3.10) becomes

$$\frac{1}{\rho}\left(\rho e^{-2\psi}\phi_\rho\right)_\rho + \left(e^{-2\psi}\phi_z\right)_z = 0 \quad \text{in} \quad D \tag{3.11}$$

$$\phi = \bar{\phi} \quad \text{on} \quad \Gamma_1, \qquad \phi = 0 \quad \text{on} \quad \Gamma_2 \tag{3.12}$$

$$\frac{1}{\rho}\left(\rho e^{-2\psi}\theta_\rho\right)_\rho + \left(e^{-2\psi}\theta_z\right)_z = 0 \quad \text{in} \quad D \tag{3.13}$$

$$\theta = \frac{\bar{\phi}^2}{2} + \frac{1}{2}\left(1 - e^{2\bar{\psi}}\right) \quad \text{on} \quad \Gamma_1, \qquad \theta = 0 \quad \text{on} \quad \Gamma_2, \tag{3.14}$$

where θ, ϕ and ψ are related by the functional relation

$$\theta = \frac{\phi^2}{2} + \frac{1}{2}\left(1 - e^{2\psi}\right). \tag{3.15}$$

In the next Lemma we prove the equivalence between these two formulations.

Lemma 3.1. *If*

$$\frac{\bar{\phi}^2}{2} + \frac{1}{2}\left(1 - e^{2\bar{\psi}}\right) < \frac{1}{2} \tag{3.16}$$

every solution of the problem (3.7)-(3.10) is a solution of (3.11)-(3.15) and vice-versa.

Proof. Let $(\phi(\rho,z),\psi(\rho,z))$ be a solution of problem (3.7)-(3.10). Define

$$\theta(\rho,z) = \frac{\phi^2(\rho,z)}{2} + \frac{1}{2}\left(1 - e^{2\psi(\rho,z)}\right). \tag{3.17}$$

With direct calculation we have (3.13) using (3.7) and (3.9). Moreover $\theta(\rho,z)$, defined by (3.17), satisfies the boundary condition (3.14). Vice-versa, let us assume $(\phi(\rho,z),\theta(\rho,z))$ to be a solution of problem (3.11)-(3.15). In view of the assumptions made on D it is possible to apply to (3.11), (3.12) the maximum principle for elliptic equation ([13] page 61). Thus from (3.11), (3.12) we have the "a priori" bound

$$-\frac{\bar{\phi}^2}{2} \le -\frac{\phi^2(\rho,z)}{2} \le 0. \tag{3.18}$$

If

$$\frac{\bar{\phi}^2}{2} + \frac{1}{2}\left(1 - e^{2\bar{\psi}}\right) > 0 \tag{3.19}$$

we obtain from (3.13) and (3.14), by the maximum principle,

$$0 \le \theta(\rho,z) \le \frac{\bar{\phi}^2}{2} + \frac{1}{2}\left(1 - e^{2\bar{\psi}}\right) \quad \text{in} \quad D. \tag{3.20}$$

If, instead,

$$\frac{\bar{\phi}^2}{2} + \frac{1}{2}\left(1 - e^{2\bar{\psi}}\right) \le 0 \tag{3.21}$$

holds, we have, again from (3.13) and (3.14),

$$\frac{\bar{\phi}^2}{2} + \frac{1}{2}\left(1 - e^{2\bar{\psi}}\right) \le \theta(\rho,z) \le 0 \quad \text{in} \quad D. \tag{3.22}$$

Adding (3.20) and (3.18) we obtain, if (3.19) holds,

$$-\frac{\bar{\phi}^2}{2} \leq \theta(\rho, z) - \frac{\phi^2(\rho, z)}{2} \leq \frac{\bar{\phi}^2}{2} + \frac{1}{2}\left(1 - e^{2\bar{\psi}}\right) \tag{3.23}$$

or, if we have (3.21),

$$-\frac{1}{2}\left(1 - e^{2\bar{\psi}}\right) \leq \theta(\rho, z) - \frac{\phi^2(\rho, z)}{2} \leq 0. \tag{3.24}$$

By (3.15) we obtain

$$\frac{1}{2}\left(1 - e^{2\psi}\right) = \theta - \frac{\phi^2}{2}. \tag{3.25}$$

In view of (3.16) the functional relation (3.25) can be solved with respect to ψ in both the cases (3.19) and (3.21). Therefore, the function

$$\psi(\rho, z) = \frac{1}{2}\log\left[1 + \phi^2(\rho, z) - 2\theta(\rho, z)\right] \tag{3.26}$$

is well-defined. We prove now that $\psi(\rho, z)$ satisfies (3.7). For, from (3.25) we have

$$\psi_\rho(\rho, z) = e^{-2\psi}\left(\phi\phi_\rho - \theta_\rho\right), \quad \psi_z(\rho, z) = e^{-2\psi}\left(\phi\phi_z - \theta_z\right). \tag{3.27}$$

Using (3.11) and (3.13) we have, after simple calculations,

$$\frac{1}{\rho}\left(\rho\psi_\rho\right)_\rho + \psi_{zz} = e^{-2\psi}\left(\phi_\rho^2 + \phi_z^2\right)$$

i.e. (3.7). On the other hand, $\psi(\rho, z)$ given by (3.26) satisfies also the boundary condition (3.8). This proves the equivalence of problem (3.7)-(3.10) with (3.11)-(3.15). □

We show now that the problem (3.11)-(3.15) can be solved in terms of the solution of an auxiliary Dirichlet's and it is therefore advantageous with respect to the formulation (3.7)-(3.10). For definiteness we assume hereafter $\bar{\phi} > 0$. From (3.15) we have

$$e^{-2\psi} = \frac{1}{1 + \phi^2 - 2\theta}. \tag{3.28}$$

Thus, the problem (3.11)-(3.14) can be written in terms of θ and ϕ only as follows:

$$\frac{1}{\rho}\left(\frac{\rho}{1 + \phi^2 - 2\theta}\phi_\rho\right)_\rho + \left(\frac{1}{1 + \phi^2 - 2\theta}\phi_z\right)_z = 0 \quad \text{in} \quad D \tag{3.29}$$

$$\phi = \bar{\phi} \quad \text{on} \quad \Gamma_1, \quad \phi = 0 \quad \text{on} \quad \Gamma_2 \tag{3.30}$$

$$\frac{1}{\rho}\Big(\frac{\rho}{1+\phi^2-2\theta}\theta_\rho\Big)_\rho + \Big(\frac{1}{1+\phi^2-2\theta}\theta_z\Big)_z = 0 \quad \text{in} \quad D \tag{3.31}$$

$$\theta = \frac{\bar{\phi}^2}{2} + \frac{1}{2}\Big(1-e^{2\bar{\psi}}\Big) \quad \text{on} \quad \Gamma_1, \quad \theta = 0 \quad \text{on} \quad \Gamma_2. \tag{3.32}$$

Let us define

$$k = \frac{\bar{\phi}^2 + 1 - e^{2\bar{\psi}}}{\bar{\phi}} \tag{3.33}$$

and consider the problem for a single equation

$$\frac{1}{\rho}\Big(\frac{\rho}{1+\phi^2-k\phi}\phi_\rho\Big)_\rho + \Big(\frac{1}{1+\phi^2-k\phi}\phi_z\Big)_z = 0 \quad \text{in} \quad D \tag{3.34}$$

$$\phi = \bar{\phi} \quad \text{on} \quad \Gamma_1, \quad \phi = 0 \quad \text{on} \quad \Gamma_2. \tag{3.35}$$

It is immediately seen that if $\phi(\rho,z)$ is a solution of the problem (3.34), (3.35) then

$$\big(\phi(\rho,z),\theta(\rho,z)\big) = \big(\phi(\rho,z),\frac{k}{2}\phi(\rho,z)\big) \tag{3.36}$$

solves (3.29)-(3.32). Now, the solution of problem (3.34), (3.35) can be easily found. To this end, let us define

$$w = F(\phi) =: \int_0^\phi \frac{dt}{1+t^2-kt}. \tag{3.37}$$

For the present method to work we need $F(\phi)$ to be invertible as a function from $[0,\bar{\phi}]$ to $[0,F(\bar{\phi})]$. Since

$$t^2 - kt + 1 = \frac{1}{4}(4-k^2) + \frac{1}{4}(k-2t)^2 \tag{3.38}$$

to have invertibility we assume

$$|k| < 2. \tag{3.39}$$

In view of (3.33), we can rewrite (3.39) in terms of $\bar{\phi}$ and $\bar{\psi}$ as

$$(\bar{\phi}-1)^2 < e^{2\bar{\psi}} < (\bar{\phi}+1)^2. \tag{3.40}$$

Hereafter we assume (3.40) in addition to (3.16). Computing the integral which defines $F(\phi)$ we find

$$F(\phi) = \frac{2}{4-k^2}\left[\operatorname{atan}\left(\frac{2\phi-k}{\sqrt{4-k^2}}\right) - \operatorname{atan}\left(\frac{-k}{\sqrt{4-k^2}}\right)\right]. \tag{3.41}$$

In term of w the problem (3.34), (3.35) becomes

$$\frac{1}{\rho}(\rho w_\rho) + w_{zz} = 0 \quad \text{in} \quad D \tag{3.42}$$

$$w = F(\bar{\phi}) \quad \text{on} \quad \Gamma_1 \tag{3.43}$$

$$w = 0 \quad \text{on} \quad \Gamma_2. \tag{3.44}$$

The solution w of the Dirichlet's problem (3.42)-(3.44) exists and is unique, (see [6]). Moreover $\phi = F^{-1}(w)$ is given by

$$\phi = \frac{\sqrt{4-k^2}}{2}\tan\left[\left(\frac{4-k^2}{2}\right)w - \operatorname{atan}\left(\frac{k}{\sqrt{4-k^2}}\right)\right] + \frac{k}{2}. \tag{3.45}$$

By (3.36), (3.45) and (3.26) we find

$$\phi(\rho,z) = \frac{\sqrt{4-k^2}}{2}\tan\left[\left(\frac{4-k^2}{2}\right)w(\rho,z) - \operatorname{atan}\left(\frac{k}{\sqrt{4-k^2}}\right)\right] + \frac{k}{2} \tag{3.46}$$

and

$$\psi(\rho,z) = \frac{1}{2}\log\left[1+\phi^2(\rho,z) - k\phi(\rho,z)\right] \tag{3.47}$$

as solution of our starting problem (3.7)-(3.10).

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