LE MATEMATICHE
Vol. LXXIII (2018) – Fasc. I, pp. 115–125
doi: 10.4418/2018.73.1.8

α-STRONG APPROXIMATE SOLUTIONS TO QUASI-VARIATIONAL INEQUALITIES

M. AIT MANSOUR - J. LAHRACHE - N.- E. ZIANE

In this paper, based on the regularization of non necessarily semicontinuous set-valued maps established in Ait Mansour et al. [2] [*A lower semicontinuous regularization for set-valued maps and its applications*, J. Conv. Analysis. 15 (2008) 473-484], we prove existence results of strong approximate solutions to quasivariational inequalities, (QVI), without any continuity condition on their related operators. A condition ensuring the convergence of a sequence of such solutions to an exact one is also provided. Moreover, we observe that the regularization of [2] leads to new approximate solutions of a weak type. The latter is not in the scope of this note as it poses a new open geometrical question on the normal cone to a subset, which we underline by the conclusion section.

1. Introduction

In this paper, on the basis of the regularization of non necessarily semicontinuous set-valued maps established in [2], we establish the existence of strong approximate solutions to quasi-variational inequalities (QVI) with non necessarily semicontinuous operators. Under an appropriate regularity condition, we as well obtain the convergence of these solutions to an exact solution. This class of inequalities is very interesting as it finds many applications in different branches

Entrato in redazione: 10 gennaio 2017
AMS 2010 Subject Classification: 49J20
Keywords: Regularization, quasi-variational inequality, strong approximate solutions, convergence condition

in applied mathematics from optimization, control, calculus of variations, game theory and economics, transportation and networks, nonsmooth mechanic to engineering and structural analysis. Specific applied models formulated as (QVI) are established in many articles, quote for example: the multi-leader-follower games in Fukushima and Pang [7], superconductivity, thermoplasticity, electrostatic with implicit ionization threshold in Kunze and Rodrigues [9], obstacle problems in Mosco and Joly [8], quasiconvex programming and traffic network in Aussel and Cotrina [4], finite dimensional stable elastic traffic network in Ait Mansour and Scrimali [3] and coercive quasi-minimization of second type semistrictly quasiconvex functions in Ait Mansour, Elakri and Laghdir [1].

Let us now introduce the quasi-variational inequality problem subject to our treatment: Let $\mathcal{K}$ be a nonempty closed, convex subset of a Banach space X whose dual, norm and duality pairing are respectively denoted by $X^\star$, $\|.\|$ and $\langle .,. \rangle$. The feasibility set-valued map will be denoted by $\mathbb{K} : \mathcal{K} \rightrightarrows \mathcal{K}$. It will be assumed to have nonempty, closed and convex values. For a closed convex subset D of X, $N_D(\bar{x})$ will stand for the normal cone to D at a point $\bar{x} \in D$, which is given by

$$N_D(\bar{x}) = \{x^\star \in X^\star |\ \langle x^\star, x - \bar{x} \rangle \leq 0\ \forall x \in D\}.$$

Given a set-valued operator $T : X \rightrightarrows X^\star$, the corresponding set-valued quasi-variational inequality problem, $QVI(T, \mathbb{K})$, is defined as follows: Find $\bar{x} \in \mathcal{K}$ s.t. $\bar{x} \in \mathbb{K}(\bar{x})$ and

$$0 \in T(\bar{x}) + N_{\mathbb{K}(\bar{x})}(\bar{x}). \tag{1}$$

This inequality is satisfied if, and only if, there exists some $x^\star \in T(\bar{x})$ such that

$$-x^\star \in N_{\mathbb{K}(\bar{x})}(\bar{x}); \tag{2}$$

from the assumptions on $\mathbb{K}$, this is equivalent to

$$\langle x^\star, y - \bar{x} \rangle \geq 0,\ \forall y \in \mathbb{K}(\bar{x}). \tag{3}$$

A fundamental remark to make at a first analysis is that the problem (QVI) generates the following family of variational inequalities $VI(T, \mathbb{K}(x))_{x \in \mathcal{K}}$: For all $x \in \mathcal{K}$, find $y(x) := y \in \mathbb{K}(x)$ and $y^\star \in T(y)$ such that

$$\langle y^\star, z - y \rangle \geq 0,\ \forall z \in \mathbb{K}(x). \tag{4}$$

Then, introducing the corresponding solution map $S : \mathcal{K} \rightrightarrows \mathcal{K}$ defined for each $x \in \mathcal{K}$ by

$$S(x) := \{y \in \mathbb{K}(x),\ \exists y^\star \in T(y) \,|\, \langle y^\star, z - y \rangle \geq 0,\ \forall z \in \mathbb{K}(x)\}, \tag{5}$$

we see that the solutions of (QVI) coincide with the fixed points of S. This fixed point scheme, widely used in the literature (see for instance [4, 8] and references therein), makes use of classical fixed point theorems such as the Kakutani's theorem and its variants, which require the upper semicontinuity of the solution map S in addition to the convexity of its values. These properties are not easy to check, in general, and they always require monotonicity and continuity type conditions on T. The alternative we propose here is to look for adequate concepts of approximate solutions that may converge to an exact solution under appropriate regularity conditions on the operator T. To do that, let us first introduce two kinds of approximation of the normal cone to a closed convex subset $D \subset X$ at a point as follows:

Definition 1.1. For a fixed $\alpha > 0$,

1. $s - N_D^{\alpha}(\bar{x})$ will stand for the *strong α-approximate normal cone* to D at a point $\bar{x} \in D$, which we define by

$$s - N_D^{\alpha}(\bar{x}) = \{x^\star \in X^\star |\ \langle x^\star, x - \bar{x}\rangle \le \alpha \|x - \bar{x}\|,\ \ \forall x \in D\}$$

   (see [10], p. 6);

2. $w - N_D^{\alpha}(\bar{x})$ will stand for the *weak α-approximate normal cone* to D at a point $\bar{x} \in D$, which we define by

$$w - N_D^{\alpha}(\bar{x}) = \{x^\star \in X^\star |\ \langle x^\star, x - \bar{x}\rangle \le \alpha \|x^\star\|,\ \ \forall x \in D\}.$$

Note that the set s-$N_D^{\alpha}(\bar{x})$ is convex, but it is not a cone, in general, whereas the set w-$N_D^{\alpha}(\bar{x})$ is a cone, but not necessarily convex. The classic normal cone $N_D(\bar{x})$ is contained in both of them.
According to these definitions, let us introduce the following concepts of strong and weak approximate quasi variational inequalities as follows:

Definition 1.2. Let $\alpha > 0$.

1. The *strong* quasi-variational inequality problem associated with $QVI(T,\mathbb{K})$ is defined as follows: Find $\bar{x} \in \mathcal{K}$ such that $\bar{x} \in \mathbb{K}(\bar{x})$ and $x^\star \in T(\bar{x})$ such that

$$(s,\alpha) - QVI(T,\mathbb{K}) \qquad -x^\star \in s - N^{\alpha}_{\mathbb{K}(\bar{x})}(\bar{x}).$$

2. The *weak* quasi-variational inequality problem associated with $QVI(T,\mathbb{K})$ is defined as follows: Find $\bar{x} \in \mathcal{K}$, such that $d(\bar{x},\mathbb{K}(\bar{x})) \le \alpha$, and $x^\star \in T(\bar{x})$ such that

$$(w,\alpha) - QVI(T,\mathbb{K}) \qquad -x^\star \in w - N^{\alpha}_{\mathbb{K}(\bar{x})}(\bar{x}).$$

Here the notation (s,α) underlines that the solutions are *strong* in the sense that they should be fixed points of the map $\mathbb{K}$, while (w,α) refers to *weak* solutions because they are only approximate fixed points of $\mathbb{K}$.

Thus, for any $\alpha > 0$, we say that a point $\bar{x} \in \mathcal{K}$ is a *strong α-approximate solution* to $QVI(T,\mathbb{K})$ if, and only if, $\bar{x}$ is a solution to $(s,\alpha)-QVI(T,\mathbb{K})$, i.e., $\bar{x} \in \mathbb{K}(\bar{x})$ and there exists $x^\star \in T(\bar{x})$ such that

$$\langle x^\star, y-\bar{x}\rangle \geq -\alpha\|y-\bar{x}\|, \ \forall y \in \mathbb{K}(\bar{x}).$$

In the same way, for any $\alpha > 0$, we say that a point $\bar{x} \in \mathcal{K}$ is a *weak α-approximate solution* to $QVI(T,\mathbb{K})$ if, and only if, $\bar{x}$ is a solution to $(w,\alpha)-QVI(T,\mathbb{K})$, i.e., $\bar{x} \in \mathcal{K}$ with $d(\bar{x},\mathbb{K}(\bar{x})) \leq \alpha$ and there exists $x^\star \in T(\bar{x})$ such that

$$\langle x^\star, y-x\rangle \geq -\alpha\|x^\star\|, \ \forall y \in \mathbb{K}(x).$$

Notice that strong approximate solutions to (QVI) come from the regularization of the operator T while weak ones come rather from the regularization (in the sense of [2]) of the above solution map S of $QVI(T,\mathbb{K})$.

2. Lower semicontinuous regularization for set-valued maps

Let X and Y be two normed vector spaces whose norm are denoted by $\|\cdot\|$. We denote by B_ρ (resp. B_ρ) the open ball centered at 0 with radius ρ in any of the spaces X,Y. The symbol $\mathcal{V}(x)$ denotes the filter of neighborhoods of x.
Let us consider a set-valued map $S : X \rightrightarrows Y$. When $S(x) \neq \emptyset$ we say that x is in the domain of S i.e., $x \in \mathrm{Dom(S)}$.
The lower limit of S at a point $\bar{x} \in \mathrm{Dom}\, S$ is defined as follows:

$$\liminf_{x\to\bar{x}} S(x) = \{y \in Y \mid \forall x_n \to \bar{x}, \exists y_n \to y, \ y_n \in S(x_n), \forall n \geq n_0 \text{ for some } n_0\},$$

Note that, if $\bar{x} \notin \mathrm{Dom(S)}$, then $\liminf S(\bar{x}) = \emptyset$.
A set-valued map $S : X \rightrightarrows Y$ is said to be lower semicontinuous at $\bar{x} \in \mathrm{Dom}\,(S)$ if

$$S(\bar{x}) \subseteq \liminf_{x\to\bar{x}} S(x).$$

If $\bar{x} \notin \mathrm{Dom}\,(\mathrm{S})$, then one considers that Γ is automatically l.s.c. at $\bar{x}$.
The upper limit of S at a point $\bar{x} \in \mathrm{Dom}\, S$ is defined as follows:

$$\limsup_{x\to\bar{x}} S(x) = \{y \in Y \mid \exists x_n \to \bar{x}, \exists y_n \to y, \ y_n \in S(x_n), \forall n \geq n_0 \text{ for some } n_0\}.$$

A set-valued map $S : X \rightrightarrows Y$ is said to be upper semicontinuous at $\bar{x} \in \mathrm{Dom}\,(S)$ if

$$\limsup_{x\to\bar{x}} S(x) \subset S(\bar{x}).$$

For a discussion on lower and upper semicontinuity, continuity as well as closedness for set-valued maps we refer to [6].
The lower limit $\liminf S : X \rightrightarrows Y$ is not a lower semicontinuous set-valued map, in general. If $Y = \mathbb{R}$ and S is convex-valued, $\liminf S$ inherits this property. Other examples and counter-examples are exposed in [2].
To recall the set-valued lower semicontinuous regularization presented in [2], we need to fix some notations.
First, the space Y will be partially ordered by a closed, convex (not necessarily pointed) cone C with nonempty interior. Then, for a set-valued mapping $S : X \rightrightarrows Y$ and $\varepsilon \geq 0$, we consider $S^{\varepsilon} : X \rightrightarrows Y$ defined for each $x \in X$ by $S^{\varepsilon}(x) = S(x) - C_{\varepsilon}$, where $C_{\varepsilon} := C \cap \bar{B}_{\varepsilon}$. We use the convention $\emptyset - C_{\varepsilon} = \emptyset$, so $\mathrm{Dom}\, S = \mathrm{Dom}\, S^{\varepsilon}$. Let us introduce $L_S^{\varepsilon} : X \rightrightarrows Y$ as a set-valued map defined for each $\bar{x} \in X$ by:

$$L_S^{\varepsilon}(\bar{x}) = \liminf_{x \to \bar{x}} S^{\varepsilon}(x). \tag{6}$$

If $\bar{x} \notin \mathrm{Dom} S$, $L_S^{\varepsilon}(\bar{x}) = \emptyset$. We recall a result in [2], where it is proved that L_S^{ε} satisfies a property weaker than lower semicontinuity; indeed, for every $\varepsilon \geq 0$, $\eta > 0$, one has:

$$L_S^{\varepsilon}(\bar{x}) \subseteq \mathrm{liminf}_{x \to \bar{x}} L_S^{\varepsilon+\eta}(x), \quad \forall \bar{x} \in X.$$

Let α be a positive number. We introduce the set-valued map $R_S^{\alpha} : X \rightrightarrows Y$, given in [2] by:

$$R_S^{\alpha}(\bar{x}) := \mathrm{cl}\left(\bigcup_{\mu \in [0,\alpha)} \liminf_{x \to \bar{x}} L_S^{\mu}(x) \right). \tag{7}$$

The map R_S^{α} has been proved in [2] to be lower semicontinuous, and it is called lower semicontinuous regularization or (C, α)-regularization of S.

Remark 2.1. If the map S is convex-valued then for any positive real number α, L_S^{μ} is convex-valued for all $\mu \in [0, \alpha)$. Therefore, for all $\bar{x} \in Dom S$, the associated regularization R_S^{α} at $\bar{x}$ is convex as a directed union of convex sets (see also the proof of [2, Theorem 5.3]).

The following lemma will be useful later.

Lemma 2.2. *Let $S : X \rightrightarrows Y$ be a given map and $\bar{x} \in \mathrm{dom}(S)$. If one of the following conditions is satisfied:*

1. *$S(\bar{x})$ is compact,*

2. *Y is reflexive and $S(\bar{x})$ is weakly closed,*

3. $S(\bar{x})$ is convex, closed, locally compact,

then, for every $\alpha > 0$,

$$\liminf_{x \to \bar{x}} S(x) \subset R_S^{\alpha}(\bar{x}) \subset S(\bar{x}) - C_{\alpha}.$$

Proof. The second inclusion comes directly from [2, Proposition 4.5], while the first one follows easily from the definition of R_S^{α}, which ensures that

$$\liminf_{x \to \bar{x}} S(x) = L_S^0(\bar{x}) \subseteq \liminf_{x \to \bar{x}} L_S^{\eta}(x) \subseteq R_S^{\alpha}(\bar{x}),$$

completing the proof.

□

3. Existence of strong α-approximate solutions for (QVI)

We start with the material needed for our existence result.

Theorem 3.1. *(Michael's selection theorem). Let X be a compact metric space, and Y be a Banach space. Let $F : X \rightrightarrows Y$ be a lower semicontinuous map with (nonempty) closed, convex values. Then F admits a continuous selection.*

Definition 3.2. Let K be a nonempty convex subset of X. A real-valued function $g : K \longrightarrow \mathbb{R}$ is said to be *quasiconvex* if for all $x, y \in K$,

$$g(tx + (1-t)y) \leqslant \max\{g(x), g(y)\}, \qquad \forall t \in [0,1].$$

A function $g : K \longrightarrow \mathbb{R}$ is said to be *quasiconcave* if $-g$ is quasiconvex.

In mathematics, and in particular game theory, Sion's minimax theorem is a generalization of John von Neumann's minimax theorem, named after Maurice Sion. It states:

Theorem 3.3. *Let X be a compact convex subset of a linear topological space and Y a convex subset of a linear topological space. Let f be a real-valued function on $X \times Y$ such that*

i) $f(x, \cdot)$ upper semicontinuous and quasiconcave on Y, $\forall x \in X$;

ii) $f(\cdot, y)$ is lower semicontinuous and quasiconvex on X, $\forall y \in Y$.

Then,

$$\min_{x \in X} \sup_{y \in Y} f(x,y) = \sup_{y \in Y} \min_{x \in X} f(x,y).$$

Theorem 3.4. *(Closed graph Theorem). Let B a compact subset of $K \subset X$ and $F : K \rightrightarrows X$ a set-valued map such that $F(K) \subset B$. If F is closed-valued, then it is closed if, and only if, it is upper semicontinuous.*

Proof. Let $x \in K$. Since $F(x)$ is closed then it is a compact (in view of the compactness of B). Hence, if F is closed at x then F is compact at x (in the sense of [6, (iv) Definition 2.5.7, pp 55]). Thus, from [6, (i) Proposition 2.5.9, pp 56], it follows that F is upper semicontinuous at x. The converse assertion comes also from [6, (i) Proposition 2.5.9, pp 56]. This ends the proof.

□

Theorem 3.5. *Let $\mathcal{K}$ be a convex compact and nonempty subset of X, $f : \mathcal{K} \to X^\star$ be a single-valued map and $\mathbb{K} : \mathcal{K} \rightrightarrows \mathcal{K}$ be a set-valued map. Assume that the following conditions hold:*

i) f is continuous;

ii) $\mathbb{K}$ is closed and lower semicontinuous with nonempty, convex and compact values.

Then $QVI(f,\mathbb{K})$ admits a solution.

Proof. It follows immediately from [4, Theorem 3.1], by setting $T(x) = \{f(x)\}$.

□

To state our main result we consider a closed convex cone C of $X^\star$, which we suppose with a nonempty interior.

Theorem 3.6. *Let $\mathcal{K}$ be a convex compact and nonempty subset of X and $T : \mathcal{K} \rightrightarrows X$ be a set-valued map satisfying the following conditions:*

i) $T(x)$ is nonempty, convex and compact, for all $x \in \mathcal{K}$;

ii) $\liminf_{y \to x} T(y) \neq \emptyset$ for every $x \in \mathcal{K}$;

iii) $\mathbb{K} : \mathcal{K} \rightrightarrows \mathcal{K}$ is closed and lower semicontinuous with nonempty, convex and compact values.

Then $QVI(T,\mathbb{K})$ admits a strong α-approximate solution for every $\alpha > 0$.

Proof. We show that Michael's selection theorem can be applied to the set-valued map $R_T^\alpha : \mathcal{K} \rightrightarrows X^\star$. Indeed, by [2, Theorem 4.4], R_T^α is l.s.c., and, from ii) and Lemma 2.2, its domain is the whole of $\mathcal{K}$. Moreover, it is closed-valued, by construction, and convex valued, by i). Thus, for all $\alpha > 0$, there exists a continuous function $f_\alpha : \mathcal{K} \longrightarrow X^\star$ such that, for all $x \in \mathcal{K}$, $f_\alpha(x) \in T(x) - C_\alpha$.

By Theorem 3.5, we obtain the existence of a solution to the single-valued quasi-variational inequality $QVI(f_\alpha,\mathbb{K})$ denoted by x_α, which is in turns a strong α-approximate solution to the set-valued quasi variational inequality $QVI(T,\mathbb{K})$. Indeed, $x_\alpha \in \mathbb{K}(x_\alpha)$ and

$$\langle f_\alpha(x_\alpha), y - x_\alpha\rangle \geq 0,\ \forall y \in \mathbb{K}(x_\alpha).$$

Therefore, there exists $x^\star_\alpha \in T(x_\alpha)$ and some $c \in C_\alpha$ such that

$$\langle x^\star_\alpha - c, y - x_\alpha\rangle \geq 0,\ \forall y \in \mathbb{K}(x_\alpha).$$

Thus

$$\langle x^\star_\alpha, y - x_\alpha\rangle \geq \langle c, y - x_\alpha\rangle \geq -\alpha\|y - x_\alpha\|,\ \forall y \in \mathbb{K}(x_\alpha).$$

$\square$

Now, under the following regularity condition (c_6) on T : For all $x,y \in \mathcal{K}$, for all $x_\alpha \longrightarrow x$, for all $y_\alpha \longrightarrow y$,

$$\liminf_{\alpha\to 0} \sup_{x^\star_\alpha \in T(x_\alpha)} \langle x^\star_\alpha, y_\alpha - x_\alpha\rangle \geq 0 \Longrightarrow \sup_{x^\star \in T(x)} \langle x^\star, y - x\rangle \geq 0,$$

we are able to obtain an exact solution to $QVI(T,\mathbb{K})$ as follows:

Theorem 3.7. *Assume that (c_6) and the conditions of Theorem 3.6 are fulfilled. Then $QVI(T,\mathbb{K})$ admits a solution.*

Proof. From Theorem 3.6 there exists a net $(x_\alpha)_\alpha$ of strong α-approximate solutions to $QVI(T,\mathbb{K})$. Thanks to the compactness of $\mathcal{K}$, this net admits a subnet, also denoted by (x_α), converging to some point $\bar{x}$. Given that $\mathbb{K}$ is closed and x_α is a fixed point of $\mathbb{K}$, it follows that $\bar{x}$ is a fixed point of $\mathbb{K}$. Now, let $y \in \mathbb{K}(\bar{x})$. Since $\mathbb{K}$ is lower semicontinuous, there exists a net $(y_\alpha)_\alpha$, with $y_\alpha \in \mathbb{K}(x_\alpha)$ converging to y. By the definition of $(x_\alpha)_\alpha$, for all $\alpha > 0$, there exists $x^\star_\alpha \in T(x_\alpha)$ such that

$$\langle x^\star_\alpha, y_\alpha - x_\alpha\rangle \geq -\alpha\|y_\alpha - x_\alpha\|.$$

Therefore, the condition (c_6) leads to

$$\sup_{x^\star \in T(\bar{x})} \langle x^\star, y - \bar{x}\rangle \geq 0.$$

Hence, apply the Sion's minimax theorem and see that

$$\inf_{y \in K(\bar{x})} \sup_{x^\star \in T(\bar{x})} \langle x^\star, y - x\rangle = \sup_{x^\star \in T(\bar{x})} \inf_{y \in K(\bar{x})} \langle x^\star, y - x\rangle.$$

Since the function $x^\star \mapsto \inf_{y\in K(\bar{x})}\langle x^\star, y-\bar{x}\rangle$ is upper semicontinuous, and $T(\bar{x})$ is compact, there exists $\overline{x^\star} \in T(\bar{x})$ such that

$$\langle \overline{x^\star}, y-\bar{x}\rangle \geq 0, \ \ \forall y \in K(\bar{x}),$$

which means that $\overline{x^\star}$ is a solution of $QVI(T,\mathbb{K})$.

□

Remark 3.8.

1. The condition (c_6) is inspired from the dual lower semicontinuity of T on $\mathcal{K}$ in the sense of [5, Definition 3.1] (see also [4]) i.e., for all $x,y\in\mathcal{K}$ and all $(y_n)\subset\mathcal{K}$ such that $y_n \longrightarrow y$,

$$\liminf_n \sup_{y_n^*\in T(y_n)} \langle y_n^*, x-y_n\rangle \leq 0 \Longrightarrow \sup_{y^*\in T(y)} \langle y^*, x-y\rangle \leqslant 0. \tag{8}$$

   Indeed, (c_6) is implied by the following condition which may be called by similarity *dual upper semicontinuity*: For all $x,y\in\mathcal{K}$ and all $(y_n)\subset\mathcal{K}$ such that $y_n \longrightarrow y$,

$$\limsup_n \sup_{y_n^*\in T(y_n)} \langle y_n^*, x-y_n\rangle \geq 0 \Longrightarrow \sup_{y^*\in T(y)} \langle y^*, x-y\rangle \geq 0. \tag{9}$$

   Observe that:

   - Any lower semicontinuous map on $\mathcal{K}$ is dually lower semicontinuous on $\mathcal{K}$ but the converse is not true as shows the counter-example provided in [5].
   - Similarly to the previous claim of [5], any upper semicontinuous map on $\mathcal{K}$ is dually upper semicontinuous on $\mathcal{K}$ but the converse is not true as shows the following counter-example: Take $\mathcal{K}=[-1,1]$ and $T:\mathcal{K}\rightrightarrows\mathcal{K}$ defined by :

$$T(x)=\begin{cases}\{-1,1\} & \text{if } x=0\\ \{-1,0,1\} & \text{otherwise.}\end{cases}$$

     Clearly, for all $x,y\in\mathcal{K}$, $\sup_{y^*\in T(y)}\langle y^*, x-y\rangle = |y-x|$. Then, T is dually upper semicontinuous on $\mathcal{K}$. However, T is not upper semicontinuous at 0 since for any sequence (x_n) converging to 0 with $x_n\neq 0$ for all n, $T(x_n)=\{-1,0,1\}$ for all n, which implies that $0\in\limsup_{x\to 0} T(x)$ but $0\notin T(0)$.

2. If $T : \mathcal{K} \longrightarrow X^\star$ is a single-valued map such that T is strongly-weakly* continuous $\Big($i.e., for all (x_n) strongly converging to some $x \in X$, $(T(x_n))$ converges to $T(x)$ with respect to the weak star topology of $X^\star\Big)$ then T is by the meantime dually lower semicontinuous and dually upper semicontinuous.

Remark 3.9. In Theorem 3.6 we assume that $\mathbb{K}(\mathcal{K}) \subset \mathcal{K}$, then in view of Theorem 3.4, with the closeness of the values of $\mathbb{K}$, $\mathbb{K}$ is closed if, and only if, it is upper semicontinuous.

4. Conclusion and a further research question

We have seen that the regularization introduced and studied in [2] suggests to define two concepts of approximate solutions to quasi-variational inequalities in strong and weak format by means of strong and weak approximation of the normal cone to a closed and convex subset D of the considered space: $s - N_D^\alpha(\bar{x})$ and $w - N_D^\alpha(\bar{x})$. While $s - N_D^\alpha(\bar{x})$ is known in the literature (see [10]), $w - N_D^\alpha(\bar{x})$ seems to be new and needs more analysis to give it a geometrical meaning in a forthcoming research. Then the next step is to investigate weak approximate solutions to quasi-variational inequalities (with non necessarily semicontinuous operators) as well as their convergence to an exact solution similarly to the above results on strong ones.

Acknowledgements

The authors wish to thank an anonymous referee for his careful reading, useful remarks and constructive comments which helped them to improve the paper.

REFERENCES

[1] M. Ait Mansour, R-.A, Elakri and M. Laghdir, *Equilibrium and quasi-equilibrium problems under φ-quasimonotonicity and φ-quasiconvexity*, Minimax Theory and its Applications 2016 (On line).

[2] M. Ait Mansour, M. Durea and M. Théra, *A lower semicontinuous regularization for set-valued maps and its applications*, J. Conv. Analysis. 15 (2008) 473-484.

[3] M. Ait Mansour and L. Scrimali, *Hölder continuity of solutions to elastic traffic network*. J. Global Optimization. 40 (2008) 175–184.

[4] D. Aussel and J. Cotrina, *Quasimonotone Quasivariational inequalities: Existence Results and Applications*, J. Optim. Theory. Appl. 158 (2013) 637-652.

[5] D. Aussel and J. Cotrina, *Stability of quasimonotone variational inequalities under sign-continuity*. JOTA. 158 (2013) 653–667.

[6] A. Göpfert, H. Riahi, C. Tammer and C. Zălinescu, (2003), *Variational methods in partially ordered spaces*. Springer-Verlag, New York.

[7] M. Fukushima and J-S. Pang, *Quasi-variational inequalities, generalized Nash equilibria, and multi-leader-follower games*, *Computational Management Science*, 1 (2005) 21-56.

[8] J-L. Joly and U. Mosco *A propos de l'existence et de la régularité des solutions de certaines inéquations quasi-variationnelles,* J. Func. Anal. 34 (1979) 107-137.

[9] M. Kunze and J.F. Rodrigues, 2000. *An elliptic quasi-variational inequality with gradient constraints and some of its applications,* Math. meth. Appl. Sci 897-908.

[10] B.S. Mordukhovich, *Variational analysis and generalized differentiation. I: Basic Theory*, (2006) Springer-Verlag.

M. AIT MANSOUR
Département de Mathématiques et informatiques
Faculté polydisciplinaire, Safi, Université Cadi Ayyad, Morocco
e-mail: ait.mansour.mohamed@gmail.com

J. LAHRACHE
Département de Mathématiques
Faculté des Sciences, Université Chouaib Doukkali, B.P 20, El Jadida Morocco
e-mail: jaafarlahrache@yahoo.fr

N.- E. ZIANE
Département de Mathématiques
Faculté des Sciences, Université Chouaib Doukkali, B.P 20, El Jadida Morocco
e-mail: jaafarlahrache@yahoo.fr