LE MATEMATICHE
Vol. LXXIV (2019) – Issue I, pp. 3–11
doi: 10.4418/2019.74.1.1

INVARIANT SUBRING OF THE COX RING OF K3 SURFACES

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In this paper, we consider the invariant subring of the Cox ring by the automorphism group of the projective variety X under some assumption. We prove that the ring is finitely generated if X is a K3 surface.

1. Introduction

Cox rings were introduced by D.Cox in [C] and are important rings which appeared in algebraic geometry, which is defined as follows:

Definition 1.1 (Multi-section rings and Cox rings)**.** Let X be a normal projective variety over an algebraically closed field k. For a semigroup Γ of Weil divisors on X, the Γ-graded ring

$$R_X(\Gamma) = \bigoplus_{D \in \Gamma} H^0(D)$$

is called the *multi-section ring* of Γ.

Suppose that $\mathrm{Cl}(X)$ is finitely generated. For such X, choose a group Γ of Weil divisors on X such that $\Gamma_{\mathbb{Q}} \to \mathrm{Cl}(X)_{\mathbb{Q}}$ is an isomorphism. Then we say that the multi-section ring $R_X(\Gamma)$ is a *Cox ring* of X.

Submission received : 9 June 2017

AMS 2010 Subject Classification: Primary 14J32; Secondary 14J45, 14B05, 14E30.
Keywords: K3surfaces, automorphism, invariant subring, Cox ring

One of the main topics related with Cox rings is their finite generation . The main theorem of this paper is the following:

Theorem 1.2. *Let X be a smooth algebraic K3 surface defined over $\mathbb{C}$. Assume that there is an* $\mathrm{Aut}(X)$*-invariant semigroup* Γ *of Weil divisors on X. Then the invariant subring of* $R_X(\Gamma)$ *by* $\mathrm{Aut}(X)$ *is finitely generated over* $\mathbb{C}$.

The theorem has the following meaning. At first, by [BCHM] and [TVAV], we know the Cox rings are finitely generated if the base varieties are of Fano type or smooth surfaces with big anticanonical divisor. This means that the positivity of the anticanonical divisors makes the Cox rings finitely generated. Secondly, we consider the boundary of this case, namely, the case in which the anticanonical divisor is trivial. If the variety is an Abelian variety, then the Picard group is not finitely generated, so that we can not define the Cox ring. Therefore, the next case to be considered is the case of the K3 surfaces.
By [K] and [AHL], we know the Cox ring of a K3 surface is finitely generated if and only if the automorphism group is a finite group. Therefore, among K3 surfaces we have both of examples of finitely generated and not finitely generated Cox ring. This suggests that if the Cox ring is finitely generated and if the automorphism group acts on the Cox ring, then the invariant subring of the Cox ring is finitely generated . It is natural to ask whether this holds in general or not. This question is first asked in the draft of ”lectures on K3 surfaces” by D. Huybrechts [H]. As we see in the proof, finite generation of the invariant subring by the automorphism group is related to the Kawamata-Morrison cone conjecture.

Acknowledgments. A part of this paper was established during the author was participating in the Pragmatic 2013, Promotion of Research in Algebraic Geometry for Mathematicians in Isolated Centres. The author would like to thank Professors Giuseppe Zappala and Alfio Ragusa, who were the local organizers of the school, for their hospitality . The author would also like to thank Professors Paolo Cascini, Yoshinori Gongyo, Yujiro Kawamata and Keiji Ogiso for stimulating discussions and advises. The author is grateful to the referees for valuable comments. The author is partially supported by the Grant-in-Aid for JSPS Fellows (24-0745).

2. Cox rings and grading

Definition 2.1 (Multi-section rings and Cox rings)**.** Let X be a normal projective variety over an algebraically closed field k. For a semigroup Γ of Weil divisors on X, the Γ-graded ring

$$R_X(\Gamma) = \bigoplus_{D\in\Gamma} H^0(D)$$

is called the *multi-section ring* of Γ.

Suppose that $\mathrm{Cl}(X)$ is finitely generated. For such X, choose a group Γ of Weil divisors on X such that $\Gamma_{\mathbb{Q}} \to \mathrm{Cl}(X)_{\mathbb{Q}}$ is an isomorphism. Then we say that the multi-section ring $R_X(\Gamma)$ is a *Cox ring* of X. It is known that the finite generation of such ring is independent of the choice of Γ (See [GOST, Remark 2.17] for example).

We want to define the action of $\mathrm{Aut}(X)$ to the *Cox ring* of X. But as the following example shows, $\mathrm{Aut}(X)$ does not acts on a *Cox ring* of X in general.

Example 2.2. Let X be a n-dimensional projective space. Then we can take the *Cox ring* of X as the ring of the polynomials of $n+1$ valuables, on which the projective general linear group $\mathbb{P}GL((n+1),\mathbb{C})$ never acts. This is because $\mathbb{P}GL((n+1),\mathbb{C})$ is projective group variety and the spectrum of the ring of the polynomials of $n+1$ valuables is affine.

Remark 2.3. In this case, we can take the *Cox ring* of X as the second Veronese subring of the polynomial of $n+1$ valuables. Then we can define the action of $\mathrm{Aut}(X)$ by using the action of the special linear group.

Hence from now on, we assume the existence of Γ which is stable under the action of $\mathrm{Aut}(X)$, namely $g^*\Gamma = \Gamma$ for any $g \in \mathrm{Aut}(X)$ in this paper.

Remark 2.4. If $g^*\Gamma = \Gamma$ holds for any $g \in \mathrm{Aut}(X)$, then $\mathrm{Aut}(X)$ naturally acts on$R_X(\Gamma)$ by pulling back the section and its grading.

To prove the main theorem, we introduce the notion of divisor of finite orbit.

Definition 2.5. A divisor $D \in \Gamma$ (resp. a section f) is said to have finite orbit if the set $\{g^*D \,|\, g \in \mathrm{Aut}(X)\}$(resp. $\{g^*f \,|\, g \in \mathrm{Aut}(X)\}$) is a finite set.

Proposition 2.6. *Let X be a normal projective variety over an algebraically closed field k with freely finitely generated divisor class group. Let $f \in R_X(\Gamma)^{\mathrm{Aut}(X)}$. Let $f = \Sigma_{i=1}^n f_i$ be the homogenous decomposition by Γ, where $f_i \in H^0(D_i)$ and $D_i \in \Gamma$. Then D_i has finite orbit by the action of* $\mathrm{Aut}(X)\}$ *for any i.*

Proof. Since f is $\mathrm{Aut}(X)$-invariant, $f = g^*f = \Sigma_{i=1}^n g^*f_i$. Hence g^*f_i appears in the homogenous decomposition of f, namely $g^*f_i = f_j$ for some j. Since $g^*f_i \in H^0(D_j)$, $g^*D_i = D_j$. This implies that for any $g \in \mathrm{Aut}(X)$, there exists j such that g^*D_i coincides with D_j. This implies the assersion. □

Remark 2.7. We denote the semigroup of the divisors of finite orbit in Γ by Γ_f and the semigroup of nef divisors of finite orbit in Γ by $\mathrm{Nef}_f(X)$. Then the proposition above means that $R_X(\Gamma)^{\mathrm{Aut}(X)}$ is contained in $R_X(\Gamma_f)$.

Proposition 2.8. *Let X be a normal projective surface over an algebraically closed field k with freely finitely generated divisor class group. Let Γ be a semigroup of Weil divisors on X such that $\Gamma \to \mathrm{Cl}(X)$ is an isomorphism. Then $R_X(\Gamma_f)$ is generated by the global sections of negative curves of finite orbit and $R_X(\mathrm{Nef}_f(X))$.*

Proof. Let D be a divisor in Γ_f and $f \in H^0(D)$. Take an ample divisor A on X. We claim that f is the product of the global sections of a nef divisor and negative curves by induction on $A.D = n$. Assume $A.D = n$. If D is nef, then there is nothing to prove. If D is not nef, there is a negative curve C such that $D.C < 0$, therefore $H^0(D-C) = H^0(D)$. If C has finite orbit, then $D-C$ has finite orbit and $A.(D-C) < n$. The claim follows from the inductive hypothesis. Therefore, it is enough to show that C has finite orbit. If C does not have finite orbit, then there is an infinite sequence $\{g_n\}$ in $\mathrm{Aut}(X)$ such that $g_i^*C \neq g_j^*C$ for any $i \neq j$. Since D has finite orbit, there is an infinite subsequence $\{g_{n_i}\}$ such that $g_{n_i}^*D = g_{n_j}^*D$ for any i, j. We fix n_0. Since $g_{n_0}^*D.g_{n_i}^*C = D.C < 0$, $g_{n_i}^*C$ is contained in the base locus of $|g_{n_0}^*D|$. This contradicts the finiteness of the irreducible components of the base locus.

□

3. the cone conjecture for K3 surfaces

In this section, we continue to assume the existence of Γ which is stable under the action of $\mathrm{Aut}(X)$, namely $g^*\Gamma = \Gamma$ for any $g \in \mathrm{Aut}(X)$.

Definition 3.1. A K3 surface over k is a smooth projective surface X such that $\omega_{X/k} = \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$.

In this section, we work on a K3 surface over an algebraically closed field k. We first remark that for a K3 surface X, linear equivalence and numerical equivalence coincide (See [H, Proposition 2.3] for example). Hence, we naturally see $\Gamma \otimes \mathbb{R} \cong N^1(X)_{\mathbb{R}}$. Secondly, the divisor class group of a K3 surface is freely finitely generated and its rank is at most 22. Hence we can consider the Cox rings of K3 surfaces. We fix Γ as in the Definition 2.1.

Note 3.2. In this section, let us denote the cone generated by the divisors of finite orbit (resp. the nef divisors of finite orbit) in $N^1(X)_{\mathbb{R}}$ by $\Gamma_{f,\mathbb{R}}$ (resp. $Nef_f(X)_{\mathbb{R}}$). For a cone C in $N^1(X)_{\mathbb{R}}$, we denote $R_X(C \cap \Gamma)$ simply by $R_X(C)$. It is easy to see that this notation is compatible with the notation in the previous section.

We show that $R_X(\Gamma_f)$ is finitely generated. By Proposition 2.8, it is enough to see that the number of (-2)-curves of finite orbit is finite and that $R_X(\mathrm{Nef}_f(X))$ is finitely generated. The first assertion follows from a result due to H. Sterk.

Proposition 3.3. *Let X be a K3 surface over an algebraically closed field k. Then the number of (-2)-curves of finite orbit is finite.*

Proof. By [S, Proposition 2.5], the number of $\mathrm{Aut}(X)$-orbits of (-2)-curves is finite. The assertion follows from this fact. □

To see the finite generation of $R_X(\mathrm{Nef}_f(X))$, we need the cone conjecture.

Definition 3.4. Let C be a cone in $N^1(X)_{\mathbb{R}}$ and G be a group acting on C. A rational polyhedral cone $\rho \subset C$ is said to be a rational polyhedral fundamental domain for the action of G if the following two conditions are satisfied:

1. $C = \cup_{g \in G} g(\rho)$
2. $g(\rho) \cap h(\rho)$ does not contains inner points for $g \neq h$

Let $\mathrm{Nef}^e(X)_{\mathbb{R}}$ (resp. $\mathrm{Nef}^e_f(X)_{\mathbb{R}}$) denote the cone generated by nef and effective divisors (resp. the cone generated by nef and effective divisors of finite orbit). The cone conjecture states that $\mathrm{Nef}^e(X)_{\mathbb{R}}$ has a fundamental domain.

Theorem 3.5. *[S][H, Chapter 4, Theorem 4.2.] Let X be a K3 surface over an algebraically closed field k of characteristic $\neq 2$. The action of* $\mathrm{Aut}(X)$ *on* $\mathrm{Nef}^e(X)_{\mathbb{R}}$ *admits a rational polyhedral fundamental domain ρ.*

Let ρ_f denote the cone generated by nef divisors in ρ of finite orbit.

Proposition 3.6. *Let X be a K3 surface over an algebraically closed field k of characteristic $\neq 2$. Then $R_X(\rho_f)$ is finitely generated over k. In particular, ρ_f is a rational polyhedral cone.*

Proof. Consider the natural inclusion $i : R_X(\rho_f) \to R_X(\rho)$. Since $R_X(\rho)$ is a multisection ring associated to a rational polyhedral cone generated by finitely many semiample divisors, $R_X(\rho)$ is finitely generated over k by [HK, Lemma 2.8]. Since a pure subring of a Noetherian ring is Noetherian, it is enough to prove that the map i splits as $R_X(\rho_f)$-module homomorphism. We have the projection $\pi : R_X(\rho) \to R_X(\rho_f)$ as k-vector spaces, namely for a homogenous element s in $R_X(\rho)$, $\pi(s)$ is s if s is in $R_X(\rho_f)$ and 0 otherwise. We see that π is $R_X(\rho_f)$-linear. To confirm this, we show that for any element $a \in R_X(\rho) - R_X(\rho_f)$ and any $r \in R_X(\rho_f)$, $ra \in R_X(\rho) - R_X(\rho_f)$. We may assume a and r are homogenous, namely $a \in H^0(N_1)$ and $r \in H^0(N_2)$ for some divisor of infinite orbit N_1 and divisor of finite orbit N_2. Then $ra \in H^0(N_1 + N_2)$ and $N_1 + N_2$ is of infinite orbit. Hence $ra \in R_X(\rho) - R_X(\rho_f)$. The second assertion follows from the first assertion.

□

Corollary 3.7. *Let X be a K3 surface over an algebraically closed field k of characteristic $\neq 2$. Let Γ be a group of Weil divisors on X such that $\Gamma \to \mathrm{Cl}(X)$ is an isomorphism.Then $R_X(\mathrm{Nef}_f(X))$ is finitely generated over k. Hence $R_X(\Gamma_f)$ is finitely generated over k.*

Proof. By Proposition 3.6, ρ_f is generated by finitely many semiample divisors of finite orbit. Since ρ_f spans $\Gamma_{f,\mathbb{R}}$ by actions of $\mathrm{Aut}(X)$, $\Gamma_{f,\mathbb{R}}$ is generated by finitely many semiample divisors. This implies the first assertion. The second assertion follows from the first assertion and Proposition 2.8 and Proposition 3.3. □

4. log canonical representations

In this section, we work on algebraic varieties over $\mathbb{C}$.

Definition 4.1. Let X be a normal variety over $\mathbb{C}$ and let Δ be an $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $\pi : Y \to X$ be a log resolution of (X,Δ). We set $K_Y = \pi^*(K_X+\Delta) + \Sigma a_i E_i$, where E_i is a prime divisor on Y. The pair (X,Δ) is called *sub kawamata log terminal* (*subklt*, for short) if the coefficients of Δ are smaller than 1 and $a_i > -1$ for all i. Futhremore, if Δ is effective, we simply call (X,Δ) *klt*.

Definition 4.2. [FG, Definition 3.1] Let (X,Δ) be an n-dimensional projective subklt pair such that X is smooth and Δ has a simple normal crossing support. We denote the group of birational mappings of X onto itself by $\mathrm{Bir}(X)$. We define

$$\widetilde{\mathrm{Bir}}_m(X,\Delta) = \left\{ g \in \mathrm{Bir}(X) \,|\, g^*\omega \in H^0(m(K_X+\Delta)) \,\text{for every}\, \omega \in H^0(m(K_X+\Delta)) \right\}.$$

It is easy to see that $\widetilde{\mathrm{Bir}}_m(X,\Delta)$ is a subgroup of $\mathrm{Bir}(X)$. An element $g \in \widetilde{\mathrm{Bir}}_m(X,\Delta)$ is called a $\widetilde{B}$-birational map of (X,Δ). By definition, we have the group homomorphism

$$\widetilde{\rho}_m : \widetilde{\mathrm{Bir}}_m(X,\Delta) \to \mathrm{Aut}(H^0(X, m(K_X+\Delta))).$$

The homomorphism $\widetilde{\rho}_m$ is called the $\widetilde{B}$-pluricanonical representation of $\widetilde{\mathrm{Bir}}_m(X,\Delta)$.

Theorem 4.3. *[FG, Theorem 3.9.] Let (X,Δ) be a projective subklt pair such that X is smooth, Δ has a simple normal crossing support, and $m(K_X+\Delta)$ is Cartier where m is a positive integer. Then $\widetilde{\rho_m}(\widetilde{\mathrm{Bir}}_m(X,\Delta))$ is a finite group.*

We can regard any effective nef divisor N on a K3 surface X as a log pluricanonical divisor of a klt pair $(X, 1/mN)$ for an enough large m. And the following theorem (abundance theorem) tells us the semiampleness of this divisor.

Proposition 4.4. *(See, for example [H, Corollary 3.11.]) Let L be a nef divisor on a smooth algebraic K3 surface defined over $\mathbb{C}$, then L is semiample.*

Theorem 4.5. *Let X be a smooth algebraic K3 surface defined over $\mathbb{C}$. Let Γ be a group of Weil divisors on X such that $\Gamma \to \mathrm{Cl}(X)$ is an isomorphism. Then the invariant subring of $R_X(\Gamma)$ by $\mathrm{Aut}(X)$ is finitely generated over $\mathbb{C}$.*

Proof. By Corollary 3.7, $R_X(\Gamma_f)$ is finitely generated over $\mathbb{C}$. By Proposition 2.8, we put $e_1,...,e_m$ and $s_1,...,s_n$ be homogenous generators of $R_X(\Gamma_f)$ which belong to $H^0(E_1),...,H^0(E_m)$ and $H^0(D_1),...,H^0(D_n)$ respectively where E_i is a (-2)-curve and D_i is a nef divisor. Put $V = \bigoplus H^0(E_i) \bigoplus (\bigoplus_{g\in \mathrm{Aut}(X)} H^0(g^*D_i))$. Since the orbit of each D_i is finite, V is finite dimensional. The action on the sections gives the representation $\pi : \mathrm{Aut}(X) \to End(V)$. We claim the image of this map is a finite group. By the Burnside's theorem (see, for example, [U, Theorem 14.9]), it is enough to show that any element $\pi(g)$ has uniformly finite order. $\pi(g^{n!})$ induces the map $\pi(g^{n!}) : H^0(D_i) \to H^0(D_i)$ for each i and D_i. By Proposition 4.4 and Bertini's theorem, we may assume $(X, 1/mD_i)$ is klt and D_i is simple normal crossing for a positive integer m. By Theorem 4.3, we have $\pi(g^{n!})$ has finite order. Hence $\pi(g)$ has uniformly finite order. And we can also do same thing to E_i and e_i.Then $\mathrm{Cox}(X)^{\mathrm{Aut}(X)} = R_X(\Gamma_f)^{\pi(\mathrm{Aut}(X))}$ and since $R_X(\Gamma_f)$ is finitely generated and $\pi(\mathrm{Aut}(X))$ is a finite group, the right hand side is finitely generated over $\mathbb{C}$.

□

The following example implies that the invariant subring of the Cox ring by the automorphism group is not finitely generated in general.

Example 4.6. Let X be a blow up of the projective plane over k at general 9 points. Let Γ be a group of Weil divisors on X such that $\Gamma \to \mathrm{Cl}(X)$ is an isomorphism. Then the invariant subring of $R_X(\Gamma)$ by $\mathrm{Aut}(X)$ is not finitely generated over k.

Since X has infinitely many (-1)-curves, the Cox ring of X is not finitely generated. Moreover, the automorphism group of X is trivial (see, [Ko]).

REFERENCES

[AHL] M. Artebani, J. Hausen, and A. Laface, On Cox rings of K3 surfaces, Compos. Math. **146** (2010), no. 4, 964-998.

[BCHM] C. Birkar, P. Cascini, C. Hacon, and J. M^{c}Kernan, Existence of minimal models for varieties of log general type, J. Amer. Math. Soc. **23**, no. 2, 405-468.

[BZ] F. A. Bogomolov, Y. G. Zarhin, Ordinary reduction of K3 surfaces, Central European Journal of Mathematics, **7**, Issue 2, 206-213, (2009)

[C] D. A. Cox, The homogeneous coordinate ring of a toric variety, J. Algebraic Geometry **4** 17-50 (1995), .

[F] O. Fujino, Abundance theorem for semi log canonical threefolds, Duke Math. J. **102** (2000), no. 3, 513-532.

[FG] O. Fujino, Y. Gongyo, Log pluricanonical representations and the abundance conjecture, Compositio Math. **150**, No.4, (2014) 593–620.

[GOST] Y. Gongyo, S. Okawa, A. Sannai and S. Takagi, Characterizaton of varieties of Fano type via singularities of Cox rings J. Algebraic Geom. **24** (2015), no. 1, 159–182.

[H] D. Huybrechts, lectures on K3 surfaces, available at http://www.math.uni-bonn.de/people/huybrech/K3Global.pdf

[HH2] M. Hochster and C. Huneke, Tight closure in equal characteristic zero, preprint.

[HK] Y. Hu and S. Keel, Mori Dream Spaces and GIT, Michigan Math. J. **48** (2000), 331–348.

[JR] K. Joshi and C. S. Rajan, Frobenius splitting and ordinary, Int. Math.Res. Not. (2003), no. 2, 109-121.

[Ko] M. Koitabashi, Automorphism group of generic rational surfaces, J. Algebra, **116** (1) :130142, 1988

[K] S. Kovács, The cone of curves of a K3 surface. Math. Ann., **300** (4):681-691, 1994.

[MS] M. Mustaţă and V. Srinivas, Ordinary varieties and the comparison between multiplier ideals and test ideals, Nagoya Math. J. **204** (2011), 125–157.

[S] H. Sterk, Finiteness Results for Algebraic K3 surfaces, Math. Z. **189**, 507-513(1985)

[TVAV] D. Testa, A. Varilly-Alvarado, and M. Velasco, Big rational surfaces, Math. Ann. **351** (2011), no. 1, 95-107.

[U] K. Ueno, Classification theory of algebraic varieties and compact complex spaces, Lecture Notes in Math., **439**, Springer, Berlin, 1975.

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