LE MATEMATICHE
Vol. LXXIII (2018) – Fasc. I, pp. 99–114
doi: 10.4418/2018.73.1.7

OPTIMALITY CONDITIONS FOR SHARP MINIMALITY OF ORDER γ IN SET-VALUED OPTIMIZATION

TIJANI AMAHROQ - IKRAM DAIDAI - AICHA SYAM

Sharp minimizers of order γ are defined for set-valued optimization problems. Necessary and sufficient conditions are given for such minimizers, this allows us to extend the well known results obtained in the scalar and vectorial cases by Auslender [6], Studniarski [21], Ward [24] and Jiménez [12, 13].

1. Introduction

Let X be a normed space, $f : X \to \mathbb{R}$ a real-valued function and S be a subset of X. For a real number γ, a point $\bar{x} \in S$ is said to be γ-order sharp local minimizer with modulus $c > 0$ for f if there exists a neighborhood U of $\bar{x}$ such that

$$f(x) \geq f(\bar{x}) + c \parallel x - \bar{x} \parallel^{\gamma}, \quad for\, all\, x \in U \cap S. \tag{1}$$

With the notation $\mathcal{S}(f,\bar{x},U) := \{x \in U : \quad f(x) = \inf_{u \in U} f(u)\}$, a point $\bar{x} \in S$ is said to be γ-order **weak** sharp local minimizer with modulus $c > 0$ for f if there exists a neighborhood U of $\bar{x}$ such that

$$f(x) \geq f(\bar{x}) + cd(x, \mathcal{S}(f,\bar{x},U))^{\gamma}, \quad for\, all\, x \in U \cap S, \tag{2}$$

Entrato in redazione: 2 giugno 2017

AMS 2010 Subject Classification: 54C60, 58E17, 49Kxx, 46N10

Keywords: Sharp minimality of order γ, Optmality conditions, Strong convexity, Set-valued maps.

where U is the same as in (1) and, as usual, $d(x,S) = \inf_{u \in S} \parallel x - u \parallel$.
Recently the sharp minimizer and the weak sharp minimizer, have received extensive research. It has been found that weak sharp minimizers are closely related to the error bound in optimization, see Zheng et al. [26] and the references therein. Furthermore, sharp minimizers of order 2 for real-valued functions are known in Ait Mansour et al. [1] under "c-eignevalue minimizers", where the authors provide the bounds of the modulus c $(i.e., c \in]0, \frac{1}{4}])$ in the case of strong quasi-convexity on the objective function.
Sharp and weak sharp minimums are very useful in numerical optimization, error bound theory as well as parametric stability and sensitivity analysis of variational problems. In [8] Cromme considers this notion in the context of the convergence of iterative numerical procedures. Auslender [6] obtains necessary and sufficient conditions for a local minimizer of orders 1 and 2, supposing that the objective function f is locally Lipschitzian and the feasible set S is closed. Studniarski [21] extends Auslender's results to any extended real-valued function f, any subset S of $\mathbb{R}^n$ (not necessarily closed) and encompassing minimizers of order greater than 2. For this aim he used directional derivatives that are generalizations of the lower and upper Hadamard derivatives. Ward [24] follows the line of Studniarski using other derivatives and tangent cones.
Jiménez [11] extends the notion of minimizer of order γ to vector optimization problems. In several papers, he extends the notion of strict minimizer to vector optimization problems. In [12, 13], Jiménez and Jiménez and Novo develop a theory on minimizers of order γ ($\gamma \geq 1$ integer) considering different frameworks.
In all of these papers, the objective functions are real-valued or vector-valued. To our knowledge, a very limited attention has been dedicated to sharp and weak sharp minimality for set-valued optimization. In [9], Durea and Strugariu proposed the weak sharp minimiser for a set-valued optimization problem by means of the oriented distance function and discussed some necessary optimality conditions with the aid of the Mordukhovich generalized differentiation. Very recently, Zhu et al. [25] extended the Fermat rules for the local minimiser of the constrained set-valued optimization problem to the sharp and the weak sharp minimimser of order 1 in Banach spaces or Asplund spaces, by means of the Mordukhovich generalized differentiation and the normal cone.
In this paper, we extend this notion to set-valued optimization problems without recourse to the use of distances adopted in [9, 25]. Also we establish necessary and sufficient optimality conditions of a sharp minimizer of order γ for set-valued optimization problems. Sufficient optimality conditions requiring a type of strong convexity are also given.
On the other hand a type of Fritz John necessary and sufficient optimality named

sharp Fritz John necessary and sufficient optimality conditions are established. Our paper is written as follows. In Sec. 2, we present some basic definitions such that γ-strongly convex set-valued maps and provide its characterization in terms of Clarke derivative. This characterization leads us to derive optimality conditions for sharp minimality of order γ. In Sec. 3, using separation theorem we present Fritz John optimality conditions for sharp minimality of order γ. In Sec. 4, we give optimality condition for a sharp minimizer of order γ without convexity assumption and a useful sufficient condition for a sharp minimizer of order 1 in finite dimensional space.

2. Preliminaries

Throughout this paper X and Y are Banach spaces, X^* and Y^* will denote the continuous duals of X and Y, respectively, and we write $<.,.>$ for the canonical bilinear forms with respect to the dualities $<X^*,X>$ and $<Y^*,Y>$. In the sequel $\mathbb{B}_Y$ denotes the open unit ball in Y and $\overline{\mathbb{B}}_Y$ its closure and $cl(A)$ will be the topological closure of a subset A of X.
In this paper the following cones will be used.

Definition 2.1. Let $A \subset X$ and $\bar{x} \in cl(A)$.
(i) The Clarke tangent cone to A at $\bar{x}$ is

$$T_c(A,\bar{x}) = \{u \in X : \forall (x_n) \to_A \bar{x},\ \forall\ (t_n) \downarrow 0^+,\ \exists (u_n) \to u\ with\ x_n + t_n u_n \in A\ \forall n\},$$

where $x \to_A \bar{x}$ means $x \to \bar{x}$ with $x \in A$.
(ii) The tangent cone to A at $\bar{x}$ is the set

$$T(A,\bar{x}) = \{v \in X :\ \forall\ (t_n) \to 0^+,\ \exists (x_n) \to_A \bar{x},\quad such\ that\ t_n^{-1}(x_n - \bar{x}) \to v\}.$$

(iii) The contingent cone to A at $\bar{x}$ is the set

$$\begin{aligned} K(A,\bar{x}) &= \{v \in X :\ \exists (x_n) \subset A,\ \ \exists (t_n) \to 0^+,\ \ such\ that\ t_n^{-1}(x_n - \bar{x}) \to v\}, \\ &= \{v \in X :\ \exists (v_n) \to v,\ \exists (t_n) \to 0^+,\quad such\ that\ \ \bar{x} + t_n v_n \in A, \forall n \in \mathbb{N}\}. \end{aligned}$$

We denote by $N_c(A,\bar{x})$ the Clarke normal cone to A at $\bar{x}$, that is,

$$N_c(A,\bar{x}) = \{x^* \in X^* : < x^*, v > \leq 0\ \ for\ all\ v \in T_c(A,\bar{x})\}.$$

Let $F : X \rightrightarrows Y$ be a set-valued map. In the sequel we denote the domain and the graph of F respectively by

$$dom(F) = \{x \in X,\ \ F(x) \neq \emptyset\},$$

$$gr(F) = \{(x,y) \in X \times Y,\ \ y \in F(x)\}.$$

If A is a subset of X, then

$$F(A) = \underset{x \in A}{\cup} F(x).$$

Definition 2.2. Let $(\bar{x},\bar{y}) \in gr(F)$. The Clarke derivative $D_cF(\bar{x},\bar{y})$ of F at $(\bar{x},\bar{y})$ is the set-valued map from X into Y defined by

$$y \in D_cF(\bar{x},\bar{y})(x) \text{ if and only if } (x,y) \in T_c(gr(F);(\bar{x},\bar{y})).$$

Definition 2.3. [5] Let $(\bar{x},\bar{y}) \in gr(F)$. The tangent derivative $DF(\bar{x},\bar{y})$ of F at $(\bar{x},\bar{y})$ is the set-valued map from X into Y defined by

$$y \in DF(\bar{x},\bar{y})(x) \text{ if and only if } (x,y) \in T(gr(F);(\bar{x},\bar{y})).$$

Due to Definition 2.1, $y \in DF(\bar{x},\bar{y})(x)$ if and only if for all $(t_n) \to 0^+$ there exists $(x_n,y_n) \to_A (\bar{x},\bar{y})$ such that $t_n^{-1}((x_n,y_n)-(\bar{x},\bar{y})) \to (x,y)$.

Definition 2.4. Let $(\bar{x},\bar{y}) \in gr(F)$. The contingent derivative $CF(\bar{x},\bar{y})$ of F at $(\bar{x},\bar{y})$ is the set-valued map from X into Y defined by

$$y \in CF(\bar{x},\bar{y})(x) \text{ if and only if } (x,y) \in K(gr(F);(\bar{x},\bar{y})).$$

Remark 2.5. Let $(\bar{x},\bar{y}) \in gr(F)$. It is well known to see that

1. $gr(D_cF(\bar{x},\bar{y})) \subset gr(DF(\bar{x},\bar{y})) \subset gr(CF(\bar{x},\bar{y}))$.
2. $gr(D_cF(\bar{x},\bar{y})) = gr(CF(\bar{x},\bar{y}))$, whenever gr(F) is convex in $X \times Y$.

The following notions of optimality will be used in the sequel. Let Y^+ be a closed convex cone pointed in Y (that is $Y^+ \cap (-Y^+) = \{0\}$) with nonempty interior $int(Y^+)$.

Let A be a nonempty subset of Y and $\bar{y} \in A$. Then $\bar{y}$ is said to be a minimizer (respectively a weak minimizer) of A with respect to Y^+ if

$$(A-\bar{y}) \cap (-Y^+) = \{0\} \qquad (resp.\ (A-\bar{y}) \cap (-int(Y^+)) = \emptyset),$$

or, equivalently

$$A \cap (\bar{y}-Y^+) = \{\bar{y}\} \qquad (resp.\ A \cap (\bar{y}-int(Y^+)) = \emptyset).$$

We denote by $Min(A)$ the set of all minimizer points of A with respect to Y^+ and by $W.Min(A)$ the set of all weak minimizer points of A with respect to Y^+.

Let S be a nonempty subset of X and consider the multiobjective optimization problem

$$(\mathcal{P}) \qquad Minimize\, F(x)$$
$$subject\ to\ \ x \in S.$$

Let $\bar{x} \in S$ and $(\bar{x},\bar{y}) \in gr(F)$. The pair $(\bar{x},\bar{y}) \in gr(F)$ is said to be a local (respectively a local weak) minimizer of $(\mathcal{P})$ with respect to Y^+ if there exists a neighborhood U of $\bar{x}$ such that $\bar{y} \in MinF(S \cap U)$ (resp. $\bar{y} \in W.MinF(S \cap U)$). This means that for all $x \in S \cap U$

$$F(x) \subset \bar{y} + (Y \backslash (-Y^{+})) \cup \{0\}$$

$$(resp. \quad F(x) \subset \bar{y} + Y \backslash (-int(Y^{+}))).$$

We now introduce a new notion of sharp minimizer of order γ for set-valued optimization problems.

Definition 2.6. Let $\gamma > 0$ and $F : X \rightrightarrows Y$ be a set-valued map. We say that $(\bar{x}, \bar{y}) \in gr(F)$ is a local sharp minimizer of order γ for $(\mathcal{P})$ with respect to Y^{+} if there exist $c > 0$ and a neighborhood U of $\bar{x}$ such that for all $x \in S \cap U$

$$F(x) + c \parallel x - \bar{x} \parallel^{\gamma} \overline{\mathbb{B}}_Y \subset \bar{y} + (Y \backslash (-Y^{+})) \cup \{0\}. \tag{3}$$

When (3) holds for all $x \in S$, we say that $(\bar{x}, \bar{y})$ is a global sharp minimizer of order γ for $(\mathcal{P})$.

Definition 2.7. Let $\gamma > 0$ and $F : X \rightrightarrows Y$ be a set-valued map. We say that $(\bar{x}, \bar{y}) \in gr(F)$ is a local weak sharp minimizer of order γ for $(\mathcal{P})$ with respect to Y^{+} if there exist $c > 0$ and a neighborhood U of $\bar{x}$ such that for all $x \in S \cap U$

$$F(x) + c \parallel x - \bar{x} \parallel^{\gamma} \overline{\mathbb{B}}_Y \subset \bar{y} + Y \backslash (-int(Y^{+})). \tag{4}$$

When (4) holds for all $x \in S$, we say that $(\bar{x}, \bar{y})$ is a global weak sharp minimizer of order γ for $(\mathcal{P})$.

Remark 2.8. Definition 5 and Definition 6 above seem to be natural extension of the notion of sharp minimizer to set-valued maps. Indeed,
a) Definition 2.6 becomes the usual notion of sharp minimizer of order γ, when Y, $Y^{+} = \mathbb{R}^{+}$ and $F : X \to \mathbb{R}$ is a real-valued function, that is, (1). This means that Definition 2.6 generalizes the corresponding scalar notion.
b) Clearly, if $F : X \to Y$ is a vector-valued mapping, then (3) is equivalent to the definition introduced by Jiménez [11, 12], that is, there exist $c > 0$ and a neighborhood U of $\bar{x}$ such that

$$(F(x) + Y^{+}) \cap \mathbb{B}(F(\bar{x}), c \parallel x - \bar{x} \parallel^{\gamma}) = \emptyset, \quad \forall x \in S \cap U \backslash \{\bar{x}\}. \tag{5}$$

c) If $(\bar{x}, \bar{y})$ is a local (respectively a local weak) sharp minimizer of order γ for $(\mathcal{P})$ with respect to Y^{+}, then $(\bar{x}, \bar{y})$ is a local (respectively a local weak) minimizer of $(\mathcal{P})$ with respect to Y^{+}.

Now, we recall the following Definition introduced by Amahroq et al. [2]. If X is a reflexif Banach space, a real-valued function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be γ-strongly convex, if there exist $c > 0$ and a function $g : [0,1] \to \mathbb{R}^+$ with

$$\lim_{\theta \to 0} \frac{g(\theta)}{\theta} = 1 \quad and \quad g(0) = g(1) = 0 \tag{6}$$

such that for all $x,\ y \in X$ and $\theta \in [0,1]$

$$f(\theta y + (1-\theta)x) \leq \theta f(y) + (1-\theta) f(x) - cg(\theta) \parallel x - y \parallel^{\gamma}. \tag{7}$$

It has been proved in [2] , that if f is γ-strongly convex then f admits a global sharp minimizer of order γ.
Note that in [1], Ait Mansour et al. proved that a strongly quasiconvex real-valued function admits a global sharp minimizer of order 2 even in the more general setting of constrained quasi-minimization coercive problems.
As in [10], we introduce the following definition with a slight modification.

Definition 2.9. Let $\gamma > 0$ and $F : X \rightrightarrows Y$ be a set-valued map. We will say that F is γ-strongly convex if there exist a constant $c > 0$ and a function $g : [0,1] \to \mathbb{R}^+$ with

$$\lim_{\theta \to 0} \frac{g(\theta)}{\theta} = 1 \quad and \quad g(0) = g(1) = 0 \tag{8}$$

such that for all $x,\ y \in X$ and $\theta \in [0,1]$

$$\theta F(y) + (1-\theta) F(x) + cg(\theta) \parallel x - y \parallel^{\gamma} \overline{\mathbb{B}}_Y \subset F(\theta y + (1-\theta)x). \tag{9}$$

Example 2.10. Consider Example 1 in [14]. Let

$$F_1(x) = [f(x), +\infty[, \quad F_2(x) =]-\infty, g(x)], \quad F_3(x) = [f(x), g(x)], with x \in C,$$

where $f,\ g : X \to \mathbb{R} \cup \{+\infty\}$ are two extended real-valued functions and C is a convex subset of X. It has been proved that if f is 2-strongly convex and that g is 2-strongly concave (that is $-g$ satisfies (7) with $\gamma = 2$) such that $f \leq g$ on C. Then F_1, F_2 and F_3 are 2-strongly convex set-valued maps. It is not difficult to prove that the result is still valid for any $\gamma > 0$.

In order to give a characterization of $\gamma-$strong convexity of set-valued maps, let us recall the following result due to Huang [10].

Theorem 2.11. *[10] Let $F : X \rightrightarrows Y$ be a closed-graph set-valued map. If for each $(x,y),\ (x',y') \in gr(F)$ and $(x^*,y^*) \in N_c(gr(F),(x,y))$ the following inequality*

$$< x^*, x' - x > + < y^*, y' - y > + c \parallel x - x' \parallel^{\gamma} \parallel y^* \parallel \leq 0, \tag{10}$$

holds for some $c > 0$. Then F is γ-strongly convex on X.

Theorem 2.12. *Let $F : X \rightrightarrows Y$ be a γ-strongly convex set-valued map with constant $c > 0$ and $(\bar{x},\bar{y}) \in gr(F)$. Then for all $x \in X$*

$$F(x) - \bar{y} + c \parallel x - \bar{x} \parallel^{\gamma} \overline{\mathbb{B}}_Y \subset D_c F(\bar{x},\bar{y})(x - \bar{x}). \tag{11}$$

Conversely, suppose that (11) holds for any $(\bar{x},\bar{y}) \in gr(F)$ and that $gr(F)$ is closed. Then F is $\gamma-$strongly convex.

Proof. Let $y \in F(x)$ and $b \in \overline{\mathbb{B}}_Y$. Since $gr(F)$ is convex, it suffices to prove that

$$F(x) - \bar{y} + c \parallel x - \bar{x} \parallel^{\gamma} \overline{\mathbb{B}}_Y \subset DF(\bar{x},\bar{y})(x - \bar{x}).$$

Let $\lambda_n \to 0^+$, we may suppose that $\lambda_n \in]0,1]$, for all n. Put

$$\begin{cases} x_n := \bar{x} + \lambda_n(x - \bar{x}) \\ y_n := \bar{y} + \lambda_n(y - \bar{y}) + cg(\lambda_n) \parallel x - \bar{x} \parallel^{\gamma} b, \end{cases}$$

where $g : [0,1] \to \mathbb{R}^+$ is given by (8) and (9). Due to the γ-strong convexity of F, we obtain that $(x_n, y_n) \in gr(F)$ for all n. Further,

$$\lambda_n^{-1}(x_n - \bar{x}, y_n - \bar{y}) \to (x - \bar{x}, y - \bar{y} + c \parallel x - \bar{x} \parallel^{\gamma} b).$$

Thus, $(x - \bar{x}, y - \bar{y} + c \parallel x - \bar{x} \parallel^{\gamma} b) \in T(gr(F);(\bar{x},\bar{y}))$ for all $y \in F(x)$ and $b \in \overline{\mathbb{B}}_Y$. What means that

$$F(x) - \bar{y} + c \parallel x - \bar{x} \parallel^{\gamma} \overline{\mathbb{B}}_Y \subset DF(\bar{x},\bar{y})(x - \bar{x}).$$

Conversely, if (11) holds for any $(\bar{x},\bar{y}) \in gr(F)$, then

$$(x - \bar{x}, y - \bar{y} + c \parallel x - \bar{x} \parallel^{\gamma} b) \in T_c(gr(F);(\bar{x},\bar{y}))$$

for all $y \in F(x)$ and $b \in \overline{\mathbb{B}}_Y$. Thus for all $(x^*,y^*) \in N_c(gr(F);(\bar{x},\bar{y}))$, $y \in F(x)$ and $b \in \overline{\mathbb{B}}_Y$ we obtain

$$< (x^*,y^*),(x - \bar{x}, y - \bar{y} + c \parallel x - \bar{x} \parallel^{\gamma} b) > \leq 0.$$

Since $b \in \overline{\mathbb{B}}_Y$ is arbitrary, so that for all $(\bar{x},\bar{y})$, $(x,y) \in gr(F)$ and $(x^*,y^*) \in N_c(gr(F);(\bar{x},\bar{y}))$

$$< x^*, x - \bar{x} > + < y^*, y - \bar{y} > + c \parallel x - \bar{x} \parallel^{\gamma} \parallel y^* \parallel \leq 0.$$

This implies that F is γ-strongly convex set-valued map by Theorem 2.11. □

Corollary 2.13. *Let $F : X \rightrightarrows Y$ be a closed-graph set-valued map. Then F is γ-strongly convex on X if and only if*

$$F(x) - \bar{y} + c \parallel x - \bar{x} \parallel^{\gamma} \overline{\mathbb{B}}_Y \subset D_c F(\bar{x},\bar{y})(x - \bar{x}), \tag{12}$$

for all $x \in X$ and $(\bar{x},\bar{y}) \in gr(F)$.

When the set-valued map F is γ-strongly convex, the following proposition proves that any local sharp minimizer of order γ for $(\mathcal{P})$ is a global sharp minimizer of order γ for $(\mathcal{P})$.

Proposition 2.14. *Let $(\bar{x},\bar{y}) \in gr(F)$, $\gamma \geq 1$ and $S \subset X$ be nonempty convex subset of X. Suppose that $F : X \rightrightarrows Y$ is γ-strongly convex on S. Then $(\bar{x},\bar{y})$ is a local sharp minimizer of order γ for $(\mathcal{P})$ with respect to Y^+ if and only if it is a global sharp minimizer of order γ for $(\mathcal{P})$ with respect to Y^+.*

Proof. Since $(\bar{x},\bar{y})$ is a local sharp minimizer of order γ with respect to Y^+ for $(\mathcal{P})$, then there exist $c_1 > 0$ and $\delta > 0$ such that for all $z \in S \cap \mathbb{B}(\bar{x},\delta)$

$$F(z) + c_1 \parallel z - \bar{x} \parallel^{\gamma} \overline{\mathbb{B}}_Y \subset \bar{y} + (Y \backslash (-Y^+)) \cup \{0\}. \tag{13}$$

Now let $z \in S \backslash \mathbb{B}(\bar{x},\delta)$, one gets for $t > 0$ sufficiently small $w := \bar{x} + t(z - \bar{x}) \in S \cap \mathbb{B}(\bar{x},\delta)$. So that

$$F(w) + c_1 \parallel w - \bar{x} \parallel^{\gamma} \overline{\mathbb{B}}_Y \subset \bar{y} + (Y \backslash (-Y^+)) \cup \{0\}.$$

Using the $\gamma-$strong convexity of F, we obtain for some $c_2 > 0$

$$tF(z) + (1-t)\bar{y} + c_2 g(t) \parallel z - \bar{x} \parallel^{\gamma} \overline{\mathbb{B}}_Y \subset \bar{y} + (Y \backslash (-Y^+)) \cup \{0\}.$$

Hence

$$F(z) - \bar{y} + c_2 \frac{g(t)}{t} \parallel z - \bar{x} \parallel^{\gamma} \overline{\mathbb{B}}_Y \subset (Y \backslash (-Y^+)) \cup \{0\}.$$

Letting $t \to 0^+$, we obtain

$$F(z) - \bar{y} + c_2 \parallel z - \bar{x} \parallel^{\gamma} \overline{\mathbb{B}}_Y \subset (Y \backslash (-Y^+)) \cup \{0\}. \tag{14}$$

With $c = \min(c_1, c_2)$, from (13) and (14) it follows that for all $z \in S$

$$F(z) - \bar{y} + c \parallel z - \bar{x} \parallel^{\gamma} \overline{\mathbb{B}}_Y \subset (Y \backslash (-Y^+)) \cup \{0\}.$$

□

From Remark 2.8 necessary optimality condition for sharp minimizer of order γ can be derived from the well known results as in Corley and Taa [7, 22]. Also in the strong convex setting one can derive sufficient optimality condition for sharp minimizer of order γ as in the convex case, but for the convenience of the reader we give an easy and direct proof.

Theorem 2.15. *Let $(\bar{x},\bar{y}) \in gr(F)$ and $\gamma \geq 1$. If $(\bar{x},\bar{y})$ is a local weak sharp minimizer of order γ for $(\mathcal{P})$, then*

$$CF_S(\bar{x},\bar{y})(x) \cap (-int(Y^+)) = \emptyset \quad for\, all \;\; x \in S. \tag{15}$$

Where F_S denotes the restriction of F to S, that is $F_S(x) = F(x)$ for $x \in S$ and $F_S(x) = \emptyset$ for $x \notin S$.

Proof. Suppose the contrary that, for some $x \in S$, there exists $y \in CF_S(\bar{x}, \bar{y})(x) \cap (-int(Y^+))$.
It follows that there exist $(t_n) \to 0^+$ and $(x_n, y_n) \to (x, y)$ with $(x_n) \subset S$ such that

$$\bar{y} + t_n y_n \in F(\bar{x} + t_n x_n) \quad for\ all\ n \in \mathbb{N}.$$

Let $c > 0$. Since $y \neq 0$, then there exist $b_n \in \overline{\mathbb{B}}_Y \cap (-int(Y^+))$ and $n_0 \in \mathbb{N}$ such that

$$\bar{y} + t_n y_n + c t_n^\gamma \| x_n \|^\gamma b_n \in -int(Y^+) + \bar{y}, \quad \forall\, n \geq n_0.$$

Putting $w_n := \bar{y} + t_n y_n$ and $u_n := \bar{x} + t_n x_n$, it follows that

$$w_n \in F(u_n),$$

and

$$w_n - \bar{y} + c \| u_n - \bar{x} \|^\gamma b_n \in -int(Y^+) \quad for\ all\ \ n \geq n_0,$$

in contradiction to $(\bar{x}, \bar{y})$ is a local weak sharp minimizer of order γ for $(\mathcal{P})$.

□

Sufficient conditions based on γ-strong convexity are now stated for problem $(\mathcal{P})$.

Theorem 2.16. *Let $(\bar{x}, \bar{y}) \in gr(F)$. If S is a convex set, F is a γ-strongly convex set-valued map on S and*

$$DF(\bar{x}, \bar{y})(x - \bar{x}) \cap (-Y^+) = \{0\}, \quad for\ all\ x \in S, \tag{16}$$

$$(respectively, \qquad DF(\bar{x}, \bar{y})(x - \bar{x}) \cap (-int(Y^+)) = \emptyset, \quad for\ all\ x \in S) \tag{17}$$

then $(\bar{x}, \bar{y})$ is a sharp minimizer (respectively, weak sharp minimizer) of order γ for $(\mathcal{P})$.

Proof. Since F is a γ-strongly convex set-valued map on S, it follows by Theorem 2.12, that for all $x \in S$

$$(F(x) - \bar{y} + c \| x - \bar{x} \|^\gamma \overline{\mathbb{B}}_Y) \cap (-Y^+) \subset DF(\bar{x}, \bar{y})(x - \bar{x}) \cap (-Y^+) = \{0\}. \tag{18}$$

Thus $(\bar{x}, \bar{y})$ is a global sharp minimizer of order γ for $(\mathcal{P})$. □

3. Sharp Fritz John multipliers

In this section we establish sharp Fritz John necessary and sufficient optimality conditions for sharp minimizer of order γ for the problem $(\mathcal{P})$.

In the next, the following set

$$Y^{+i} = \{\varphi \in Y^*, \quad \varphi(y) \geq 0, \;\; for\, all\;\; y \in Y^+\},$$

denotes the nonnegative dual cone of Y^+. We say that $\varphi \in Y^{+i}$ is definitely positive, if $\varphi(y) > 0$ for all $y \in int(Y^+)$, and strictly positive, if $\varphi(y) > 0$ for all $y \in Y^+ \backslash \{0\}$.

In the following, we show that weak sharp minimizers of order γ for problem $(\mathcal{P})$ are exactly minimizers for the following real-valued function :

$$\psi(x,y) = \varphi(y) - c \parallel x - \bar{x} \parallel^{\gamma} \parallel \varphi \parallel,$$

for some elements $\varphi \in Y^{+i}$. Such element φ will be called sharp multipliers.

Theorem 3.1. (**a**) *Suppose that* $(\bar{x},\bar{y})$ *is a weak sharp minimizer of order* γ *for* $(\mathcal{P})$ *and that* $F(x)$ *is convex for all* $x \in S$*. Then for all* $x \in S$*, there exist a definitely positive* $\varphi \in Y^{+i}$ *and* $c > 0$ *such that*

$$\varphi(y) - \varphi(\bar{y}) \geq c \parallel x - \bar{x} \parallel^{\gamma} \parallel \varphi \parallel, \quad for\, all\; y \in F(x)\; with\; x \in S. \tag{19}$$

(**b**) *If there exist a strictly (resp. definitely) positive* $\varphi \in Y^{+i}$ *and* $c > 0$ *such that*

$$\varphi(y) - \varphi(\bar{y}) \geq c \parallel x - \bar{x} \parallel^{\gamma} \parallel \varphi \parallel, \quad for\, all\; y \in F(x)\;\; with\;\; x \in S,$$

then $(\bar{x},\bar{y})$ *is a (resp. weak) sharp minimizer of order* γ *for* $(\mathcal{P})$.

Proof. (**a**) Let $x \in S$, then there exists $c > 0$ such that

$$(F(x) - \bar{y} + c \parallel x - \bar{x} \parallel^{\gamma} \overline{\mathbb{B}}_Y) \cap -int(Y^+) = \emptyset.$$

We have $F(x) - \bar{y} + c \parallel x - \bar{x} \parallel^{\gamma} \overline{\mathbb{B}}_Y$ is a convex set. By separation theorem, there exists $\varphi \in Y^*$ that does not vanish identically and $\alpha \in \mathbb{R}$ such that

$$\varphi(z) \leq \alpha, \quad for\, all\; z \in -int(Y^+), \tag{20}$$

and

$$\varphi(z) \geq \alpha, \quad for\, all\; z \in (F(x) - \bar{y} + c \parallel x - \bar{x} \parallel^{\gamma} \overline{\mathbb{B}}_Y). \tag{21}$$

But since $\frac{1}{n}z \in -int(Y^+)$ for all $z \in -int(Y^+)$ and $n \in \mathbb{N}^*$, the continuity of φ gives from (20) that $\varphi \in Y^{+i}$. Similarly, we obtain that $\alpha \geq 0$. Finally, let $y \in F(x)$ and $b \in \overline{\mathbb{B}}_Y$, from (21) we obtain

$$\varphi(y) - \varphi(\bar{y}) \geq c \parallel x - \bar{x} \parallel^{\gamma} \parallel \varphi \parallel, \quad for\, all \;\; y \in F(x).$$

On the other hand, φ is a definitely positive functional. Otherwise, there exists $\hat{y} \in int(Y^+)$ such that $\varphi(\hat{y}) = 0$. So that for some $r > 0$, we get

$$\varphi(\hat{y} + rb) \geq 0, \quad for\, all \;\; b \in \overline{\mathbb{B}}.$$

As a consequence $\parallel \varphi \parallel = 0$, which is in contradiction with φ is not a zero functional.
(**b**) Let $x \in S$, by assumption we have

$$\varphi(y - \bar{y} + c \parallel x - \bar{x} \parallel^{\gamma} b) \geq 0, \quad for\, all\, b \in \overline{\mathbb{B}} \;\; and\; y \in F(x).$$

Since φ is strictly positive, so that, if $v \in F(x) - \bar{y} + c \parallel x - \bar{x} \parallel^{\gamma} \overline{\mathbb{B}}$, we obtain that $v \notin (-Y^+)\backslash\{0\}$. Thus

$$(F(x) - \bar{y} + c \parallel x - \bar{x} \parallel^{\gamma} \overline{\mathbb{B}}_Y) \cap (-Y^+)\backslash\{0\} = \emptyset.$$

The last assertion shows that $(\bar{x}, \bar{y})$ is a sharp minimizer of order γ for $(\mathcal{P})$. □

4. Necessary and Sufficient Optimality Conditions without convexity assumption

The aim of this section is to give sufficient condition for a point $\bar{x}$ to be a sharp minimizer of order 1 for the problem $(\mathcal{P})$.

Proposition 4.1. *Let $F : X \rightrightarrows Y$, $(\bar{x}, \bar{y}) \in gr(F)$ with $\bar{x} \in S$ and $\gamma \geq 1$. Then $(\bar{x}, \bar{y})$ is not a local sharp minimizer of order γ for $(\mathcal{P})$ with respect to Y^+ if and only if there exist sequences $x_n \in S\backslash\{\bar{x}\}$, $y_n \in F(x_n)$ and $d_n \in Y^+\backslash\{0\}$ such that $x_n \to \bar{x}$ and*

$$\lim_{n \to +\infty} \frac{y_n - \bar{y} + d_n}{\parallel x_n - \bar{x} \parallel^{\gamma}} = 0. \tag{22}$$

Proof. Part "if". Let (x_n), (y_n) and (d_n) satisfying (22).
Reasoning "ad absurdum", suppose that $(\bar{x}, \bar{y})$ is a local sharp minimizer of order γ for $(\mathcal{P})$ with respect to Y^+. Then there exist δ, $c > 0$ such that

$$F(x) - \bar{y} + c \parallel x - \bar{x} \parallel^{\gamma} \overline{\mathbb{B}}_Y \subset (Y\backslash(-Y^+)) \cup \{0\}, \quad for\, all\, x \in \mathbb{B}_X(\bar{x}, \delta) \cap S.$$

Now for $\varepsilon = \min(\delta, c)$, from (22) there exists $n_0 = n_0(\varepsilon)$ such that for each $n \geq n_0$, we have $x_n \in \mathbb{B}_X(\bar{x}, \varepsilon) \cap S$, $y_n \in F(x_n)$ and

$$y_n + d_n \in \overline{\mathbb{B}}_Y(\bar{y}, \varepsilon \parallel x_n - \bar{x} \parallel^\gamma) \subset \overline{\mathbb{B}}_Y(\bar{y}, c \parallel x_n - \bar{x} \parallel^\gamma).$$

Hence for each $n \geq n_0$, there exists $b_n \in \overline{\mathbb{B}}_Y$ such that

$$-d_n = y_n - \bar{y} + c \parallel x_n - \bar{x} \parallel^\gamma b_n \in (Y \backslash (-Y^+)) \cup \{0\},$$

we have a contradiction since $d_n \in Y^+ \backslash \{0\}$.
Part "only if". By assumption, for all $\delta > 0$ and for all $c > 0$, there exist $x \in \mathbb{B}_X(\bar{x}, \delta) \cap S \backslash \{\bar{x}\}$, $b \in \overline{\mathbb{B}}_Y$ and $y \in F(x)$ such that

$$y - \bar{y} + c \parallel x - \bar{x} \parallel^\gamma b \in -Y^+ \backslash \{0\}.$$

In particular, for all $n \in \mathbb{N}^*$, taking $\delta = \frac{1}{n}$ and $c = \frac{1}{n}$, there exist $x_n \in \mathbb{B}_X(\bar{x}, \frac{1}{n}) \cap S \backslash \{\bar{x}\}$, $y_n \in F(x_n)$, $b_n \in \overline{\mathbb{B}}_Y$ and $d_n \in Y^+ \backslash \{0\}$ such that

$$-d_n := y_n - \bar{y} + \frac{1}{n} \parallel x_n - \bar{x} \parallel^\gamma b_n,$$

that is,

$$\frac{\parallel y_n - \bar{y} + d_n \parallel}{\parallel x_n - \bar{x} \parallel^\gamma} < \frac{1}{n},$$

and the claim follows. □

In the remaining of this work, for a vector $y \in \mathbb{R}^p$ its components are denoted by y^i, with $i \in \{1, ..., p\}$.

Proposition 4.2. *Let $F : X \rightrightarrows Y$, $(\bar{x}, \bar{y}) \in gr(F)$ with $\bar{x} \in S$ and $\gamma > 0$.*
(a) If there exist $d \in Y^+ \backslash \{0\}$, sequences $x_n \in S \backslash \{\bar{x}\}$ and $y_n \in F(x_n)$ such that

$$\lim_{n \to +\infty} \frac{y_n - \bar{y}}{\parallel x_n - \bar{x} \parallel^\gamma} = -d \in -Y^+ \backslash \{0\}, \tag{23}$$

then $(\bar{x}, \bar{y})$ is not a local sharp minimizer of order γ for $(\mathcal{P})$ with respect to Y^+.
(b) Let $Y = \mathbb{R}^p$ and $Y^+ = \mathbb{R}^p_+$. If $(\bar{x}, \bar{y})$ is not a local sharp minimizer of order γ for $(\mathcal{P})$ with respect to Y^+, then there exist sequences $x_n \in S \backslash \{\bar{x}\}$ and $y_n \in F(x_n)$ such that

$$\lim_{n \to +\infty} \frac{y_n - \bar{y}}{\parallel x_n - \bar{x} \parallel^\gamma} = \overline{d} \in [-\infty, 0]^p. \tag{24}$$

(c) Conversely to (b), if there exist sequences $x_n \in S \backslash \{\bar{x}\}$, $x_n \to \bar{x}$, $y_n \in F(x_n)$ and $\overline{d} \in [-\infty, 0]^p$ such that

$$\lim_{n \to +\infty} \frac{y_n - \bar{y}}{\parallel x_n - \bar{x} \parallel^\gamma} = \overline{d}, \tag{25}$$

then $(\bar{x}, \bar{y})$ is not a local sharp minimizer of order γ for the problem $(\mathcal{P})$ with respect to Y^+.

Proof. (a) By assumption, we have that

$$\lim_{n\to+\infty} \frac{y_n - \bar{y} + \| x_n - \bar{x} \|^\gamma d}{\| x_n - \bar{x} \|^\gamma} = 0.$$

Let $d_n := \| x_n - \bar{x} \|^\gamma d$, since $d_n \in Y^+ \backslash \{0\}$ for all n, we can apply Proposition 4.1, and the conclusion follows.

(b) By Proposition 4.1, there exist sequences $x_n \in S \backslash \{\bar{x}\}$, $y_n \in F(x_n)$ and $\tilde{d}_n \in \mathbb{R}^p_+ \backslash \{0\}$ such that $x_n \to \bar{x}$ and

$$\lim_{n\to+\infty} \frac{y_n - \bar{y} + \tilde{d}_n}{\| x_n - \bar{x} \|^\gamma} = 0. \tag{26}$$

Let

$$d_n = \frac{\tilde{d}_n}{\| x_n - \bar{x} \|^\gamma}, \qquad a_n = \frac{y_n - \bar{y}}{\| x_n - \bar{x} \|^\gamma}, \qquad and \quad b_n = a_n + d_n, \quad for\ all\ n.$$

With this notation, the equation (26) establishes that $\lim\limits_{n\to+\infty} b_n = 0_p \in \mathbb{R}^p$. Since $d_n \in \mathbb{R}^p_+ \backslash \{0\}$, we may construct a subsequence (d_{k_n}) such that $d = \lim\limits_{n\to+\infty} d_{k_n} \in [0, +\infty]^p$. As $b_{k_n} = a_{k_n} + d_{k_n}$, taking the limit, we get

$$0 = \lim_{n\to+\infty} b_{k_n} = \lim_{n\to+\infty} a_{k_n} + \lim_{n\to+\infty} d_{k_n}.$$

Therefore, $\overline{d} := \lim\limits_{n\to+\infty} a_{k_n} = -d \in [-\infty, 0]^p$, and the result is proved.

(c) Let $\overline{d} \in [-\infty, 0]^p$ satisfying (25).

If $-\overline{d} \in \mathbb{R}^p_+ \backslash \{0\}$ the result follows from (a).

If $-\overline{d} \notin \mathbb{R}^p_+ \backslash \{0\}$, some of the component of $\overline{d}$ is $-\infty$. Reordering, we can suppose that $\overline{d} = (\overline{d}^1, \ldots, \overline{d}^k, \overline{d}^{k+1}, \ldots, \overline{d}^p)$ with $\overline{d}^i = -\infty$ for $i = 1, \ldots, k$ and $\overline{d}^i \in]-\infty, 0]$ if $i > k$ with $k \geq 1$.

We have that

$$\lim_{n\to+\infty} \frac{y_n^i - \bar{y}^i}{\| x_n - \bar{x} \|^\gamma} = -\infty,$$

for $i = 1, \ldots, k$. Thus, since for n large enough and for $i = 1, \ldots, k$, $d_n^i := -(y_n^i - \bar{y}^i) > 0$. Let

$$d_n = (d_n^1, \ldots, d_n^k, -\| x_n - \bar{x} \|^\gamma \overline{d}^{k+1}, \ldots, -\| x_n - \bar{x} \|^\gamma \overline{d}^p) \in \mathbb{R}^p_+ \backslash \{0\}.$$

Clearly,

$$\lim_{n\to+\infty} \frac{y_n - \bar{y} + d_n}{\| x_n - \bar{x} \|^\gamma} = 0,$$

and using Proposition 4.1, we conclude the result. □

From the above result, we obtain the following sufficient optimality condition of a local sharp minimizer of order 1 for problem $(\mathcal{P})$.

Theorem 4.3. *Let* $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$, $(\bar{x},\bar{y}) \in gr(F)$ *with* $\bar{x} \in S$ *and* $Y^+ = \mathbb{R}^p_+$. *If*

$$CF(\bar{x},\bar{y})(x) \cap \mathbb{R}^p_- = \emptyset, \quad \forall\, x \in K(S,\bar{x})\backslash\{0\}. \tag{27}$$

Then $(\bar{x},\bar{y})$ *is a local sharp minimizer of order 1 for the problem* $(\mathcal{P})$.

Proof. Suppose that $(\bar{x},\bar{y})$ is not a local sharp minimizer of order 1 for the problem $(\mathcal{P})$. Then, by Proposition 4.2 there exist sequences $x_n \in S\backslash\{\bar{x}\}$, $y_n \in F(x_n)$ such that $x_n \to \bar{x}$ and

$$\lim_{n\to+\infty} \frac{y_n - \bar{y}}{\| x_n - \bar{x} \|} = w \in [-\infty, 0]^p.$$

Put $v_n := (x_n - \bar{x})/t_n$ with $t_n = \| x_n - \bar{x} \|$ and $w_n := (y_n - \bar{y})/t_n$. By extracting subsequence if necessary, we may suppose that

$$\lim_{n\to+\infty} v_n = v \in K(S,\bar{x})\backslash\{0\}.$$

Since $\bar{y} + t_n w_n \in F(\bar{x} + t_n x_n)$ for all n, it follows that

$$w \in CF(\bar{x},\bar{y})(v) \cap \mathbb{R}^p_-,$$

which is a contradiction to the hypothesis (27). □

Conclusion. For a perspective research, it could be interesting to express our definitions of sharp minima in an equivalent way by the use of distances from a point to a set, which could link our results to quantitative stability for set-valued optimization.

Acknowledgements

The authors are grateful to an anonymous referee and the managing editor for their valuable comments, especially for providing them with useful references related to the topic, which helped them to improve the paper.

REFERENCES

[1] M. Ait Mansour, R.-A. Elakri and M. Laghdir, *Equilibrium and quasi-equilibrium problems under φ-quasimonotonicity and φ-quasiconvexity*, Minimax Theory and its Appl. 2 (2017), 175-229.

[2] T. Amahroq, I. Daidai and A. Syam, *γ-strongly convex functions, γ-strong monotonicity of their presubdifferential and γ-subdifferentiability, application to nonlinear PDE*. Journal of Nonlinear and convex analysis, to appear (2017).

[3] T. Amahroq and A. Taa, *Sufficient conditions of optimality for multiobjective optimization problems with γ-paraconvex data*, Studia Math. 124 (1997), 239-247.

[4] T. Amahroq and A. Taa, *On Lagrange-Kuhn-Tucker multipliers for multiobjective optimization problems*, Optimization 41 (1997), 159-172.

[5] J. P. Aubin and H. Frankowska, *Set-valued analysis*, Birkhauser, Boston, 1990.

[6] A. Auslender, *Stability in mathematical programming with nondifferentiable data*, SIAM J. Control Optim. 22 (1984), 239-254.

[7] H. W. Corley, *Optimality conditions for maximization of set-valued functions*, J. Optim. Theory Appl. 58 (1988), 1-10.

[8] L. Cromme, *Strong uniqueness: a far reaching criterion for the convergence of iterative procedures*, Numerische Mathematik 29 (1978), 179-193.

[9] M. Durea and R. Strugariu, *Necessary optimality conditions for weak sharp minima in set-valued optimization*, Nonlinear Analysis 73 (2010), 2148-2157.

[10] H. Huang, *Global error bounds with exponents for multifunctions with set constraints*, Commun. Contemp. Math. 12 (2010), 417-435.

[11] B. Jiménez, *Strict efficiency in vector optimization*, J. Math. Anal. Appl. 265 (2002), 264-284.

[12] B. Jiménez, *Strict minimality conditions in nondifferentiable multiobjective programming*, Journal of Optimization Theory and Applications 116 (2003), 99-116.

[13] B. Jiménez and V. Novo, *Higher-order optimality conditions for strict local minima*, Ann. Oper. Res., 157 (2008), 183-192.

[14] H. Leiva, N. Merentes and K. Nikodem, *Strongly convex set-valued maps*, J. Glob. Optim. 57 (2013), 695-705.

[15] D. T. Luc and C. Malivert, *Invex optimization problems*, Bull. Aust. Math. Soc. 46 (1992), 47-66.

[16] D. T. Luc, Contingent derivatives of set-valued maps and applications to vector optimization, Math. Programming, 50 (1991), 99-111.

[17] J. -P. Penot, *Differentiability of relations and differential stability of perturbed optimization problems*, SIAM J. Contr. Optim. 22 (1984), 529-551.

[18] B. T. Polyak, *Existence theorems and convergence of minimizing sequences in extremum problems with restrictions*, Dokl. Akad. Nauk. SSSR, 166 (1966), 287-290.

[19] R. T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optim. 14 (1976), 877-898.

[20] D. S. Shi, *Contingent derivative of the perturbation map in multiobjective optimization*, Journal of Optimization theory and Applications 70 (1991), 385-395.

[21] M. Studniarski, *Necessary and sufficient conditions for isolated local minima of nonsmooth functions*, SIAM J. Control Optim. 24 (1986), 1044-1049.

[22] A. Taa, *Necessary and sufficient conditions for multiobjective optimization problems*, Optimization 36 (1996), 97-104.

[23] A. Taa, *Set-valued derivative of multifunctions and optimality conditions*, Numer. Func. Anal. and Optimz. 19 (1998) 121-140.

[24] D. E. Ward, *Characterizations of strict local minima and necessary conditions for weak sharp minima*, J. Optim. Theory Appl. 80 (1994), 551-571.

[25] S. K. Zhu, S. J. Li and X. W. Xue, *Strong Fermat Rules for Constrained Set-Valued Optimization Problems on Banach Spaces*, Set-Valued Var. Anal. 20 (2012), 637-666.

[26] X.-Y. Zheng and K. F. NG, *Hölder stable minimizers, tilt stability and Hölder metric regularity of subdifferentials*, SIAM Journal. Optimization 25 (2015), 416-438.

TIJANI AMAHROQ
Department of mathematics, University Cadi Ayyad
Faculté des sciences et techniques, Marrakech, Morocco
e-mail: amahroq@gmail.com

IKRAM DAIDAI
Department of mathematics, University Cadi Ayyad
Faculté des sciences et techniques, Marrakech, Morocco
e-mail: ikram_daidai_57@yahoo.fr

AICHA SYAM
Department of mathematics, University Cadi Ayyad
Faculté des sciences et techniques, Marrakech, Morocco
e-mail: aicha.syam@gmail.com